Two-bridge links and Heegaard splittings

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Abstract

We survey work of Sakuma related to two-bridge links and tunnel numbers, and some subsequent developments. We see that the main classical tools are automorphisms of Heegaard splittings and disc complexes. We attempt to find a common large-scale picture that explains why there is an isosophy between the study of simple unknotting tunnels and the representation theory of rank two Kleinian groups.

Contents

§1. Maximum in interior

(1.1) Proposition. *Let L be a 2-bridge link with at least two twist regions. Then the volume map V : $A(T)$ cannot have a maximum on $\partial A(T)$ for special choice of T as given earlier talk.*

Proof is technical and relies on following argument. First restrict tetrahedra shapes that can occur using certain algebraic restrictions. On remainder show that there exist paths from boundary into the interior along which V is strictly increasing.

§2. Unknotting tunnels

In this section we survey work of Kobayashi [Kob99], Morimoto and Sakuma [MS91], Adams and Reid [AR96], and Kuhn [Kuh96], following in part the nice survey of Sakuma [Sak98].

Let k be a link in $\mathbb{S}^3$, and let M be its complement (topological) manifold. The **tunnel number** $t(k)$ is the smallest number of properly embedded arcs in M (i.e. endpoints on k) such that the complement of a tubular neighbourhood of the arcs in M is a handlebody. We are particularly interested

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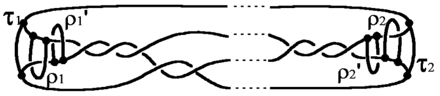
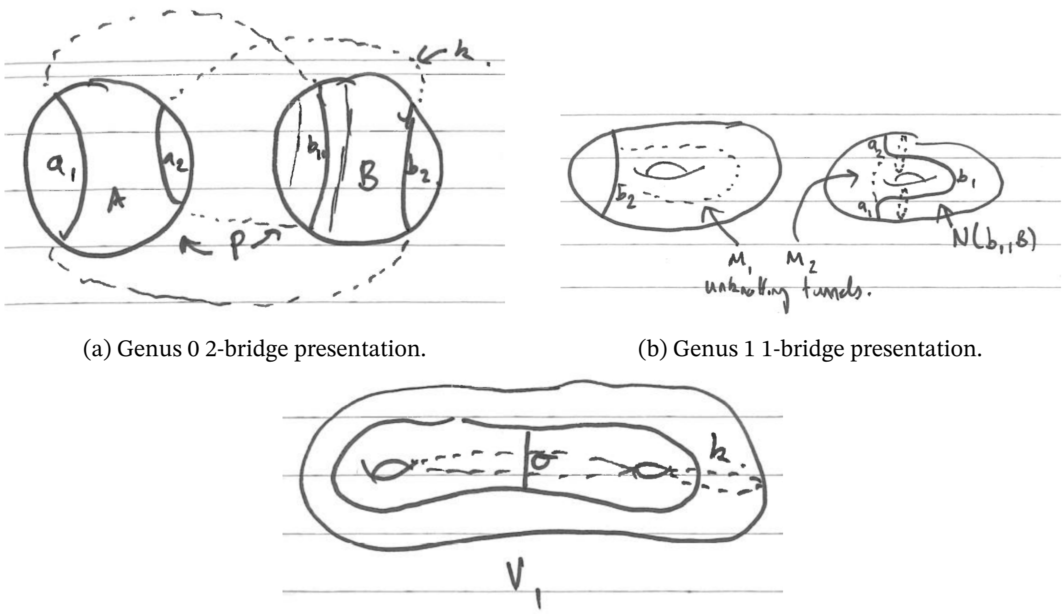


Figure 1: The six unknotting tunnels for a two-bridge knot. For a two-bridge link only τ_1 and τ_2 (the **upper and lower tunnels**) are unknotting tunnels. Figure 1.1 of [Kob99].



(a) Genus 0 2-bridge presentation.

(b) Genus 1 1-bridge presentation.

(c) Genus 2 0-bridge presentation.

Figure 2: Heegaard splittings of $\mathbb{S}^3 \setminus k$ occuring in the proof of Kobayashi's theorem

in the case $t(k) = 1$. A properly embedded arc τ in M such that $M \setminus N(\tau)$ is a handlebody is called an **unknotting tunnel**. Since the complement of a handlebody in $\mathbb{S}^3$ is also a handlebody, we see that if k admits an unknotting tunnel then it is either a knot or a two-component link with the components joined by τ.

(2.1) Theorem (Kobayashi, 1999). *Every unknotting tunnel for a non-trivial two-bridge knot is isotopic (in the knot complement) to one of the six shown in figure 1.*

Sketch of proof. Let $k \subset \mathbb{S}^3$ be 2-bridge and fix a sphere P in $\mathbb{S}^3$ which cuts the knot complement M into two Conway balls $A = \mathbb{B}^3 \setminus a_1 \cup a_2$ and $B = \mathbb{B}^3 \setminus b_1 \cup b_2$, figure 2a. Consider $A \cup N(b_1, B)$: this is a solid torus with an arc drilled out and its complement in M is also a solid torus with an arc (b_2) drilled out, figure 2b. This is called a genus 1 one-bridge presentation of k.

Given such a decomposition there are two obvious unknotting tunnels μ_1 and μ_2. In addition it is easy to see from the construction that one of the visible unknotting tunnels is τ_1 or τ_2 and the other is one of $\rho_1, \rho_1', \rho_2, \rho_2'$.

A result of Kobayashi and Saeki [KS00] is that every genus 1 one-bridge presentation of a two-bridge knot is obtained from a rational splitting as above. Hence the result is shown if we can show

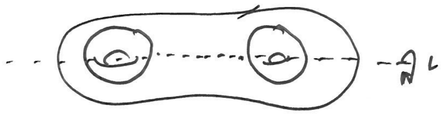


Figure 3: The hyperelliptic involution.

that every unknotting tunnel arises as the unknotting tunnel of a genus 1 one-bridge splitting, for then by the Kobayashi–Saeki theorem it is one of the six known tunnels.

Suppose k admits an unknotting tunnel σ and the induced genus two Heegaard splitting of $\mathbb{S}^3$ is $V_1 = N(k \cup \sigma)$, $V_2 = \mathbb{S}^3 \setminus V_1$, c.f. figure 2c. We say the splitting is **weak** if there exist k-compressing discs D_1 and D_2 respectively properly embedded in V_1 and V_2 with disjoint boundaries ∂D_1 and ∂D_2. If there are no such discs then the splitting is **strong**.

If the splitting is weak, it can be shown that the discs D_1 and D_2 can be choosen to be non-separating in their respective handlebodies[1] There are two possibilities now: either D_1 intersects k or it doesn't (c.f. the two dotted discs in figure 2c).

- If D_1 does not intersect k, then cut the handlebody along D_1 to give a solid torus T. Since D_2 is non-separating in V_2, ∂D_2 cannot bound a disc in T for then the union of this disc with D_2 would be a non-separating 2-sphere in $\mathbb{S}^3$. We see that ∂D_2 must be a latitude of T, because if it was more twisted then we would be able to construct a lens space inside $\mathbb{S}^2$ from a piece of V_2 and T. Hence ∂D_2 is isotopic in M to the knot k, in particular k is the boundary of an embedded disk, and k is trivial.
- If D_1 intersects k, then let N be a regular neighbourhood of D_1 in V_1. Set $T_1 = \overline{V_1 \setminus N}$, $T_2 = V_2 \cup N$. Then T_2 is a solid torus with $N \cap k$ a trivial embedded arc[2] (i.e. it is of the form the left image of figure 2b with $b_2 = N \cap k$), and $T_1 \cup T_2$ is a genus 1 one-bridge presentation for k with σ the unknotting tunnel associated with T_1 in the one-bridge presentation.

If the splitting is strong, then a detailed study of the embedded discs in V_1 (carried out in §4 of [Kob99]) shows that if σ is not isotopic to τ_1 or τ_2, then there is an essential annulus in the manifold M. Roughly speaking, the idea of this analysis is to consider the interaction between the sphere P inducing the rational decomposition and the genus two surface Q that induces the genus two Heegaard splitting arising from $k \cup \sigma$: the simple closed curves of $P \cap Q$ bound a number of discs which intersect k in various configurations and the proof proceeds by (i) induction on the number of discs and (ii) cases on the different combinatorics of the discs.

Once it is known that there is an essential annulus then the knot is a torus knot, for which the result is known due to Boileau, Rost, and Zieschang [BRZ88] (and in fact the six unknotting tunnels reduce to only three up to isotopy). Since these are not hyperbolic we ignore this case.

(2.2) Theorem (Adams and Reid, 1996; Kuhn, 1996)**.** *Every unknotting tunnel for a non-trivial two-bridge two-component link is isotopic (in the knot complement) to either τ_1 or τ_2 shown in figure 1.*

Sketch of proof. We follow the proof in [AR96] which uses hyperbolic geometry. Suppose first that the link k is not a closed 2-braid (i.e. it has at least two twist regions). Then the complement is hyperbolic. Let ι be the hyperelliptic involution of the handlebody containing $k \cup \sigma$, σ being the unknotting tunnel, figure 3. Lifting to the universal cover $\mathbb{H}^3$, the involution becomes a $\pi_1(k)$-invariant set of elliptic

[1]Claim 1 in the proof of Proposition 2.15 of [Kob99].
[2]Claim 2 in the proof of Proposition 2.15 of [Kob99].

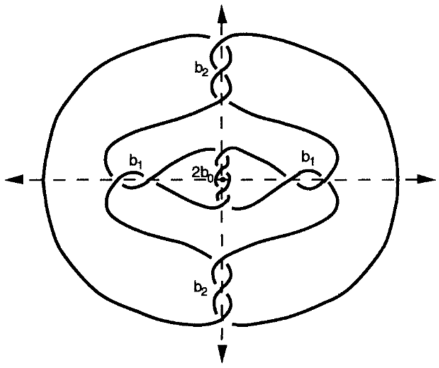


Figure 4: The symmetry group of $[2b_0, 2b_1, 2b_2]$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The hyperelliptic involution is the rotation by π around the horizontal line. Figure 1 of [AR96].

involutions of order 2 and the unknotting tunnel lifts to some subset of the axes of these involutions. This shows that every unknotting tunnel is isotopic to a geodesic which ends on the cusps.

The link k always has orientation preserving symmetry group $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, figure 4. Only one of the involutions preserves the two components. Hence the only possible unknotting tunnels are the four arcs in the knot complement which make up the axis of this unique involution. Two of these are the upper and lower tunnels. Let α and β be the other arcs, and suppose for a contradiction that α is an unknotting tunnel. Let M be the link complement. The hyperelliptic involution ι extends to the whole of $\mathbb{S}^3$; let $p : \mathbb{S}^3 \to \mathbb{S}^3$ be the quotient, and consider a small neighbourhood N of $\alpha \cup k$. By assumpution, this is a handlebody. Its complement $\mathbb{S}^3 \setminus N$ is also a handlebody. One now shows that $p(N)$ is a ball (here we use that the involution is hyperelliptic, so e.g. the image of the handlebody surface is a 6-marked sphere), hence its complement $p(\mathbb{S}^3 \setminus N)$ is a ball, and $p(\beta)$ is unknotted in this ball. This means that $p(\alpha \cup k \cup \beta)$ is unknotted (i.e. a trivially embedded genus three trivalent graph). But consideration of the diagram figure 4 shows that the quotient is actually $p(\alpha \cup k \cup \beta) = [b_0, 4b_1, b_2, 4b_3, \ldots]$ which is knotted.

§3. Kleinian groups generated by two parabolics

We now proceed to study the family of Kleinian groups $\langle X, Y \rangle$ where X and Y are parabolic with distinct fixed points. Suitable normalisation allows us to assume that our group is

$$\Gamma_\rho = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \, Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle.$$

The fundamental questions are:

1. For what ρ are these groups discrete? and,
2. When Γ_ρ is discrete, what is the isometry type of $\mathbb{H}^3/\Gamma_\rho$?

These questions have a long history (the earliest papers which I am aware of are by Sanov in 1947 [San47] and Brenner in 1955 [Bre55], see [EMS23]), but the modern point of view was initiated by Riley [Ril72; Ril75a; Ril75b; Ril13; BJS13; Ril79] who found these groups from his study of the hyperbolisation of two-bridge links. In the following, $\mathfrak{b}(q,p)$ denotes the q/p 2-bridge link.

(3.1) Proposition ([Ril72, Proposition 1]). *Fix a 2-bridge link $\mathfrak{b}(q,p)$. For any $s \neq q$ write $\bar{s}$ for the reprentative of s mod 2q in the interval $(-q, q)$. For each i set $\varepsilon_i = -\operatorname{sgn}(\overline{ip})$. Define a word $R_{p/q}$ in the* *symbols X and Y by*

$$R_{p/q} = X^{\varepsilon_1} Y^{\varepsilon_2} \cdots (X \text{ or } Y \text{ depending on } q)^{\varepsilon_{q-1}},$$

so $R_{p/q}$ is a word of length $q-1$. Then, if q is odd (so the link is a knot) we have

$$\pi_1(\mathfrak{b}(q,p)) \simeq \langle X, Y \,:\, R_{p/q}X = YR_{p/q}\rangle;$$

if q is even (so the link has two components) then

$$\pi_1(\mathfrak{b}(q,p)) \simeq \langle X, Y \,:\, R_{p/q}Y = YR_{p/q}\rangle.$$

The essence of the proof of proposition (3.1) is to compute a Wirtinger representation for the link such that two of the generators are the bridge arcs, and then eliminate all other generators. *The single relator which remains is exactly the word which represents a loop around the upper unknotting tunnel of the knot.* We denote this formal word in X and Y by $W_{p/q}$: under the actual representation $F(X,Y) \to \Gamma_\rho$, the word $W_{p/q}$ is sent to the identity.

Thus we have a large family of discrete groups Γ_ρ: for each 2-bridge link $\mathfrak{b}(q,p)$ there is some ρ (exactly four choices which give either the same knot or highly related knots) such that $W_{p/q} = I_2$, in which case the group is not only discrete but $\mathbb{H}^3/\Gamma_\rho$ is the 2-bridge link complement.[3] This family of groups all have representations

$$\langle X, Y \,:\, X^\infty = Y^\infty = W_{p/q} = 1\rangle$$

where the relation G^∞ is to be read as "G is parabolic".

Riley studied the representations of the strongly related groups

$$\langle X, Y \,:\, X^\infty = Y^\infty = W_{p/q}^n = 1\rangle,$$

where $n \in \mathbb{Z}$. Geometrically these correspond to 2-bridge knot complements where the unknotting tunnel is replaced by a cone arc (with endpoints on the knot) with cone angle $2\pi/n$: the element $W_{p/q}$ is an elliptic of order n. He called these groups **Heckoid groups** [Ril92].

As $n \to \infty$ the cone angle decreases to zero, and in the limit we obtain free groups

$$\langle X, Y \,:\, X^\infty = Y^\infty = W_{p/q}^\infty = 1\rangle$$

where $W_{p/q}$ has gone parabolic. When these groups are discrete they correspond to **cusp groups**, where the unknotting tunnel is deleted and replaced by a rank one cusp (the vertices where it meets the knot are thrice-punctured spheres). When ρ is increased further, $W_{p/q}$ becomes loxodromic and the manifold $\mathbb{H}^3/\Gamma_\rho$ is homeomorphic to a 3-ball with two drilled arcs.

The geometric procedure is shown in figure 5.

In 2002 Agol [Ago02] sketched an incomplete proof of the following theorem; two complete proofs were given by Aimi, Lee, Sakai, and Sakuma [Aim+20], and Akiyoshi, Ohshika, Parker, Sakuma, and Yoshida [Aki+21].

(3.2) Theorem. *A non-free non-Fuchsian Kleinian group G is generated by two non-commuting parabolic elements if and only if one of the following holds:*

1. *G is conjugate to some hyperbolic 2-bridge link group; or*
2. *G is conjugate to the Heckoid group $\langle X, Y \,:\, W_{p/q}^n = 1\rangle$ for some $p/q \in \mathbb{Q}$ and some $n \in \mathbb{Z}_{>1}$; or*

[3] It is a theorem of Riley which can be found in [Ril72; Ril75b] and [KAG86, Problem 86] that so long as a knot group representation is faithful and has parabolics in the correct places then it is actually giving the correct action on $\mathbb{H}^3$, for some examples and applications see [KAG86, Examples 59, 60].

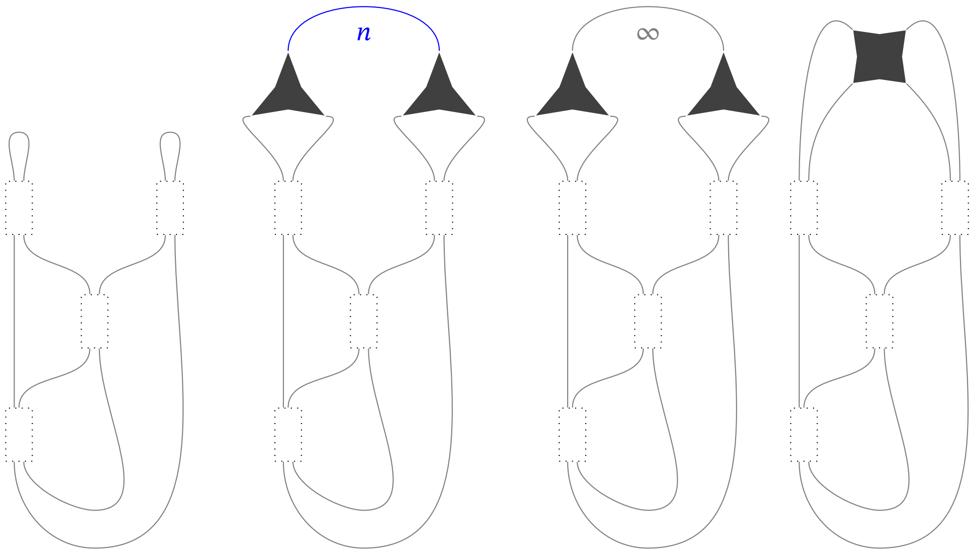


Figure 5: The four kinds of orbifolds found along an extended pleating ray. Left to right: a 2-bridge link or knot complement; a Heckoid orbifold; a cusp group; and a Riley group. In the elliptic case, all can occur and the two cone orders a and b correspond to the two arcs of the link separated by the unknotting tunnel. If $a \neq b$ and the ray corresponds to a knot then the procedure must stop at the Heckoid group with unknotting tunnel a cone arc of order 2. From [EMS24].

3. *G is conjugate to the orbifold holonomy of a quotient of a Heckoid manifold by order two involutions of the p/q-knot.*

If G is a hyperbolic 2-bridge link group then it has exactly two parabolic generating pairs, up to conjugacy. If G is a Heckoid group then it has a unique parabolic generating pair up to conjugacy.

In other words, all the non-free non-Fuchsian groups on two parabolic generators arise by taking a two-bridge link, replacing the upper unknotting tunnel with a singular locus of some order, and possibly taking an order two quotient as in theorem (2.2).

Remark. The characterisation of groups Γ_ρ with quotients cusped finite-volume manifolds as hyperbolic 2-bridge link groups is due to Adams [Ada96]. The Fuchsian groups generated by two non-commuting parabolics are fully classified by Knapp [Kna68]. The case that X and Y are allowed to be finite order is qualitatively very similar, and has been fully studied by Chesebro, Martin, and Schillewaert [CMS24].

The Kleinian groups that are *freely* generated by parabolic elements are called **Riley groups** and have been studied in detail by a wide range of people as they are the easiest example of quasi-Fuchsian groups of the second kind. The primary sources in this direction are the work of Keen and Series [KS94], Komori and Series [KS98], and Ohshika and Miyachi [OM10]. Additional work is surveyed in [EMS23].

The study of the Riley groups is still essentially derived from the 2-bridge link structures: although an unknotting tunnel has been deleted, the data associated to the knot (encoded in the family of words $W_{p/q}$) is still visible in the conformal structure of the four-times punctured sphere on the conformal boundary. One point of view is that the conformal structure comes from the way that a braid group acts on the Conway ball to produce a rational tangle: this action extends to the bounding sphere $S_{0,4}$ and hence induces a twisting of the conformal structure. Details may be found in [Elz23, §3.1].

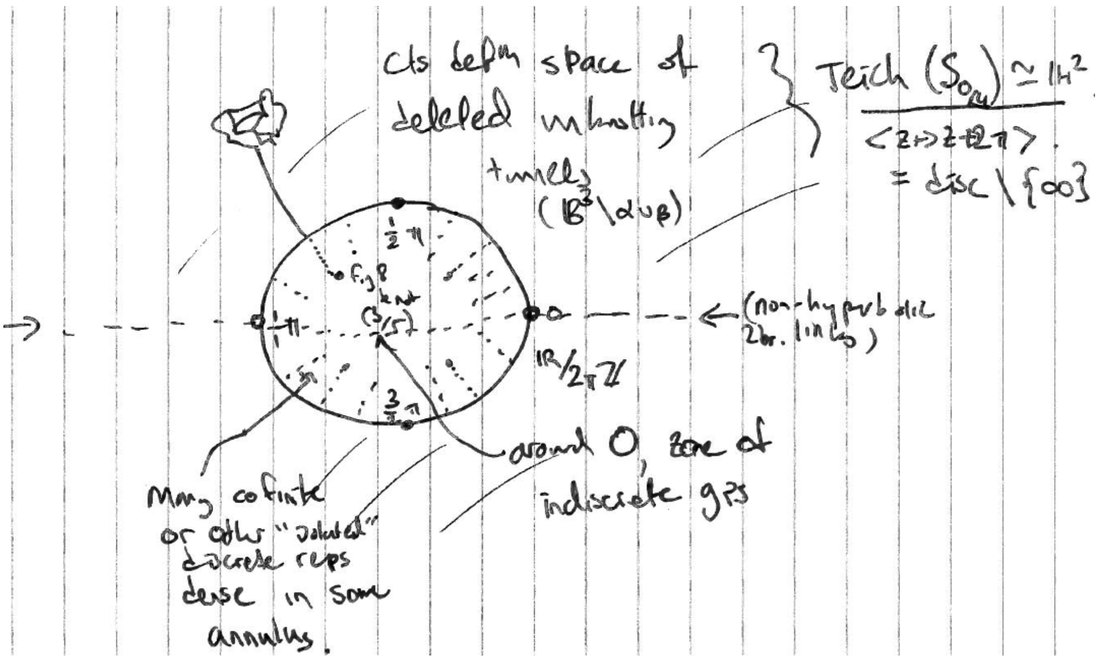


Figure 6: Rough map of the Kleinian groups generated by a pair of parabolics.

A full picture of the entire deformation space of ρ is seen in figure 6. It should be noted that this is only heuristic and in fact the boundary is highly non-circular (one can show it is not even a quasicircle).

§4. The enumeration of unknotting tunnels

Another unifying point of view, this time from 2-bridge knots to arbitrary knots with tunnel number 1, may be found in work of Cho and McCollough [CM09]. Roughly speaking their point of view is similar to the proof of theorem (2.1) outlined above: knot tunnels are detected by using complexes of embedded discs in the genus two Heegaard splitting. This point of view is philosophically isomorphic (isosophic?) to an extension of the study of upper/lower unknotting tunnels of two-bridge knots as carried out by Sakuma et. al., indeed one can view the enumeration of two-generated Kleinian groups as an enumeration of non-separating discs in genus two surface if two compressing discs have already been chosen (dual to the bridges). In [CM09] this setting is the setting of 'simple tunnels'. Just like how the isolated groups (the non-free ones, i.e. two-bridge link groups, Heckoid groups, and certain quotients) in the space of discrete representations are indexed by vertices of the Farey triangulation and permuted around semi-transitively (transitive on triangles not vertices) by $\mathrm{SL}(2,\mathbb{Z}) \approx \mathrm{Mod}(S_{0,4})$ where the indeterminacy comes from some global symmetry (the $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ group which is the symmetry group of the knot complement), in general there is a large complex of discs called the **disc complex** $\mathcal{D}(H)$ and some large group $\mathcal{G}$ acting on $\mathcal{D}(H)$ such that $\mathcal{D}(H)/\mathcal{G}$ is a complex indexing all the possible tunnels. There is even a natural spanning tree akin to the Stern-Brocot tree. One can extend this picture to define combinatorial structures on which to hang trace polynomials for general Schottky groups and general function groups.

§5. Skein algebras, character varieties, and unknotting tunnels

> A certain novice monk in Inaba was rumoured to have a beautiful daughter, and many men came asking for her hand. But the girl ate nothing but chestnuts and never touched grains, so her father declared that she was too eccentric to be marriagable, and rejected them all. [Ken13, No. 40]

> Now it dawned on me that what shocked me most—shocked me as an insult: not a word was there about our resemblance; not only was it not criticized (for instance they might have said, at least: 'Yes, an admirable resemblance, yet such and such markings show it to be not his body) but it was not mentioned at all—which left one with the impression that it was some wretch whose appearance was quite different to mine... This affected ignorance of what, to me, was most precious and all-important, struck me as an extremely cowardly trick, implying as it did that, from the very first, everybody knew perfectly well that it was not I, that it simply could not have entered anybody's head to mistake the corpse for mine. [Nab16, p. 142]

We have now observed that there is a strong link between the representation theory of Kleinian groups and the study of unknotting tunnels. It is *a priori* surprising that the classification of *all* Kleinian groups on two parabolic generators (and in fact on two *elliptic* generators, after applying some qualitatively simple deformations [EMS24; CMS24]) essentially follows from considering the nature of this class of knots.

In this section, we will show that skein theory as developed originally by Jones, Witten, Lickorish, Turaev, and many others provides a unifying theme between Heegaard splitting theory, representations of surface groups, and the combinatorics of the curve complex.

We follow first a paper of Przytycki and Skiora [PS00]. Let R be a ring (with unit 1), let $q \in R$ be invertible. Let M be a (connected oriented) 3-manifold. The **Kauffman skein module** $\mathcal{S}(M, R, q)$ is the algebra defined by taking the R-module generated freely by isotopy classes of framed links in M (for notion of a **frame** on a link see e.g. [PS97] but roughly speaking it is a choice of infinitesimally wide ribbon in M which follows the link) and quotienting by the ideal $\mathfrak{s}$ generated by the relations

$$\times = q \asymp + q^{-1})(\quad \text{and} \quad D \cup \bigcirc = -(q^2 + q^{-2})D.$$

Remark. In the important case $q = -1$, the image $D + \mathfrak{s}$ of a framed link in the skein module is independent of the framing on D.

More generally, we allow M to have boundary and we take only framed links which match some specified configuration on the boundary. This is the setting that is studied by Lickorish, Yokota, and others. In papers of Yokota in particular [MSY96; Yok95] skein theory is done on subsurfaces of $\mathbb{S}^2$ with marked points on the boundary and the reconciliation with the 3-manifold theory was slightly unclear to the author at first glance; in this matter a paper of Lickorish [Lic93] was the kindler of True Understanding.

§5.1. Optional background in character theory

For later use we relate this discussion to representation theory. We recall that a **representation** of a group G over a vector space V is a homomorphism $\rho : G \to \mathrm{Aut}(V)$. We will restrict to the setting $V = \mathbb{C}^2$ and ask for images $\rho(G)$ to lie in $\mathsf{SL}(2,\mathbb{C})$. Suppose G is finitely presented on the generators $g_1, ..., g_n$ and relators $R_1 = \cdots = R_m = 1$. Then a map $\check{\rho} : \{g_1, ..., g_n\} \to \mathsf{SL}(2,\mathbb{C})$ extends to a representation if and only if for every relator $R_j = \prod_{i=1}^{\ell_j} g_{k_j}$ the equation $\prod_{i=1}^{\ell_j} \check{\rho}(g_{k_j}) = I_2$ for all j. This sets up three polynomial equations (one entry is determined by the determinant condition) on the entries of the $\check{\rho}(g_k)$ for all relators, giving a total of $3m$ equations in a space of dimension $3n$. The set of all representations is therefore generically dimension $3(n-m)$. For example take a surface group which has $2g$ generators and one relation and look at representations in $\mathsf{SL}(2,\mathbb{R})$; we end up with final dimension $3(2g-1) = 6g-3$. This is three too many since we recall from Teichmüller theory that the correct number of real dimensions should be $6g-6$. The remaining three are lost if we mod out conjugacy in $\mathsf{PSL}(2,\mathbb{R})$.

Given a representation $\rho : G \to \mathsf{SL}(2,\mathbb{C})$ the **character** is the map $\chi_\rho : g \mapsto \operatorname{tr}\rho(g)$. If we know the value of sufficiently many characters, this gives us a number of polynomial equations in the parameters of the representation ρ. It is intuitive then that the character map should determine the representation fully. This is not always the case[4] but it is true in this setting. Again the group presentation gives us a number of polynomial equations in the entries of the images of the generators (that this is true is a consequence of the famous theorem that all traces in a finitely generated subgroup of $\mathsf{SL}(2,\mathbb{C})$ are integer polynomials in the traces of words in the generators of some bounded length, c.f. [MR03, §3.5]). The subvariety of $\mathbb{C}^{3n}$ cut out by these equations, (GIT) quotient by the conjugation action of $\mathsf{SL}(2,\mathbb{C})$, is called the **character variety** $X(G)$. The seminal work on these varieties is the work of Culler and Shalen [CS83].

It is easy to see [GMM98] that exactly three traces determine the representations of a group on two generators $\langle X, Y\rangle$: $\operatorname{tr} X$, $\operatorname{tr} Y$, and $\operatorname{tr}[X,Y]$. (We have two generators and no relations and indeed $3(2-0) = 6$ minus 3 for conjugacy gives that we should only need three traces). In the case of the knot groups above, $\operatorname{tr} X = \operatorname{tr} Y = 2$ and so we have exactly one complex dimension of freedom, $\operatorname{tr}[X,Y] = \rho^2 + 2$. (This reflects the fact that the groups Γ_ρ and $\Gamma_{-\rho}$ are indistinguishable, differing only in choice of generators).

Warning. As the audience of this note is not composed solely of algebraic geometers, we assume all things that we see are varieties even when they are blatently not (e.g. they might be non-reduced, certainly we have already taken a GIT quotient without asking theological questions). In the cases of interest to us, everything will turn out to be a variety (i.e. reduced over $\mathbb{C}$, though maybe not irreducible).

§5.2. Representation theory

Let G be a group and define $\operatorname{Ten} R[G]$ to be the tensor algebra over the group ring $R[G]$. Define an ideal $\mathfrak{u} \lhd \operatorname{Ten} R[G]$ generated by the elements

$$\mathrm{id}_G - 2, \quad g \otimes h - gh - gh^{-1}, \text{ and } g \otimes h - h \otimes g$$

where $g, h \in G$. Observe the formal similarity with the trace identities in $\mathsf{SL}_2(\mathbb{C})$

$$\operatorname{tr} I_2 = 2, \quad (\operatorname{tr} g)(\operatorname{tr} h) = \operatorname{tr} gh + \operatorname{tr} gh^{-1}, \text{ and } (\operatorname{tr} g)(\operatorname{tr} h) = (\operatorname{tr} h)(\operatorname{tr} g).$$

Set $\mathcal{S}(G,R) := \operatorname{Ten} R[G]/\mathfrak{u}$; this is the **skein module of** G. Let $[\cdot] : G \to \mathcal{S}(G,R)$ be the natural inclusion-projection map. Then $[g] = [g^{-1}]$ (c.f. $\operatorname{tr} g = \operatorname{tr} g^{-1}$). The map $\mathcal{S}(\cdot, R)$ is functorial in the

[4]for a counterexample with a linear group see `https://en.wikipedia.org/wiki/Character_variety#Variants`

sense that any group homomorphism $f : G \to H$ induces a map $f_* : \mathcal{S}(G, R) \to \mathcal{S}(H, R)$, and f_* is an epimorphism if f is an epimorphism.

(5.1) Theorem ([PS00, Theorem 2.8]). *For any 3-manifold M and any ring R there exists an isomorphism*

$$\hat{\xi} : \mathcal{S}(M, R, -1) \to \mathcal{S}(G, R)$$

such that for any knot k in M, $\hat{\xi}(k) = -[\gamma]$ where γ is an element of π_1 representing k.

(5.2) Slogan. *The classical limit of the skein algebras in a manifold M is a trace algebra of representations $\pi_1(M) \to \mathsf{SL}_2(R)$.*

The way to formalise this is via work of Brumfiel and Hilden (Contemp. Math. 1995) summarised in [PS00, §3]. For the time being let G be any group and let R be a ring. Define an ideal $\mathfrak{bh} \lhd R[G]$ generated by elements of the form

$$g(h + h^{-1}) - (h + h^{-1})g$$

for $g, h \in G$. We define the **Brumfiel–Hilden algebra** to be the R-algebra $H_R(G) = R[G]/\mathfrak{bh}$. The point of these algebras is the following observation. Let $\iota : H_R(G) \to H_R(G)$ be the involution $[g] \mapsto [g^{-1}]$. Let $\iota : M_2(R) \to M_2(R)$ be the involution $(a, b|c, d) \mapsto (d, -b| - c|a)$ (so ι restricts to inversion in the special linear group). Then any homomorphism $f : G \to \mathsf{SL}_2(R)$ extends to a homomorphism $\hat{f} : H_R(G) \to M_2(R)$ such that $\iota\hat{f} = \hat{f}\iota$, and every homomorphism $H_R(G) \to M_2(R)$ that is involution-preserving is an extension of an $\mathsf{SL}_2(R)$ representation of G in this way.

Let $TH_R(G)$ be the subalgebra of $H_R(G)$ generated by elements $[g] + [g^{-1}]$ for $g \in G$.

(5.3) Theorem ([PS00, Theorem 3.2, Corollary 3.5, Theorem 3.6, Corollary 3.7]). *The map $\psi : \mathcal{S}(G, R) \to TH_R(G)$ defined by extending $\psi([g]) = [g] + [g^{-1}]$ linearly is well-defined and an epimorphism. If one of the following holds:*

1. *G is Abelian*
2. *G is free*
3. *G is a surface group*
4. *G is a $(2, 2k + 1)$ torus knot group for $k \geq 0$*

then ψ is an isomorphism and both $\mathcal{S}(G, R)$ and $TH_R(G)$ are free R-modules. If $1/2 \in R$ then ψ is an isomorphism but we get no information about freeness.

Observe that the groups in theorem (5.3) are exactly the (torsion free) groups which can arise as representations in the Riley slice,

Fix now an oriented surface S. Later we will restrict to the two-punctured disc in analogy with the decomposition of every Riley group into a pair of Fuchsian groups uniformising such discs (equivalently decomposition of the corresponding knot-with-deleted-unknotting-tunnel into two clean pieces along a disc with either zero or two deleted points). Let $I = [-1, 1]$. We will consider $\mathcal{S}(S \times I, R, \mathscr{q})$. Observe first that these modules admit an algebra structure, where for two framed links D_1 and D_2, $D_1 \cdot D_2$ is defined to be 'embed D_1 in $S \times [0, 1]$ and D_2 in $S \times [-1, 0]$ and concatenate'. This is clearly associative but not necessarily commutative and it extends bilinearly to the whole skein module.

(5.4) Proposition ([PS00, Fact 4.1, Lemma 4.2, Corollary 4.3]). *For any surface S, any ring R, and any $\mathscr{q} \in R$ invertible:*

1. *(R-module structure) $\mathcal{S}(S \times I, R, q)$ is free over R with basis $\mathcal{B}(S)$ that is one representative from every isotopy class of link in S with no homotopically trivial components with the framing parallel to S.*

2. *(R-algebra structure) If S is not the annulus and has the d boundary components $\partial_1 S, ..., \partial_d S$, then there is a natural isomorphism of R-algebras*[5]

$$R[\partial_1, ..., \partial_d] \to \mathcal{S}(N_1 \rtimes I \cup \cdots \cup N_d \times I, R, q)$$

where each N_i is a regular neighbourhood of $\partial_i S$ in S, which is defined by sending ∂_i to the knot in N_i which is parallel to $\partial_i S$. The inclusion $N_1 \rtimes I \cup \cdots \cup N_d \times I \subset S \times I$ induces a monomorphism

$$R[\partial_1, ..., \partial_d] \to \mathcal{S}(S \times I, R, q)$$

3. *A basis for $\mathcal{S}(S \times I, R, q)$ over $R[\partial_1, ..., \partial_d]$ is the set of all links in $\mathcal{B}(S)$ which have no boundary-parallel components.*

4. *If q is not a root of unity, then the image of $R[\partial_1, ..., \partial_d]$ in $\mathcal{S}(S, R, q)$ is the centre of the algebra.*

We now come slowly to the first point. We are now in §7 of [PS00].

(5.5) Theorem. *Suppose that $\mathcal{S}(G, \mathbb{C})$ has no nilpotent elements.*[6] *In particular this is true when G:*

1. *is free non-Abelian; or*

2. *is the knot group of a 2-bridge knot.*

Then there is an isomorphism of $\mathbb{C}$-algebras $\mathcal{S}(G, \mathbb{C}) \to \mathbb{C}[X(G)]$, where $\mathbb{C}[X(G)]$ denotes the affine coordinate ring of $X(G)$, that arises from the map

$$TH_{\mathbb{C}}(G) \ni [g] + [g^{-1}] \mapsto (\chi \mapsto \chi(g)) \in \mathbb{C}[X(G)]$$

and theorem (5.3).

Let us recall some algebraic geometry in a hand-wavy way. If V is a variety over $\mathbb{C}$, then its coordinate ring $\mathbb{C}[V]$ is the ring of polynomial maps $V \to \mathbb{C}$. There is a duality whereby algebraic maps $\phi : V \to W$ translate to $\mathbb{C}$-algebra homomorphisms $\phi_* : \mathbb{C}[W] \to \mathbb{C}[V]$: if $f : W \to \mathbb{C}$ is a map then $\phi_*(f)(v) = f(\phi(v))$ is the corresponding element of $\mathbb{C}[V]$. Homomorphisms $\mathbb{C}[V] \to \mathbb{C}$ are in duality with inclusion maps of points into V, since $\mathbb{C}$ is the coordinate ring of a single point. Details may be found e.g. in [Sha13].

Now suppose $\chi : G \to \mathbb{C}$ is some $\mathsf{SL}(2, \mathbb{C})$-character of G, i.e. a point of $X(G)$. By the dualities just described, we obtain a homomorphism $\mathbb{C}[X(G)] \to \mathbb{C}$, and therefore a homomorphism $h_\chi : \mathcal{S}(G, \mathbb{C}) \to \mathbb{C}$,

$$h_\chi(g) = \chi(g).$$

Conversely every homomorphism $\mathcal{S}(G, \mathbb{C}) \to \mathbb{C}$ arises from a character of G. This really formalises slogan (5.2).

Let $\Gamma_\rho = \langle X, Y \rangle$ be free and represented in $\mathsf{SL}(2, \mathbb{C})$ with $\operatorname{tr} X = \operatorname{tr} Y = 2$. We claimed above that the geometry of $\mathbb{H}^3/\Gamma_\rho$ is dependent only on the combinatorial data $\{(p/q, \operatorname{tr} W_{p/q}) : p/q \in \mathbb{Q}\}$. It follows from the theory of Keen and Series that for each p/q there is a smooth real curve $\mathcal{P}_{p/q}$ such

[5]The right-hand algebra here is over a non-connected manifold. If $M = M_1 \coprod M_2$ then $\mathcal{S}(M, R, q) = \mathcal{S}(M_1, R, q) \otimes \mathcal{S}(M_2, R, q)$.

[6]If there are nilpotents, we don't get an affine cordinate ring, and instead we get the non-reduced coordinate ring of the character scheme.

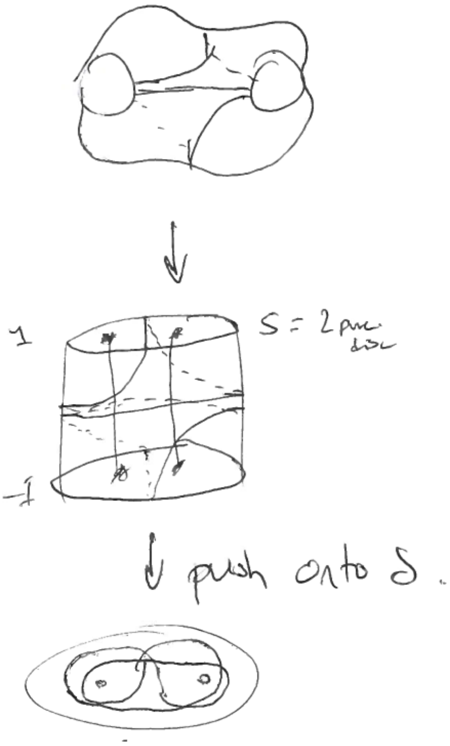


Figure 7: Pushing $\gamma_{1/2}$ onto a twice-punctured disc gives a projection of the trefoil knot (but wrong crossing choices to be a diagram).

that for $\rho \in \mathcal{P}_{p/q}$, $\operatorname{tr} W_{p/q} \in \mathbb{R}$ and the group Γ_ρ decomposes into two Fuchsian groups P_1 and P_2, each generated by two parabolics $P_1 = \langle f_1, g_1 \rangle$ and $P_2 = \langle f_2, g_2 \rangle$, with $W_{p/q} = f_1 g_1 = f_2 g_2$. Each of these groups acts on some round disc in $\Omega(\Gamma_\rho)$ to glue it up into a twice-punctured disc, c.f. §2.3 of [EMS24] for details. Let S be the topological disc with two holes. We are motivated to study $\mathcal{S}(S \times I, \mathbb{C}, q)$ since the manifold $S \times I$ is homeomorphic to $\mathbb{H}^3/\Gamma_\rho$ for all ρ in the Riley slice. Since $\Gamma_\rho \simeq \pi_1(S \times I) \simeq F(2)$ is free we may utilise all the theory above without hassle. We find that

$$\mathcal{S}(S \times I, \mathbb{C}, -1) = \mathcal{S}(F(2), \mathbb{C}) = \mathbb{C}[X(F(2))].$$

That is, formal $\mathbb{C}$-linear combinations of framed links in $S \times I$ are in bijection with maps $X(F(2)) \to \mathbb{C}$. As described above a basis for $\mathcal{S}(S \times I, \mathbb{C}, -1)$ over $\mathbb{C}$ comes from links in S with non homotopically trivial components. The corresponding maps $X(F(2)) \to \mathbb{C}$ are the maps 'write down a $F(2)$-representative in π_1 and take its character map'. Observe that the Farey words in general cannot be pushed onto a single surface S. In figure 7 we show the example of the 1/2 word.

It does not work to take the vertical framing. Does taking the frame normal to $S_{0,4}$ give the Farey word out of the formal skein algebra (we should, the simple closed curves on the disc are $[X], [Y], [XY]$)?

Anyway the Farey words do not appear *special* here. This makes sense because we are working purely topologically, we are not detecting any of the residual knot structure. We also have not got any information about discreteness. We need to therefore impose more structure than just the topological/compression disc decomposition, and it seems like looking at skein algebras related to the 2-punctured disc is somehow not carrying enough data to recover the Riley slice. Perhaps we need to

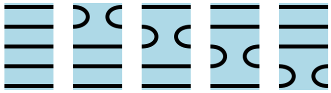


Figure 8: The unit element $1 \in R$ and the four generators of $TL_5(q)$.
Kilom691, CC BY-SA 3.0, via Wikimedia Commons.

consider all the structure carried in Teich($S_{0,4}$). Alternatively, instead of projecting to the disc perhaps we should be studying the piece of $\gamma_{p/q}$ which actually lies on the disc and fix the combinatorial structure of the intersections with the boundary—of course this only recovers q and not p.

§5.3. Unknotting tunnels

Enigmatic remarks of Sakuma [Sak98] together with hints in other parts of the literature [Elz+24; DT24; MOV24; BS22] suggest that there is a concrete pathway between the representation theory outlined above and the quantum invariants of 3-manifolds obtained from unknotting tunnels of 2-bridge links and that there should be some equivalence with the theory of closed 3-braids [Bir85]. More precisely, quantum knot invariants should be explicitly obtained in terms of the words $W_{p/q}$ via the latter's interpretation as unknotting tunnels. Sakuma points us in particular towards work of Yokota, and we begin by following [MSY96]. However, at this stage I have been unable to identify much representation theory out of this direction.

Let D_l denote an oriented disc with an even number l of distinguished points. More generally $D_{l_1+\dots+l_n}$ denotes a disc with n distinct arcs on its boundary ordered according to the orientation, each with the specified number of points. We can now study the skein algebra $\mathcal{S}(D_l, R, q)$ where we allow not just multicurves with loops in D_l but also those with framed arcs which begin and end at the specified boundary points. Yokota requires $R = \mathbb{C}$ and q to be norm one.

The **Temperley-Lieb algebra** $TL_n(q)$ is the algebra $\mathcal{S}(D_{n,n}, R, q)$ where R is commutative. It is generated by $n-1$ elements $e_1, \dots, e_{n-1}$ (shown for $TL_5(q)$ in figure 8) satisfying the relations

- $e_i^2 = q e_i$ for all $1 \leq i \leq n-1$;
- $e_i e_{i+1} e_i = e_i$ for all $1 \leq i \leq n-2$;
- $e_i e_{i-1} e_i = e_i$ for all $2 \leq i \leq n-1$;
- $e_j e_i = e_i e_j$ for all $1 \leq i, j \leq n-1$ with $|i-j| \neq 1$.

Observe the similarity with the braid relations.

We need to know that there exist such things as the **Jones-Wenzel idempotents**. These are the unique non-zero elements $f_n \in TL_n(q)$ such that

$$f_n^2 = f_n \text{ and } e_i f_n = 0 = f_n e_i \text{ for all } i.$$

The exact construction is not important, it can be found e.g. in [Yok95, p. 546]. In diagrams it is represented by a box with n inputs and outputs (of course, you line up a number of the elements e_i and 1 next to each other).

Suppose you connect f_n up to itself by n parallel strands and embed the resulting diagram into $\S^2$ (left image of figure 9). The Kauffman bracket of the result is denoted by Δ_n, and one can compute that it is

$$\Delta_n = (-1)^n \frac{q^{2(n+1)} - q^{-2(n+1)}}{q^2 - q^{-2}}$$

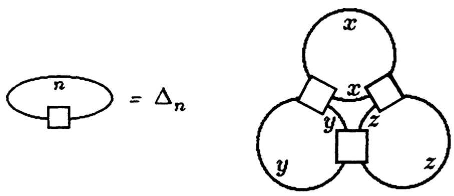


Figure 9: The diagrams for Δ_n (left) and $\Gamma(x, y, z)$ (right). From figures 2 and 3 of [Lic93].

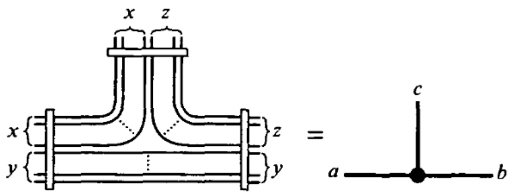


Figure 10: From weighted trivalent graphs to the Temperley-Lieb algebra. From figure 9 of [Yok95].

(this looks like the factor out the front of the Alexander polynomial of the closed braid, which is no accident). If $a, b, c \in \mathbb{Z}_{\geq 0}$ satisfy the **admissibility condition** that there exist $x, y, z \in \mathbb{Z}_{\geq 0}$ with $x + y = a$, $y + z = b$, $z + x = c$, then we can connect up three Jones-Wenzel idempotents f_a, f_b, f_c by three loops of x, y, z parallel lines (right image of figure 9). The resulting Kauffman bracket is

$$\Delta_{a,b,c} = \Gamma(x, y, z) = \frac{\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!}{\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!}$$

where $\Delta_n! = \Delta_n \Delta_{n-1} \cdots \Delta_0$ and $\Delta_{-1}! = 0$.

Now let Γ be a trivalent graph embedded in $\mathbb{S}^3$. Assign to each edge $e \in E(\Gamma)$ a weight $\omega(e)$ such that for every vertex the weights $\omega(e_1)$, $\omega(e_2)$, $\omega(e_3)$ assigned to the three incident edges e_1, e_2, e_3 satisfy the admissibility condition. If this is done then every edge e of Γ can be replaced by $\omega(e)$ parallel curves passing through a copy of f_n (figure 10). Applying the Kauffman bracket produces an invariant $Y_{\Gamma,\omega}(q)$ of Γ called the **Yamada invariant**, defined up to multiplication by $\pm q^{\pm n}$.

Suppose that $k \subset \mathbb{S}^3$ is a knot. We say that k is **strongly invertible** if there is an involution h of $\mathbb{S}^3$ which preserves k such that $\mathrm{Fix}(h)$ is a circle intersecting k in two points. (Compare with the proof of theorem (2.2).) Let $p : \mathbb{S}^3 \to \mathbb{S}^3/h = \mathbb{S}^3$ be the projection map. Set $\bigcirc = p(\mathrm{Fix}(h))$ and $\gamma = p(k)$; $\bigcirc$ is an unknot, γ is an arc which intersects $\bigcirc$ at its endpoints, and $\gamma \cup \bigcirc$ is the spine of an unknotted genus two handlebody, a θ-curve.

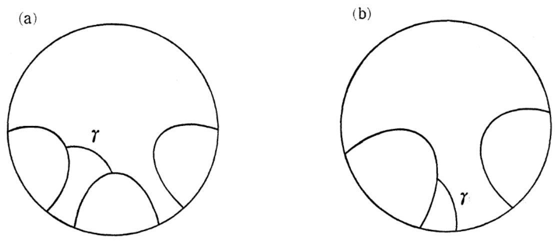


Figure 11: Left: half of a three-bridge presentation for a θ-graph. Right: half of a two-bridge presentation. From figure 1.1 of [MSY96].

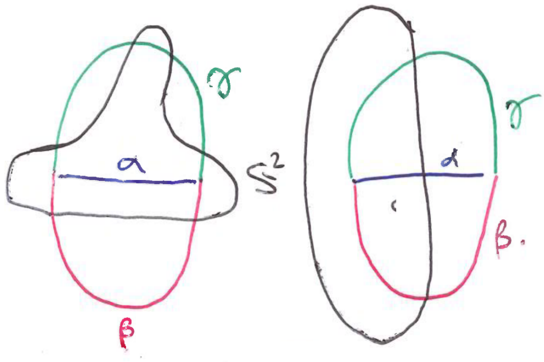


Figure 12: Left: a three-bridge presentation for a θ-graph. Right: a two-bridge presentation.

Recall that a n-bridge presentation for a link $k \subset \mathbb{S}^3$ is a genus zero Heegaard splitting $A \cup_{\mathbb{S}^2} B$ of $\mathbb{S}^3$ which intersects k transversely in exactly $2n$ points, such that $k \cap A$ (resp. $k \cap B$) can be moved via an ambient isotopy of A (resp. B) to n disjoint curves on ∂A (resp. ∂B). We define similar things now for the quotient θ-graphs of the previous paragraph.

- $\gamma \cup \bigcirc$ has a 3-bridge decomposition if $\mathbb{S}^3$ admits a genus zero Heegaard splitting which is a 3-bridge presentation for the unknot $\bigcirc$ with the property that γ is embedded as an unknotted arc inside one of the balls. That is, in one of the balls we get an isotopy of three disjoint arcs to the boundary sphere like usual, but in the other ball we get an isotopy of an unknotted arc and an 'H' to the boundary, c.f. left image of figure 11.
- $\gamma \cup \bigcirc$ has a 2-bridge decomposition if $\mathbb{S}^3$ admits a genus zero Heegaard splitting which is a 2-bridge presentation for the unknot $\bigcirc$ with the property that γ intersects $\mathbb{S}^2$ transversely exactly once and the two bridge isotopies also move the piece of γ inside each ball to the boundary; i.e. in each ball we get an isotopy of an unknotted arc and a 'T' to the boundary, c.f. right image of figure 11.

(5.6) Theorem ([MSY96, Theorems 1.2 and 3.2])**.** *1. A knot $k \subset \mathbb{S}^3$ has tunnel number one if and only if k admits a strong inversion such that the corresponding θ-curve G has a 3-bridge decomposition. If the θ curve edges are labelled α, β, γ such that $\alpha \cup \beta$ has an induced 1-bridge presentation and $\alpha \cup \gamma$ has an induced 2-bridge presentation (left of figure 12), and $\omega(\beta) = b$ and $\omega(\gamma)$ are fixed, then*

$$\sum_a \frac{\Delta_a}{\Delta_{a,b,c}^2 \Delta_b \Delta_c^3} \left| Y_{G,\omega}(e^{i\psi}) \right|^2 \le 1$$

where $a = \omega(\alpha)$ ranges over all admissable values and where $|\psi| \le \pi/2(b + 2c + 1)$.

2. A knot $k \subset \mathbb{S}^3$ admits a genus 1 one-bridge decompisition if and only if k admits a strong inversion such that the corresponding θ-curve G has a 2-bridge decomposition. If the θ curve edges are labelled α, β, γ such that $\alpha \cup \beta$ has an induced 1-bridge (right of figure 12), and $\omega(\beta) = b$ and $\omega(\gamma)$ are fixed, then

$$\left| Y_{G,\omega}(e^{i\psi}) \right| \le \Delta_{a,b,c} \Delta_c$$

where $a = \omega(\alpha)$ is some admissable value and where $|\psi| \le \pi/(a + b + 3c + 2)$.

The primary ingredient in the proof of this theorem is the introduction of a Hermitian form on a space which we will now construct. Suppose we have a marked disc $D = D_{l_1+\cdots+l_n}$. We can consider we can consider the subspace $\mathcal{H}(l_1 + \cdots + l_n)$ constructed by taking elements of the skein algebra of D and plumbing onto each arc a Jones-Wenzel idempotent f_{l_i}. There is a natural isomorphism

$$(\cdot)^* : \mathcal{S}(D_{l_1+\cdots+l_n}, \mathbb{C}, q) \to \mathcal{S}(D_{l_n+\cdots+l_1}, \mathbb{C}, q)$$

which is obtained by taking a reflecton of $D_{l_1+\cdots+l_n}$ (taking complex conjugates of coeffiecients and reflections of diagrams on the interior). One may check using the definition of f_m that $f_m^* = f_m$. Hence $*$ restricts to a map $\mathcal{H}(l_1 + \cdots + l_n) \to \mathcal{H}(l_n + \cdots + l_1)$. Given any two elements $u, v \in \mathcal{H}(l_1 + \cdots + l_n)$, we may glue u and v^* along their boundaries identified in the natural way in order to obtain a diagram in $\mathbb{S}^2$. Taking the Kauffman bracket gives a complex number. Multiplying by $i^{l_1+\cdots+l_n}$ gives a Hermitian form $\theta : \mathcal{H}(l_1 + \cdots + l_n) \times \mathcal{H}(l_1 + \cdots + l_n) \to \mathbb{C}$. One can show that it is positive definite. If B_n is the Artin braid group on n strands then $\xi \in B_n$ induces a natural isomorphism $\mathcal{H}(l_1 + \cdots + l_n) \to \mathcal{H}(l_\xi(1) + \cdots + l_\xi(n))$, and that $\theta(\xi(u), \xi(v)) = \theta(u, v)$ for all u, v, ξ.

The connection with the Heegard splitting/bridge decomposition theory is that the bridge decomposition induces an element of the (in the 3-bridge case) braid group B_6. If the labelling is as in figure 12 then this element $\xi \in B_6$ induces a unitary operator (introducing appropriate twists where needed to make the vertex labels correctly ordered)

$$\xi : \mathcal{H}(b + c + b + c + c + c) \to \mathcal{H}(b + b + c + c + c + c).$$

The inequalities in the theorem statement come from explicitly computing the Yamada invariants, which can be done in terms of a specific orthonormal basis for the Hermitian form θ. This does not motivate geometrically the exact choice of the Yamada invariant, unfortunately: maybe one is lead to the Hermitian form first, and then one asks for an orthonormal basis, writes down the inequalities $\theta(u_i, u_j) = e_{i,j}$, and then observes that this gives a graph invariant.

§5.4. *Statistical algebraic geometry and the curve complex*

We now introduce a basis of the skein algebra $\mathcal{S}(M)$ due to D. Thurston [Thu14], but it seems to have originally arisen in the world of cluster algebras. This basis is of interest to various people as the expansions of all products of basis elements are conjectured to have positive coefficients over $\mathbb{Z}$.[7] This positivity is verified in some cases e.g. for the specialisation $q = 1$ by Thurston and for $S_{0,4}$ and $S_{1,1}$ (the sporadic hyperbolic surfaces) by Bousseau [Bou23, Theorem 1.1].

Define the Chebyshev polynomials[8] by

$$Ч_0(x) = 2, \qquad Ч_1(x) = x, \quad \text{and for } n \geq 2, \quad Ч_n = xЧ_{n-1}(x) - Ч_{n-2}(x).$$

One observes that $Ч_n(\lambda + \lambda^{-1}) = \lambda^n + \lambda^{-n}$ for all n. Let γ be an isotopy class of multicurve on $S_{g,n}$. Then in $\mathcal{S}(M)$, $\gamma = \gamma_1^{n_1} \cdots \gamma_k^{n_k}$ where all the n_j are positive integers and where all the γ_j are connected. Define $\mathbf{Ч}(\gamma) := Ч_{n_1}(\gamma_1) \cdots Ч_{n_k}(\gamma_k)$. Then the set of all $\mathbf{Ч}(\gamma)$ for all multicurves γ is a $\mathbb{Z}[q^{\pm 1}]$-basis for $\mathcal{S}(M)$. In fact it is invariant under $\mathrm{Mod}(S)$.

1. Quantised character variety of $\pi_1(S_{0,4}) \to \mathsf{SL}(2, \mathbb{C})$: `https://arxiv.org/pdf/1811.09293`

2. This is related to the skein algebras through [Bou23] see also `https://rheapalakbakshi.com/wp-content/uploads/2020/10/bousseau_beamer_skein_gwu_2020.pdf`.

[7]Positivity seems to be very important generally in this area e.g. in scattering theory many people now study the positive Grassmannian and objects like amplitudihedra.

[8]We note in passing that the traces of the words $W_{p/q}$ defined above seem to be very closely related to the Chebyshev polynomials as well [EMS22].

3. Should obtain in this way some kind of path from quantised rep spaces to skein algebras. Qn: How are quantised character varieties related to the classical varieties?

4. We also cannot in this way see where B_3 comes in. For this we should consider the Heegaard splitting point of view, perhaps from https://link.springer.com/chapter/10.1007/bfb0101200. Using (1–3) we get a relation between rep theory and the skein algebra, does this book chapter also give such a relation?

https://arxiv.org/pdf/1805.06062

Cluster algebras: https://www.youtube.com/watch?v=BfVTf-3HY2E

Blanchet talk on Jones polynomials: https://www.youtube.com/watch?v=fVF_Yls4v40&t=2070s

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