Mostow rigidity etc

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§1. Mostow–Prasad rigidity theorem

Historical note. The result was proved for cocompact lattices by Mostow (1973) and improved to general (cofinite) lattices by Marden (1974). The full theorem was independently proved by Prasad (1973). See the introduction of [Mos73] for further historical context.

(1.1) Theorem (General statement, [Mos73, Theorem A']). *Let G be a semisimple Lie group with $Z(G) = 1$ and no nontrivial compact normal subgroups. Let $\Gamma \leq G$ be discrete and cofinite (i.e. Γ is a lattice).*

Given any two such pairs (G, Γ) and (G', Γ') equipped with an isomorphism $\theta : \Gamma \to \Gamma'$, there exists a global isomorphism $\tilde{\theta} : G \to G'$ which extends θ, provided thaat there is no factor G_i of G which is isomorphic to $\mathsf{PSL}(2, \mathbb{R})$ and such that ΓG_i is closed in G.

Taking $G = \text{Isom}(\mathbb{H}^n)$ $(n \geq 3)$, $\tilde{\theta}$ preserves the maximal compact subgroup of G and so descends to the quotient space $\mathbb{H}^n$ and we find that $\tilde{\theta}$ induces Γ-invariant isometries $\mathbb{H}^n \to \mathbb{H}^n$. This is the content of the hyperbolic version of the theorem. We will restrict to $n = 3$ for simplicity of exposition, but the proof which we outline goes through for arbitrary n.

(1.2) Theorem (Hyperbolic rigidity). *Suppose that we have two complete finite volume hyperbolic 3-manifolds M and N and an isomorphism $\theta : \pi_1(M) = \Gamma \to \Gamma' = \pi_1(N)$. Then there exists an isometry $f : M \to N$ such that $\theta = f_*$ (i.e. $f_* : \text{Hom}(\mathbb{S}^1, M) \to \text{Hom}(\mathbb{S}^1, N)$ is $f_*(\gamma)(z) = f(\gamma(z))$).*

We sketch the proof following §8.5 of [Kap01][1]. The proof is basically in two parts:

- Construct from θ a quasiconformal deformation which induces it at the level of the conformal boundary.
- Show that this deformation is actually a Möbius transformation and so θ is induced by an inner automorphism of $\text{Isom}(\mathbb{H}^3)$.

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[1] actually I prefer the exposition in [MT98] if you can get hold of a copy, but it's less accessible

§1.1. Some simplifying observations.

We see first that Γ and Γ' are geometrically finite (since geometric finiteness is equivalent to having a finite volume convex core [Mar16, §3.11, fact (2)]). In particular, all parabolics are rank two [Mar16, Theorem 3.3.4]. We may conclude that θ is type-preserving (if $\gamma \in \Gamma$ is parabolic then it lies in an abelian subgroup so is detectable by isomorphisms at the group level).

§1.2. Some hard theorems of Tukia.

We recall that a **quasiconformal map** is a homeomorphism $f : U \to \hat{\mathbb{C}}$ which is differentiable a.e. in the open set U such that at points where it is defined the quotient $f_{\bar{z}}/f_z$ is uniformly bounded by some $M < \infty$. These objects are familiar from Teichmüller theory, and if you are not familar the following elementary textbook has a nice exposition: [IT87]. The idea is that $f_{\bar{z}}/f_z$ measures the distorsion of circles into ellipses on each tangent space $T_z U$ and so the condition is that the infinitesimal action of f sends circles to ellipses of uniformly bounded eccentricity.

We recall next that a map $f : (X, \rho) \to (Y, \varrho)$ (not necessarily cts) is a **quasi-isometry** if there exist constants $A \geq 1$, $B \geq 0$, and $C \geq 0$ such that the following hold:

1. (coarsely isometric) for all $x, x' \in X$, $\frac{1}{A}\rho(x, x') - B \leq \varrho(f(x), f(x')) \leq A\rho(x, x') + B$.
2. (coarsely surjective) for all $y \in Y$, there exists some $x \in X$ such that $\varrho(f(x), y) \leq C$.

A good reference for Gromov-type things is [BH99]. The point is that just as isometries of $\mathbb{H}^3$ extend to conformal maps on $\hat{\mathbb{C}}$, quasi-isometries on $\mathbb{H}^3$ extend to quasi-conformal maps on $\hat{\mathbb{C}}$.

(1.3) Theorem ([Kap01, Theorem 8.16]). *If $\theta : \Gamma \to \Gamma'$ is a type-preserving isomorphism of geometrically finite groups uniformising manifolds M, N respectively, then there exists a θ-equivariant quasiconformal map $h : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. In particular $\theta(\gamma) = h\gamma h^{-1}$ for all $\gamma \in \Gamma$ so $\Gamma' = h\Gamma h^{-1}$.*

Sketch of proof. Fix $x, x' \in \mathbb{H}^3$. Consider:

$$\begin{array}{ccccc} \Gamma & \longrightarrow & \Gamma x & \hookrightarrow & \mathbb{H}^3 \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\hat{\theta}} & & \\ \Gamma' & \longrightarrow & \Gamma' x' & \hookrightarrow & \mathbb{H}^3 \end{array}$$

By Gromov hyperbolicity theory, the horizontal maps sending group elements to their vertices in the embedded Cayley graphs are quasi-isometries. Since Γ and Γ' are isomorphic they are quasi-isometric in the Gromov (path) metric via θ. Hence the induced map $\hat{\theta}$ given by $\hat{\theta}(\gamma x) = \theta(\gamma)x$ for $\gamma x \in \Gamma x$ is a quasi-isometry and we get a θ-equivariant quasiisometry $\mathbb{H}^3 \to \mathbb{H}^3$ which extends to a θ-equivariant quasiconformal map $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. 𓅃

We have seen therefore that Γ and Γ' are quasiconformal conjugates. We now observe that they must be conjugate in $\mathrm{Isom}(\mathbb{H}^3)$. Since $\Lambda(z)$ is uncountable but there are only countably many parabolic fixed points in Γ we can pick an element of $z \in \Lambda(\Gamma)$ which is not a parabolic fixed point and such that the quasiconformal map is differentiable and nonsingular at z.

(1.4) Theorem ([Kap01, Theorem 8.34], [MT98, Theorem 3.34]). *If $\Gamma \leq \mathrm{Isom}(\mathbb{H}^3)$ is discrete, non-elementary, and geometrically finite, $\zeta \in \Lambda(\Gamma)$ is a non-parabolic fixed point*[2]*, and $f\Gamma f^{-1}$ is a quasi-conformal conjugate of Γ such that f is differentiable at z with $d_\zeta f \neq 0$, then $f \in \mathbb{M}$.*

[2] Kapovich and others require ζ to be a point of approximation: if you are worried see [Mas87, §VI.C.3].

Sketch of proof. Without loss of generality, $\zeta = 0$ and $f(0) = 0$ and $f(\infty) = \infty$. Let γ_n be a sequence of elements such that $\gamma_n(j) \to 0$ with accumulation occuring inside a conical neighbourhood of the vertical axis (this is where we use that ζ is a point of approximation). For each n let λ_n be the vertical coordinate of $\gamma_n(j)$. Then the hyperbolic distance $d(\gamma_n(j), \lambda_n j)$ is bounded. from above. Consider the sequence of Möbius transformations $z \mapsto \gamma_n^{-1}(\lambda_n z)$; this sequence is relatively compact in $\mathbb{M}$ (it is closed and has bdd norm) so we can choose a convergent subsequence that goes to an element of $\mathbb{M}$.

Now observe that for all n, $f\gamma_n f^{-1} \in \mathbb{M}$ and so is

$$\eta_n(z) = \frac{f\gamma_n f^{-1}(z) - f(0)}{\lambda_n}$$

But

$$\eta_n f(\gamma_n^{-1}\lambda_n(z)) = \eta_n f\gamma_n^{-1} f^{-1} f(\lambda_n(z)) = \frac{f(\lambda_n z) - f(0)}{\lambda_n}$$

where as $\lambda_n \to 0$ the right hand side converges to the linear map $z \mapsto d_0 f(z)$ while in the left hand side $\gamma_n^{-1}\lambda_n$ goes to an element of $\mathbb{M}$ and hence f goes to a linear map modulo pre- and post-composition by elements of $\mathbb{M}$ (since the right hand side converges η_n converges and elements of $\mathbb{M}$ can converge only to constant maps or Möbius transformations): we conclude that f is an element of $\mathbb{M}$. ⅅ⥄

§1.3. *Volume as a knot invariant*

(1.5) Corollary. *If two finite-volume hyperbolic 3-manifolds are homeomorphic, then they are isometric. In particular volume is a homeomorphism invariant.* ⅅ⥄

Recall that a knot is **prime** if it does not decompose under connected sum, i.e. k is prime iff whenever $k = k' \oplus k''$ one of k' or k'' is the unknot, and a knot is **tame** if it is isotopic to a knot which is made up of finitely many straight line segments.

(1.6) Theorem (Gordon–Luecke, 1989 [GL89]).

1. *Complements are link invariants: If k and k' are links and $k \sim k'$, then $(\mathbb{S}^3 \setminus k) \simeq_{homeo.} (\mathbb{S}^3 \setminus k')$.*
2. *Knots are determined by their complement: If k and k' are tame knots, then $(\mathbb{S}^3 \setminus k) \simeq_{homeo.} (\mathbb{S}^3 \setminus k')$ if and only if $k \sim k'$.*
3. *Prime knots are determined by their fundamental group: If k and k' are tame prime knots, and $\pi_1(k) \simeq \pi_1(k')$, then $k \sim k'$.* ⅅ⥄

Remark. The Gordon–Luecke theorem *does not hold* for links [Rol03, §9.H].

(1.7) Corollary. *If k and l are hyperbolic links and the volumes of $\mathbb{S}^3 \setminus k$ and $\mathbb{S}^3 \setminus l$ are different, then k and l are not isotopic.*

It's natural to ask if the is converse true — do hyperbolic volumes determine the knots? (Clearly there is no hope for links.)

(1.8) Theorem.
1. *Given some $v \in \mathbb{R}_{>0}$, the number of hyperbolic 3-manifolds with volume v is finite.*
2. *The set of all volumes $\mathcal{F}_3$ is a well-ordered (in the induced order) non-discrete subset of $\mathbb{R}_{>0}$.*
3. *Given any $n \in \mathbb{N}$ there exists some volume $v \in \mathcal{F}_3$ such that $\left|\mathrm{Vol}^{-1}(v)\right| = n$. (Wielenberg, 1981)*

This theorem follows from Thurston's Dehn filling theorem (together with a lot of work) by taking sequences of Dehn surgeries of manifolds and looking at the convergence behaviour of their volumes. The motivation behind this is the classification of *incomplete* hyperbolic structures on hyperbolic manifolds, of which there are infinitely many.

§1.4. Generalisations of Mostow rigidity

Mostow rigidity is the statement that cofinite Kleinian groups (in particular they are of the first kind, $\Lambda(\Gamma) = \mathbb{S}^2$) are quasiconformally rigid. The proof sketched above used the following: (i) all such groups are geometrically finite; (ii) geometrically finite groups support no nontrivial quasiconformal deformations of their limit set; (iii) the entire deformation (which is quasi-isometric on the manifold) is detected by the action on the sphere as a quasiconformal motion.

The classical generalisation of (ii) is:

(1.9) Theorem (Sullivan rigidity, [Kap01, §8.6]). *If G is a finitely generated (not necessarily geometrically finite) Kleinian group, then any G-equivariant quasiconformal map supported on $\Lambda(G)$ (i.e. which is conformal everywhere except on some positive measure subset of $\Lambda(G)$) is conformal on $\hat{\mathbb{C}}$.*

(Of course if $\Lambda(G)$ has measure zero this is not useful, but there exist groups which are infinite volume but have positive measure limit sets, for example degenerate B-groups.) As a consequence, any Kleinian group of the first kind is quasiconformally rigid.

Remark. What is the set of groups covered by Sullivan's rigidity theorem but not Mostow's? These are groups with empty conformal boundary but which are not finite volume, and they are groups uniformising manifolds where all ends are geometrically infinite ends [Mar16, §5.5].

The classical generalisation of (iii) still requires the groups to be geometrically finite. The point is that you now have some domain of discontinuity $\Omega(G)$ which can be deformed along with the limit set, but that this deformation is the only possible kind of flexibility.

(1.10) Theorem (Marden-Tukia isomorphism, [Tuk85, Theorem 4.2]). *Let G and G' be geometrically finite non-elementary Kleinian groups and let $f : \Omega(G) \to \Omega(G')$ be a homeomorphism inducing an isomorphism $\phi : G \to G'$. Then f can be extended to a homeomorphism F of $\overline{\mathbb{H}^3}$ inducing ϕ such that F is a Möbius transformation if f is conformal or if $\Omega(G) = \varnothing$.*

In less formal language, if a homeomorphism from one manifold to another restricts to a conformal map on the boundary then the homeomorphism is globally Möbius and so induces an isometry.

The removal of the condition 'geometrically finite' in the Marden–Tukia isomorphism theorem is known as the ending lamination theorem [Mar16, §5.7] and uses combinatorial techniques of an entirely different flavour to the classical quasiconformal theory.

§2. Dehn fillings

Much of this section follows parts of Chapter 5 of [Pur20].

(2.1) Definition. Let M be a manifold with torus boundary component T, and let $\gamma_{p/q}$ be an isotopy class of simple closed curves on t. The manifold obtained by attaching a solid torus to T such that $\gamma_{p/q}$ bounds a disc is called the **Dehn filling** of M along $\gamma_{p/q}$.

(2.2) Definition. Let M be a manifold, let k be a knot in M, and let $p/q \in \hat{\mathbb{Q}}$. The manifold M' obtained from M by drilling out a solid torus neighbourhood of k and performing a p/q Dehn filling along the result is called the result of **Dehn surgery** along k.

(2.3) Example (Lens spaces). Let k be the unknot in $\mathbb{S}^3$. Then:

1. $0/1$ surgery gives $\mathbb{S}^1 \times \mathbb{S}^2$. Proof: we are gluing two solid tori $\mathbb{S}^1 \times \mathbb{B}^2$ meridian-to-meridian and latitude-to-latitude: gluing the boundaries of the $\mathbb{B}^2$'s gives an $\mathbb{S}^2$ in each level of the $\mathbb{S}^1$ so we end up with $\mathbb{S}^1 \times \mathbb{S}^2$.

2. p/q surgery gives the lens space $L(p/q)$ by definition. This is the quotient of $\mathbb{S}^3 \subset \mathbb{C}^2$ by $(w,z) \equiv (e^{2\pi i/p}w, e^{2\pi i q/p}z)$: to see this we consider the Clifford torus

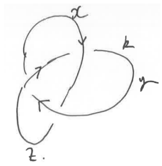
$$C = \frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 = \{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\phi}) : 0 \le \theta, \phi < 2\pi\} \subseteq \mathbb{C}^2$$

   which is preserved by the action of $\zeta : (w,z) \mapsto (e^{i2\pi/p}w, e^{i2\pi q/p}z)$ and the meridian and latitude descend under the quotient to the p/q curve on a torus in $L(p/q) = \mathbb{S}^3/\langle\zeta\rangle$. For details see [Hem76, pp. 20–23].

3. $\pm 1/n$ surgery gives $\mathbb{S}^3$. Proof: when $n = 0$ this is the 'do nothing' meridian-to-latitude gluing; more generally this is the $L(\pm 1, n)$ lens space, in which case setting $p = \pm 1$ we get the quotient of $\mathbb{S}^3$ by $(w,z) \equiv (e^{2\pi i}w, e^{2\pi i n}z) = (w,z)$.

We take the following example from [PS97, §18].

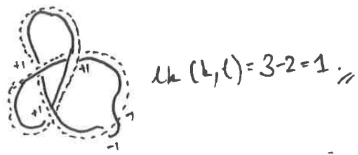
(2.4) Example (Poincaré sphere). Consider the trefoil knot k with the following diagram:



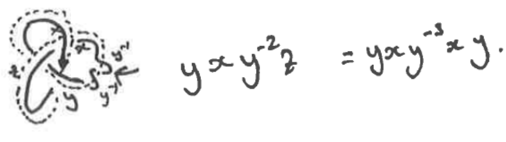
The Wirtinger presentation is

$$\begin{aligned}\pi_1(k) &= \langle x, y, z : x^{-1}zx = y, y^{-1}xy = z, z^{-1}yz = x\rangle \\ &= \langle x, y : xyx = yxy\rangle.\end{aligned}$$

We pick a meridian (easy) and a longitude of the knot l with linking number 1:



The 1/1 curve in this frame is easy to write down:



Hence the 1/1 Dehn surgery manifold D has fundamental group

$$\pi_1(D) = \langle x, y : xyx = yxy, 1 = yxy^{-3}xy\rangle.$$

Lemmista A. $H_1(D, \mathbb{Z}) = 0$. *Proof:* The homology is the abelianisation:

$$H_1(D, \mathbb{Z}) = \pi_1(D)/[\pi_1(D), \pi_1(D)] = \langle x, y : xyx = yxy, y^{-2}x^{-1} = xy^{-3}, xy = yx \rangle.$$

The relators give us a system of two equations:

$$\begin{array}{rrcrr} 2x & +\ y & = & x & +2y \\ -2x & -2y & = & x & -3y \\ \hline & -\ y & = & 2x & -\ y \end{array}$$

i.e. $x = 0$ and by back-substitution $y = 0$. 𓄿𓍿𓎡

Lemmista B. $\pi_1(D) \simeq A_5$. *Proof:* It is known that $A_5 = \langle a, b : a^5 = b^3 = (ba)^2 \rangle$; we may take $a = y$ and $b = yx$ to obtain the change of variables and it is enough to show that the $\pi_1(D)$ relations hold in A_5 and the A_5 relations hold in $\pi_1(D)$. 𓄿𓍿𓎡

As a corollary of the two lemmistae we have that the first homology of D is trivial (agreeing with $\mathbb{S}^3$) but the fundamental group is nontrivial (so D is not homeomorphic to $\mathbb{S}^3$). We know that $\dim H_1 = \dim H_2$ (Poincaré duality) and $0 = \dim H_0 = \dim H_3$ so all homologies vanish. Therefore D is a homology sphere, the **Poincaré homology sphere**.

The result of Dehn surgery in a hyperbolic manifold is usually hyperbolic. This follows from the next theorem, which we first state in a rough sense: *Let M be a 3-manifold homeomorphic to the interior of a compact manifold with boundary a single torus $\mathbb{T}^2$ such that M admits a complete hyperbolic structure. Then the space of all Dehn surgeries on M contains an open neighbourhood of the complete structure.*

More precisely, let M be any 3-manifold with torus boundary C (C is called a **cusp torus**) and suppose that an incomplete hyperbolic structure is placed on M, so the holonomy group $\pi_1(C)$ is not generated by parabolic elements. Then there is a natural map $L : \pi_1(C) = H_1(C, \mathbb{Z}) \to \mathbb{C}$ given by the complex length function, and this admits a canonical extension $L : H_1(C, \mathbb{R}) \to \mathbb{C}$. (In other words we extend from simple closed curves to arbitrary laminations of one leaf.) We must be very careful here (and it is not explained very well in either [Pur20, Chapter 6] or the primary source [Thu79, §4.5]) to define complex length not in the traditional way for $\mathsf{PSL}(2, \mathbb{C})$ but exactly by linear extension (so $L(p\alpha + q\beta) := pL(\alpha) + qL(\beta)$). The point is that this defines a lift of the usual complex length function [Mar16, Exercise 7–20] from $\pi_1(C) \to \mathbb{R} + i[0, 2\pi)$ to $\pi_1(C) \to \mathbb{R} + i\mathbb{R}$.

Remark. A more precise definition is given in Section 5 of [HK08], where the holonomy elements are considered to come from the Euclidean development and not the hyperbolic development, in which case the quantity of twisting is visible just in the same way as it is when defining Fenchel–Nielsen coordinates for the Teichmüller space of the torus [FM12, §10.6]: it is physically a translation length projected onto one axis of the torus, rather than a loxodromic element that picked up a twist while walking along a path.

The fundamental example is a homology class with complex length $2\pi i$. This corresponds to a curve which represents a meridian (since it has translation length zero along the core of the torus—which is the projection of the shared axis of the generators—and holonomy 2π around the axis). In the completion of M the curve γ bounds a smooth hyperbolic disc. Hence the completion of M with this hyperbolic structure is a manifold homeomorphic to the Dehn filled manifold along γ and we therefore get a complete hyperbolic structure. On the other hand if the imaginary part is $\theta \neq 2\pi$, in the completion the curve γ will bound a hyperbolic cone of angle θ, the metric on the completion is not smooth, and so we don't get a structure. There is a unique $c \in H_1(C, \mathbb{R})$ with $L(c) = 2\pi i$, and this c is called the **Dehn surgery coefficient** of C. The subset of $H_1(C, \mathbb{R}) \simeq \mathbb{R}^2$ consisting of all Dehn filling coefficients for all possible hyperbolic structures (that is, $H_1(C, \mathbb{R})$ is a topological invariant so does not depend on the incomplete structure on M, but L does depend on this structure, so we get different coefficients for each structure that all lie in the same $\mathbb{R}^2$) is called the **hyperbolic Dehn filling space** for M, and by convention we let ∞ be the complete hyperbolic structure on M if it exists.

(2.5) Theorem (Thurston's Dehn filling theorem)**.** *Let M be a 3-manifold homeomorphic to the interior of a compact manifold with boundary a single torus $\mathbb{T}^2$ such that M admits a complete hyperbolic structure. Then the Dehn filling space of M contains an open neighbourhood of ∞. More generally if M is the interior of a compact manifold with torus boundary components $T_1, \dots, T_n$ and if it admits a complete structure, then for each T_i the corresponding Dehn filling space contains an open neighbourhood of ∞.*

Sketch of proof. Suppose M admits a complete structure, and let $\rho : \pi_1(X) \to \mathsf{PSL}(2, \mathbb{C})$ be the representation verifying this. The fundamental group of $\mathbb{T}^2$ has image $\rho(\pi_1(\mathbb{T}^2))$ generated by two parabolics with the same fixed point. Now show that ρ admits a one-parameter family of deformations, and that each deformation sends these parabolics to a pair of loxodromics which share fixed points. Now this gives a distance measure which induces an incomplete complex structure that has Dehn filling coefficient varying continuously around ∞. In general, one can do high-dimensional deformations to get the result for n boundary components. ⛵

(2.6) Corollary. *Let X be a complete hyperbolic manifold with n torus boundary components $T_1, \dots, T_n$. For each T_i, exclude finitely many Dehn fillings. The resulting Dehn fillings yield a manifold with a complete hyperbolic structure.*

Proof. For every i there are only finitely many elements of $H_1(C, \mathbb{Z})$ that lie outside the open neighbourhood of ∞ in $H_1(C, \mathbb{R})$ given by the theorem. ⛵

Conversely, all 3-manifolds arise by Dehn surgery:

(2.7) Theorem (Lickorish/Wallace, 1960–1962)**.** *Let M be a closed orientable 3-manifold. Then M is the result of Dehn surgery along some link in $\mathbb{S}^3$.*

A slightly stronger version of this is:

(2.8) Theorem (Jørgensen)**.** *Let $C > 0$. Among all hyperbolic 3-manifolds M with volume at most C, there are only finitely many homeomorphism types of M_ε. There is a universal link $L_C \subseteq \mathbb{S}^3$ such that every complete hyperbolic manifold with volume at most C is obtained by some Dehn surgery along L.*

The combination of corollary (2.6) and theorem (2.7) implies, roughly speaking, that most 3-manifolds are hyperbolic.

References

[BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*. Grundlehren der mathematischen Wissenshaften 319. Springer, 1999. ISBN: 978-3-540-64324-1. DOI: 10.1007/978-3-662-12494-9 (cit. on p. 2).

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Princeton Mathematical Series. Princeton University Press, 2012. ISBN: 978-0-691-14794-9 (cit. on p. 6).

[GL89] C. McA. Gordon and J. Luecke. "Knots are determined by their complements". In: *Journal of the American Mathematical Society* 2.2 (1989), pp. 371–414. DOI: 10.2307/1990979 (cit. on p. 3).

[Hem76] John Hempel. *3-Manifolds*. Annals of Mathematics Studies 86. Princeton University Press, 1976. ISBN: 0-691-08178-6 (cit. on p. 5).

[HK08] Craig P. Hodgson and Steven P. Kerckhoff. "The shape of hyperbolic Dehn surgery space". In: *Geometry and Topology* 12 (2008), pp. 1033–1090. DOI: 10.2140/gt.2008.12.1033. arXiv: 0709.3566 [math.GT] (cit. on p. 6).

[IT87] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer, 1987. ISBN: 978-4-431-68174-8. DOI: 10.1007/978-4-431-68174-8 (cit. on p. 2).

[Kap01] Michael Kapovich. *Hyperbolic manifolds and discrete groups*. Progress in Mathematics 183. Birkhaüser, 2001. ISBN: 978-0-8176-4913-5. DOI: 10.1007/978-0-8176-4913-5 (cit. on pp. 1, 2, 4).

[Mar16] Albert Marden. *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title "Outer circles". Cambridge University Press, 2016. ISBN: 978-1-107-11674-0 (cit. on pp. 2, 4, 6).

[Mas87] Bernard Maskit. *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag, 1987. ISBN: 978-3-642-61590-0. DOI: 10.1007/978-3-642-61590-0 (cit. on p. 2).

[Mos73] G.D. Mostow. *Strong rigidity of locally symmetric spaces*. Annals of Mathematics Studies 78. Princeton University Press, 1973. ISBN: 0-691-08136-0 (cit. on p. 1).

[MT98] Katsuhiko Matsuzaki and Masahiko Taniguchi. *Hyperbolic manifolds and Kleinian groups*. Oxford University Press, 1998. ISBN: 0-19-850062-9 (cit. on pp. 1, 2).

[PS97] V.V. Prasolov and A.B. Sossinsky. *Knots, links, braids, and 3-manifolds*. Translations of Mathematical Monographs 154. American Mathematical Society, 1997. ISBN: 978-0-8218-0588-6 (cit. on p. 5).

[Pur20] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 978-1-4704-5499-9 (cit. on pp. 4, 6).

[Rol03] Dale Rolfsen. *Knots and links*. Corrected edition. AMS Chelsea, 2003. ISBN: 0-8218-3436-3 (cit. on p. 3).

[Thu79] William P. Thurston. *The geometry and topology of three-manifolds*. Unpublished notes. 1979. URL: http://library.msri.org/books/gt3m/ (cit. on p. 6).

[Tuk85] Pekka Tukia. "On isomorphisms of geometrically finite groups". In: *Publications mathématiques de l'I.H.É.S* 61 (1985), pp. 171–214. URL: http://www.numdam.org/item/PMIHES_1985__61__171_0/ (cit. on p. 4).