On isometric lifts of thin parts

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(0.1) Theorem (Theorem 5.20 of [Pur20]). *There exists a universal constant $\varepsilon_3 > 0$ such that for $0 < \varepsilon \leq \varepsilon_3$, the ε-thin part of any complete, orientable, hyperbolic 3-manifold M consists of tubes around short geodesics, rank-1 cusps, and/or rank-2 cusps.*

The proof as written uses Theorem 5.22 to deduce that every point x in the thin part of $\mathbb{H}^3/G$ can be lifted to an open set $U \subset \mathbb{H}^3$ which is preserved by an elementary group $H \leq G$ generated by elements of translation length less then ε at x. This means that U/H is isometric to a tube around a short geodesic, rank-1 cusp, or rank-2 cusp. In order to show that U/G is isometric to this quotient, we must show in addition that U is precisely invariant under H (i.e. that elements of $G \setminus H$ move U off itself).

Jessica provides following proof of this (via email):

Proof (of precise invariance). We compute explicitly that the hyperbolic diameter of U is less than ε. Hence if $g \in G$ moves $y \in U$ into a point of U, then $d(y, g(y)) \leq \varepsilon$ and so g lies in the elementary group which preserves a connected component of the lift of the thin part around y. But this connected component is just U and so this elementary group is just H.

But in fact we can do without the hyperbolic geometry. Here is the full proof of Theorem 5.20 which I take from [MT98, Theorem 2.24] (actually they also state 'the connected component of the lift is precisely invariant' without proof but it is more self contained than the proof given in [Pur20] and I think is structured in a way which makes it easier for us to to deconstruct it and perform a close reading analysis on the text).

Proof. I apologise for the bad prose style, but since we are performing deconstruction of the text I will give the proof as a numbered list. The setup is as above, that is we have a manifold $M = \mathbb{H}^3/G$ where $G \leq \mathbb{M}$ is discrete.

1. For any $g \in G$, set $P(g) = \{x \in \mathbb{H}^3 : d(x, g(x)) < \varepsilon\}$ where ε is some number which is at most the Margulis constant. If g is parabolic, then $P(g)$ is a small horoball. If g is loxodromic, then either $P(g)$ is empty (if g has long translation length) or $P(g)$ is a tubular neighbourhood of the axis of g (since g translates longer and longer distances, the translation length increasing monotonically the further away you get from the axis).

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2. Let N be the lift of the ε-thin part of your manifold $\mathbb{H}^3/G$ to $\mathbb{H}^3$. We first observe that N is the union of all the $P(g)$ for $g \neq 1$. Indeed, a point x lies in N if there is a nontrivial closed curve through the projection of x to the manifold which has of length at most ε, i.e. if there is some group element g which translates x a distance less than ε, i.e. x lies in $P(g)$.

3. Let N' be a connected component of N, let $J = \mathrm{Stab}_G(N')$. We now observe that N' is precisely invariant under J. This is because N is G-invariant and G is acting by homeomorphisms: if y lies in N' and $g(y)$ lies in N' then g must send the entire connected component N' to N', i.e. g stabilises N'.

4. Observe next that every $P(g)$ is preserved by g. In particular, if $P(g)$ is a subset of N' then g preserves some subset of N' and so by (3) we have that g actually stabilises the whole of N', i.e. $g \in J$. We see therefore that N' is the union of the $P(g)$ for $g \in J$.

5. Now suppose g, h are both elements of $J \setminus \{1\}$ and $P(g)$ intersects $P(h)$ nontrivially in some point y. Then g and h both lie in the subgroup $G(y, \varepsilon) \leq G$ generated by elements which do not move the radius ε ball around y off itself, and this is an elementary group (here is where we use the Margulis lemma). Since all the $P(g)$ are open and the set N' is connected, we see that all the $g \in J$ share fixed points and so the whole group J must be elementary.

We now know that J is an elementary group and N' is precisely invariant under J, so the quotient $\mathbb{H}^3/G$ is isometric to N'/J around elements of N', and this is what we wanted to show.

The proof of precise invariance is just item (3). Observe that we do not need to do any explicit hyperbolic geometry. Observe that we do not actually need Margulis' lemma (or Jørgensen's inequality) to deduce precise invariance! The other key point is the path-connectedness argument to show that the groups $G(y, \varepsilon)$ are all equal for different elements $y \in N'$: *a priori* it could be that every point in the thin part is in a neighbourhood stabilised by some elementary group, we need to show that all of these elementary groups are the same; the Margulis lemma shows that the map $y \to G(y, \varepsilon)$ is locally constant, and since the set is path-connected we see that the map is constant on the whole of N'.

References

[MT98] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford University Press, 1998. ISBN: 0198500629 (cit. on p. 1).

[Pur20] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 9781470454999 (cit. on p. 1).