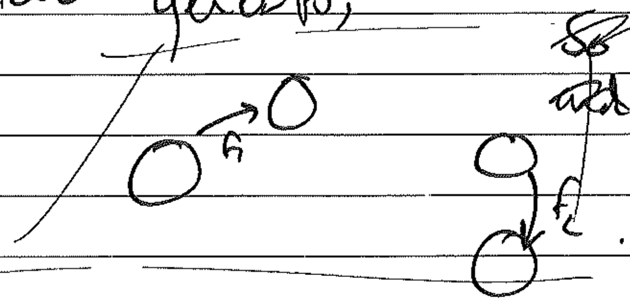
Rmk. All manifolds are smooth & oriented.

§1 Incompressible surfaces.

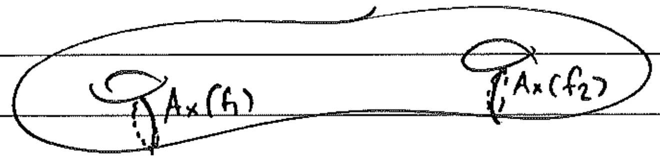
Consider a classical Schottky gp Γ on two generators,



~~So f_1, f_2 are loxo.~~

and So $\Gamma = \langle f_1, f_2 \rangle = \langle f_1 \rangle * \langle f_2 \rangle$, and Γ is purely loxodromic.

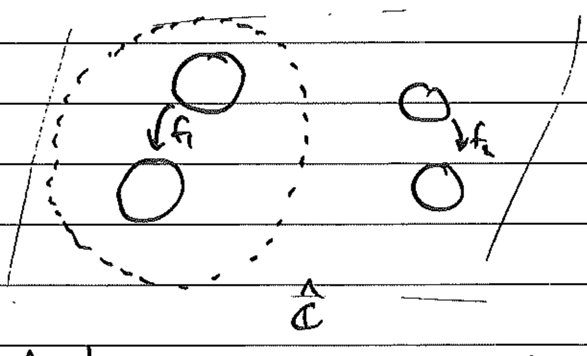
The quotient manifold is a handlebody:

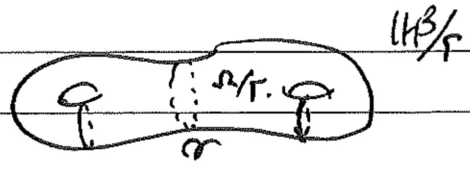


Hence $\pi_1(\mathbb{H}^3/\Gamma)$ is free on two generators, but $\pi_1(\Omega(\Gamma)/\Gamma)$ is $\pi_1(S_{2,0})$ which is not free. Hence the induced map

$$\rho: \pi_1(\Omega(\Gamma)/\Gamma) \to \pi_1(\mathbb{H}^3/\Gamma)$$

has nontrivial kernel:





$\rho(\gamma) = 1$.

Definition. An orientable surface S in an orientable 3-manifold M is incompressible if it is π_1-injective, i.e. $\pi_1(S) \to \pi_1(M)$ is mono, and if $\pi_1(S) \neq 1$.

Eg. So ~~[illegible]~~ [illegible] surfaces of $\Omega(\Gamma)/\Gamma$ are compressible.

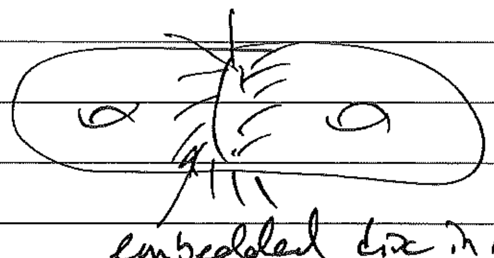
Thm. (Loop theorem, stated by Stallings, see Hempel Thm. 4.2).

Let M be a 3-manifold & $F \subset \partial M$ a connected 2-manifold. If $N \trianglelefteq \pi_1(F)$ and $\ker(\pi_1(F) \to \pi_1(M))$ ~~does not contain~~ is not contained in N, then there is a proper embedding
$$g : (\mathbb{B}^2, \partial \mathbb{B}^2) \hookrightarrow (M, F)$$
s.t. $g|_{\partial \mathbb{D}^2} \notin N$.

Cor. $(N = 1)$:
If $\ker(\pi_1(F) \to \pi_1(M))$ is nontrivial, then there is an embedded disc D in M with $\partial D \subseteq F$ nontrivial.

Prp. A surface $S \hookrightarrow M$ is incompressible iff it is not S^2 or D^2, and for every embedded disk $D \hookrightarrow M$ with $\partial D \subset F$, ∂D bounds a disc on F.

In our example:



Proof: "⇒" Suppose $S \hookrightarrow M$ is a surface which is not S^2 or D^2, and there exists an embedded disc $D \hookrightarrow M$ that has $\partial D \subset F$ but ∂D does not bound a disc. Then ∂D represents a nontrivial loop in $\pi_1(F)$ which is contractible in M along D. So $\pi_1(F) \to \pi_1(M)$ is not mono.

"⇐" Suppose $\ker(\pi_1(F) \to \pi_1(M))$ is nontrivial. Then by the loop thm, there is an embedded disc w/ the desired properties. □.

Rmk. We can generalize our discussion of the Schottky gp as follows. Suppose Γ is torsion free Kleinian, and $\Omega(\Gamma)/\Gamma$ is compressible in $\mathbb{H}^3/\Gamma$. Then the disc D certifying compressibility is incompressible, and

either $\Gamma = \Gamma_1 * \Gamma_2$ (if D separates $\mathbb{H}^3/\Gamma$, and Γ_1, Γ_2 are the π_1 of the two pieces)

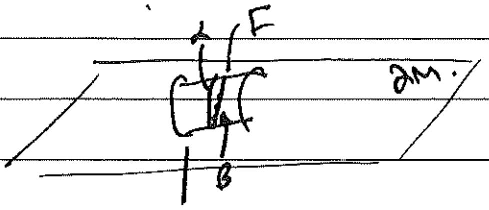
or $\Gamma = \Gamma_0 *_{\mathbb{Z}}$ (if D does not separate.)

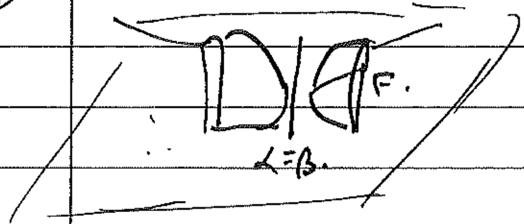
Cf. M-T Theorem 2.58.

Defn. Let F be a surface with boundary in M. F is boundary incompressible if for all discs D in M cobounded by $\alpha \cup \beta$ s.t. $\alpha = D \cap F$, $\beta = D \cap \partial M$, there is some arc γ in ∂F s.t. $\alpha \cup \gamma$ cobounds a disc in F.

(i.e. if whenever you can compress the surface across D, this is just compression along an boundary comp. up to homotopy.)

compressible

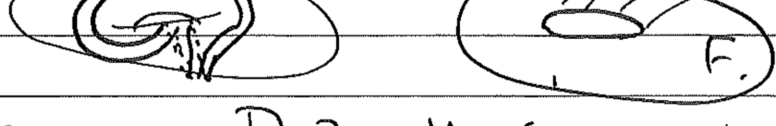




VS.

incompressible.

Let $k = T(p,q)$ be a torus knot with $|p|>1$, $|q|>1$, take $F = T^2 \setminus k$ in $S^3 \setminus k$. Then F is bdy incompressible.



Indeed suppose D is a bdy compression disc. Then ∂D lies on $F \cup \partial S^3(k) = T^2$. So D lies on one side of T^2 only. Hence D is either bdd by a meridian or a longitude of T^2. But we can't decompose a meridian and a longitude into a single arc of k and a single arc in F.

($\partial F \subset \partial M$)

Defn. A properly embedded surface $F \subseteq M$ is essential if one of the following holds:

- F is a 2-sphere that does not bound a 3-ball
- ~~F is a disc and either $\partial F \subset \partial M$ does not bound a disc on ∂M or ∂F does bound a disc E in ∂M but E~~
- F is a disc such that if $\partial F \subseteq \partial M$ bounds a disc E on ∂M then $E \cup F$ bounds a 3-ball in M;
- F is not a disc, sphere, is incompressible, bdry incompressible, & not bdry parallel.

Thm A manifold that contains an embedded essential torus cannot be hyperbolic.

Proof: not covered.

Cor. ~~Ex.~~ A satellite knot is not hyperbolic.

Thm. A manifold that has any torus bdry components (but is not $T^2 \times I$ or a solid torus) and which contains an essential embedded annulus A is not hyperbolic.

Pf. Suppose M is hyperbolic. The core curve γ of A is isotopic to ∂M. By hyperbolicity, γ is isotopic to a cusp. In particular γ is rep. by a parabolic elt of $\pi_1(M)$. We see both bdry cpts of the annulus are rep. by the same parabolic, hence A is ~~[illegible]~~.
A isotopic to a bdry curve & not essential.

Cor: Torus knots do not have hyperbolic complement.

In fact, these are the only obstructions:

~~**Thm.** A knot complement $S^3 \setminus k$ is hyperbolic iff k is not a satellite or a torus knot.~~

Thm. A knot complement $S^3 \setminus L$ is geometric (i.e. admits a (G,X)-str. for X a Thurston geometry) iff k is not satellite. It is hyperbolic iff it is neither satellite nor a torus knot.

Seifert fibred spaces.

Defn. We endow $\widetilde{SL(2,\mathbb{R})}$ with a geometric structure.

Consider the adjoint action of $SL(2,\mathbb{R})$ on $sl(2,\mathbb{R})$. The Killing form of $sl(2,\mathbb{R})$, $B(X,Y) = 4\operatorname{tr}(XY)$, has signature $(2,1)$, and so level sets of this form are $\mathbb{H}^2$'s which are orbits of the $SL(2,\mathbb{R})$-action. Point stabilisers are $O(2) \simeq S^1$, so $SL(2,\mathbb{R}) \simeq \mathbb{H}^2 \times S^1$ and $\widetilde{SL(2,\mathbb{R})}$ is a line bundle over $\mathbb{H}^2$.

Example. The torus knot is $\widetilde{SL(2,\mathbb{R})} / \widetilde{SL(2,\mathbb{Z})}$.

(Proof: Milnor, Alg. K-theory, Ex. 1.5.2).

Defn. A trivial fibred solid torus is $S^1 \times \mathbb{D}^2$ with the trivial foliation $(S^1 \times \{x\})_{x \in \mathbb{D}^2}$. A fibred solid torus is a solid torus with an S^1-foliation that is finitely covered by a trivial fibred solid torus. A Seifert fibration is a fibration of M s.t. each fibre has a nbd which is a fibred solid torus.

Thm. If $S^3 \setminus k$ (k knot) admits a Seifert fibration, then it admits an $\widetilde{SL}(2,\mathbb{R})$-geometry and a $\mathbb{H}^2 \times \mathbb{E}^1$-geometry, and neither is rigid.

Thm. - If l is a link ~~[illegible]~~ $S^3 \setminus l$ admits a Seifert fibration iff $Z(\pi_1(l)) \neq 1$. thm

- If k is a knot, then $S^3 \setminus k$ is Seifert fibred iff k is a torus knot.

Remark. We have:

~~- Hyperbolic knots ([illegible] not satellite or torus)~~

~~- torus knots (admit~~

- torus knots (admit Seifert fibrations)
- satellite knots (not geometric) (admit torus decomposition into pieces which are not satellite)
- hyperbolic knots.

$\Rightarrow$ only 3 Thurston geometries are possible.