*The lake where they had hidden the reflections**
Quasi-Fuchsian groups and their embeddings

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Abstract

We will discuss uniformisation of surfaces, deformations of Fuchsian groups, and the relationship to 3-dimensional topology. Emphasis will be placed on the construction of beautiful examples of these groups.

> The theory of Kleinian groups was founded by Schottky, Poincaré, and Klein in the 19th century. For many years it was dormant, except, of course, for the important special case of Fuchsian groups. The burst of activity during the last decade is based, directly or indirectly, on the use of quasi-conformal mappings as a working tool in complex function theory. *L. Bers, 1974* [Ber74]

Contents

*Alison Glenny. *Bird collector*, p. 62. Auckland: Compound Press, 2021.

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§1. In the before times...

One of the most important insights of Riemann was the idea that multiply-valued functions on $U \subseteq \mathbb{C}$ should be viewed as singly-valued functions on a covering space $\tilde{U} \to U$. This is one of the pivotal moments in 19th century mathematics and was an essential point in the birth of many modern fields (algebraic curve theory, geometric function theory, analytic number theory...).

Suppose you take a multiply-valued function on U, or in more modern language a function defined on a Riemann surface $\tilde{U}$, and you invert it. The result is a function from $\mathbb{C}$ to your Riemann surface $\tilde{U}$. The most classical examples are the so-called **elliptic functions**, which are the functions $f : \mathbb{C} \to \mathbb{C}$ that are doubly periodic in the sense that there exists a lattice $\Lambda \subset \mathbb{C}$ with $f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$: in other words, they are defined on the torus $\mathbb{C}/\Lambda$. [The term *elliptic* comes from the fact that their inverses arise in the computation of arc lengths on ellipses. This is the etymology of the term 'elliptic curve': elliptic curves are exactly the algebraic curves uniformised by elliptic functions.]

The construction of functions $\mathbb{C} \to S$ for arbitrary Riemann surfaces is called the (classical) **uniformisation problem**. In modern language, we ask for groups $\Gamma < \mathrm{Aut}(\mathbb{C})$ which act on $U \subset \mathbb{C}$ such that $S = U/\Gamma$ and the desired uniformisation function is the projection function $z \mapsto \Gamma z$; we say that Γ **uniformises** S.

The fundamental result in the study of uniformisation by groups is the famous

(1.1) Theorem (Riemann mapping theorem)**.** *If R is a simply connected Riemann surface, then R is biholomorphic to exactly one of the following:*

1. *The Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, if $\chi(R) = 2$;*
2. *The plane $\mathbb{C}$, if $\chi(R) = 0$;*
3. *The disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, if $\chi(R) < 0$.*

(For a proof, see the theorem of Paragraph IV.6.1 of [FK92] or Chapter 6 of [Ahl79].)

Here, the Euler characteristic of a Riemann surface R of genus g with n punctures and deleted discs is

$$\chi(R) = 2 - 2g - 2n.$$

Note that each of the three cases corresponds to the three surface geometries: spherical, Euclidean, and hyperbolic. (We recall that the unit disc Δ is conformally equivalent to the upper half-plane $\mathbb{H}^2$ and both are models for the hyperbolic plane [Bea83, Chapter 7].)

We now state a weak version of the Klein-Koebe-Poincaré uniformisation theorem (we exclude the torsion case, for simplicity; the more general theorem in the connected case can for example be found as Theorem IV.9.12 of [FK92]).

(1.2) Theorem (Klein-Koebe-Poincaré uniformisation)**.** *Let R be a connected Riemann surface. Then:*

1. *If $\chi(R) = 2$, then R is the sphere (so admits a metric of constant sectional curvature 1)*
2. *If $\chi(R) = 0$ (so R is either of genus 1 with no punctures, or is a sphere with one puncture), then there is a group G of Euclidean motions on $\mathbb{C}$ such that $R = \mathbb{C}/G$ (so admits a metric of constant sectional curvature 0);*
3. *If $\chi(R) < 0$, then there exists a discrete group Γ of conformal mappings of $\mathbb{H}^2$ which acts as a group of hyperbolic isometries, and $R = \mathbb{H}^2/\Gamma$.*

(1.3) Definition. A discrete group of isometries of $\mathbb{H}^2$ is called a **Fuchsian group**.

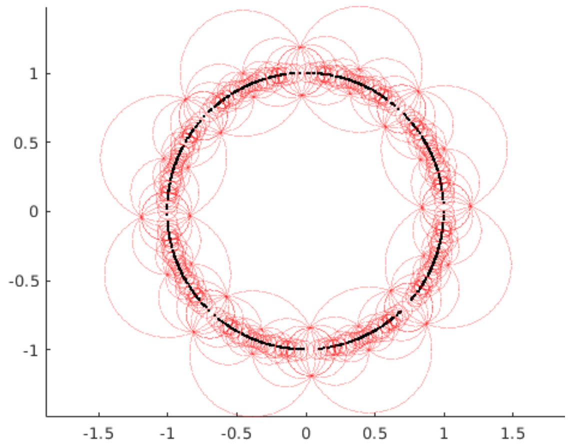


Figure 1: Isometric circles of the group F.

Meanwhile, Klein continued to object to the name "Fuchsian functions," on the grounds that Fuchs had published nothing about them whereas Schwarz had. Nor indeed had he, Klein, done anything on Kleinian functions except to bring one special case to Poincaré's attention, but Schottky's name should be mentioned, along with that of Klein's former student Dyck, who had brought out the group-theoretic aspects of the subject in his work. Klein even complained at length to his students in his seminar about the name "Fuchsian," and also about the fact that his journal, *Mathematische Annalen*, was seemingly unknown in France. [Gra13, p. 233]

§2. Fuchsian groups

We now enter the era of Bers and Ahlfors. Let us recall, a **Kleinian group** is a discrete subgroup of the group $\mathbb{M}$ of Möbius transformations of $\hat{\mathbb{C}}$. That is, it is a discrete group of conformal automorphisms of $\mathbb{S}^2$. If we take a conformal embedding of $\mathbb{S}^2$ into $\mathbb{S}^3$, e.g. the standard embedding $\{(x, y, 0) : x, y \in \mathbb{R}\} \cup \{\infty\}$, then every $g \in \mathbb{M}$ extends uniquely to a conformal automorphism of $\mathbb{S}^3$ which preserves the two halves of $\mathbb{S}^3 \setminus \mathbb{S}^2$, each of which is a $\mathbb{B}^3$; and every conformal automorphism of $\mathbb{B}^3$ is a hyperbolic isometry.

Similarly to how we just embedded $\mathbb{B}^3 \hookrightarrow \mathbb{S}^3$ and obtained $\mathrm{Isom}(\mathbb{B}^3)$ as a subgroup of the conformal automorphisms of $\mathbb{S}^3$, we can embed $\mathbb{B}^2 \hookrightarrow \mathbb{S}^2 = \hat{\mathbb{C}}$ and obtain $\mathrm{Isom}(\mathbb{B}^2)$ as a subgroup of conformal automorphisms of $\mathbb{S}^2$; this latter group is just $\mathbb{M}$, which motivates the following definition, generalising definition (1.3):-

(2.1) Definition. A Fuchsian group is a pair (G, Δ) where $\Delta \subset \hat{\mathbb{C}}$ is a round disc and G is a Kleinian group which preserves Δ.

Remark. This procedure works in all dimensions: Kleinian groups on $\mathbb{S}^n$ (i.e. discrete groups of conformal automorphisms of $\mathbb{S}^n$) are Fuchsian groups in $\mathbb{S}^{n+1}$ (i.e. discrete groups of conformal automorphisms of $\mathbb{B}^n \subset \mathbb{S}^{n+1}$) which are in turn discrete groups of isometries of the hyperbolic metric on $\mathbb{B}^n$, c.f. [KAG86; CNS13].

(2.2) Example. We give explicit matrices for a Fuchsian group acting on $\mathbb{B}^2$ that uniformises a genus two group:

$$F = \left\langle \begin{bmatrix} 1+\sqrt{2} & \omega \\ \overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (1+i)\omega \\ (1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & i\omega \\ -i\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (-1+i)\omega \\ (-1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix} \right\rangle.$$

where

$$\omega = \sqrt{-6+2\sqrt{2}-4i\sqrt{-2+2\sqrt{2}}}$$

The computation leading to this example may be found in appendix A. A fundamental domain is shown in figure 1.

(2.3) Lemma. *Let G be a Kleinian group. Then the following are equivalent:*

1. *G is Fuchsian (i.e. there exists Δ a round disc preserved by G);*
2. *G preserves an oriented round circle;*
3. *G is conjugate in $\mathsf{PSL}(2,\mathbb{C})$ to a subgroup of $\mathsf{PSL}(2,\mathbb{R})$.*

Proof. Clearly (1) implies (2): the round circle is $\partial\Delta$. To see that (2) implies (3) just conjugate the round circle to $\hat{\mathbb{R}}$. To get (1) from (3), take the round disc to be $\mathbb{H}^2$.

(2.4) Exercise. A Kleinian group $\langle X, Y\rangle$ is Fuchsian if and only if $\operatorname{tr} X$, $\operatorname{tr} Y$, and $\operatorname{tr} XY$ are real.

Clearly

$$\Lambda(G) \subseteq \partial\Delta, \tag{2.5}$$

because the limit set is the closure of the set of attracting fixed points of elements of G, and if any of these fixed points (say for an element g) lie in one of the components U of $\hat{\mathbb{C}} \setminus \partial\Delta$ then the other component U' must be mapped into U by some sufficiently high power of g. If we have equality in equation (2.5) then we say that the group is **of the first kind**, otherwise it is **of the second kind**. One is led to wonder about the etymology of this subtle and yet expressive terminology, clearly created by some mathematician of great imagination... Compact Riemann surfaces with punctures are uniformised by Fuchsian groups of the first kind. We should also mention at this point the **Bers uniformisation theorem** which says that, given any two complex structures S, S' on a hyperbolic Riemann surface Σ, there exists a quasi-Fuchsian group (G,Δ) such that $S = \Delta/G$ and $S' = (\hat{\mathbb{C}} \setminus \overline{\Delta})/G$. In fact this can be improved: given any *sequence* of hyperbolic Riemann surfaces there exists a Kleinian group which simultaneously uniformises all of them [Mas87, VIII.B].

We give another example which at first glance seems more complicated but which is somewhat easier to deal with for our purposes.

(2.6) Example (Grandma's groups, [MSW02, p. 229])**.** Let $t_a, t_b \in \mathbb{C}$ be two parameters and choose t_{ab} to be a solution of the **Markoff equation**

$$t_a^2 + t_b^2 + t_{ab}^2 = t_a t_b t_{ab}.$$

Set $z_0 = (t_{ab} - 2)t_b/(t_b t_a b - 2t_a + 2it_{ab})$ and define the matrices

$$b = \begin{bmatrix} \frac{t_b - 2i}{2} & \frac{t_b}{2} \\ \frac{t_b}{2} & \frac{t_b + 2i}{2} \end{bmatrix}, \quad ab = \begin{bmatrix} \frac{t_{ab}}{2} & \frac{t_{ab}-2}{2z_0} \\ \frac{(t_{ab}+2)z_0}{2} & \frac{t_{ab}}{2} \end{bmatrix}$$

(so the matrix $a = (ab)b^{-1}$ has trace t_a). One can check that under all these conditions the commutator $[a, b]$ is parabolic with fixed point 1.

Let $G = \langle b, ab \rangle$. By exercise (2.4) G is Fuchsian if and only if t_a, t_b, and t_{ab} are all real, and this occurs (for example) whenever t_a and t_b are both real and sufficiently large. We show six examples in figure 2, which should be read as stills from a film where we have moved (t_a, t_b) smoothly around $\mathbb{C}^2$.

(2.7) Exercise. When $t_a = t_b = 3$, the group G of example (2.6) preserves $\mathbb{B}^2$ and the quotient $\mathbb{B}^2/G$ is a once-punctured torus (hint: apply the Poncaré polyhedron theorem to the isometric circles of a, b, and ab).

What is going on here? We take small deformations of the entries of our generator matrices, and we obtain fractal deformations of the limit set. This in itself is fairly easy to understand: the limit set is (the closure of) the set of fixed points of the group, and these fixed points are obtained by solving quadratic equations in the entries of the matrices, and these entries are all algebraic functions of the parameters of the generators—so it makes sense that small deformations of the parameters induce small (or at least controlled) motions of the limit set.

The behaviour of the deformed groups *as Kleinian groups* is a little harder to understand, and the most conceptual path to enlightenment is via the λ-lemma. In the following, a **holomorphic motion** of a set $A \subset \mathbb{C}$ is like a complex-analytic version of a homotopy: it is a map $\Phi : \mathbb{B}^2 \times A \to \hat{\mathbb{C}}$ which is holomorphic in the first parameter (which plays the role of the time parameter in a homotopy, with $\Phi(0, -)$ being the identity on A) and injective in the second parameter.

(2.8) Theorem (The λ-lemma). *Let Γ be a non-elementary Kleinian group and let Γ_λ be some family of groups obtained by small deformations of matrix entries of generators of Γ in some open set. If these deformations induce holomorphic motions of $\Lambda(\Gamma)$ (the only thing to check here being injectivity), then these holomorphic motions extend to $\hat{\mathbb{C}}$ in an equivariant way: that is, we obtain a holomorphic motion*

$$\tilde{\Phi} : \mathbb{B}^2 \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

which is the same as the original motion on $\Lambda(\Gamma)$ but such that the maps

$$\tilde{\Phi}(\lambda, -) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

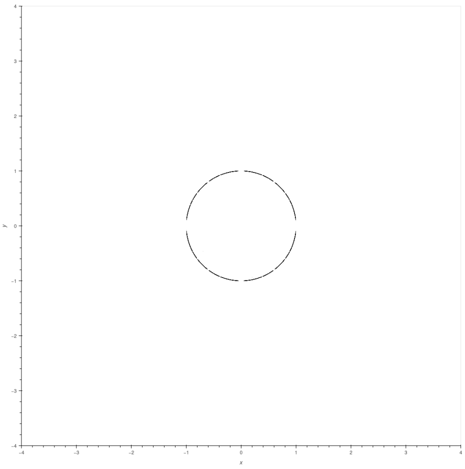
are quasiconformal *with maximal dilatation controlled only by $|\lambda|$.*

(A full technical discussion of the λ-lemma, which is due to Mañé, Sad, and Sullivan [MSS83], extended by Słodkowski [Sło91; Sło97], and equivarianced by Earle, Kra, and Krushkal' [EKK94] may be found in the surprisingly readable monograph of Astala, Iwaniec, and Martin [AIM09].)

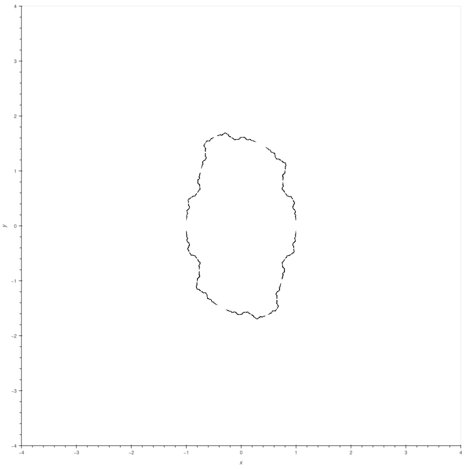
The upshot is the following corollary/slogan:

(2.9) Corollary. *If (G, Δ) is a Fuchsian group and small holomorphic deformations are made to the matrix entries of G to form a new group $\tilde{G}$, then so long as the limit set does not collide with itself there is a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{G} = fGf^{-1}$; in particular $\tilde{G}$ preserves $f\Delta$, which is a quasicircle. Since $f\Delta$ moves quasiconformally, the quotient surfaces also vary quasiconformally, and so give different elements of the Teichmüller space of G/Δ. In fact one can show that they are in bijection with the entire Teichmüller space, [MT98, §5.3]. In particular by the Marden–Tukia isomorphism theorem [Mar74; Tuk85] the resulting 3-manifolds are all the 3-manifolds quasi-isometric to the original manifold $\mathbb{H}^3/G$ by homeomorphisms.*

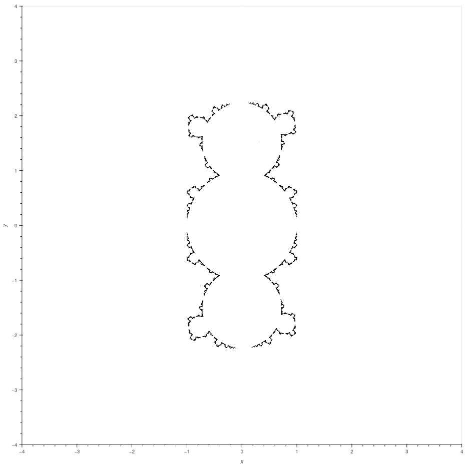
(2.10) Definition. A Kleinian group G is called **quasi-Fuchsian** if there exists some quasiconformal homeomorphism $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ that induces an isomorphism $\phi : G \to f^{-1}Gf$ such that (i) $\phi(G)$ is Fuchsian; and (ii) $g \in G$ is parabolic iff $\phi(g)$ is parabolic.



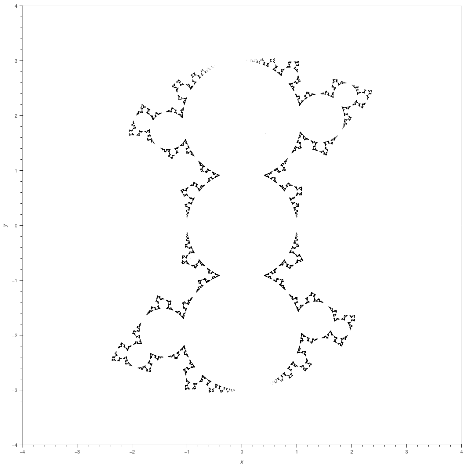
(a) $t_a = 3$, $t_b = 3$



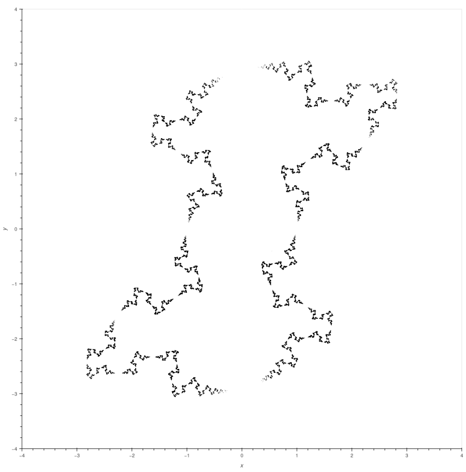
(b) $t_a = 3 + 0.9i$, $t_b = 3$



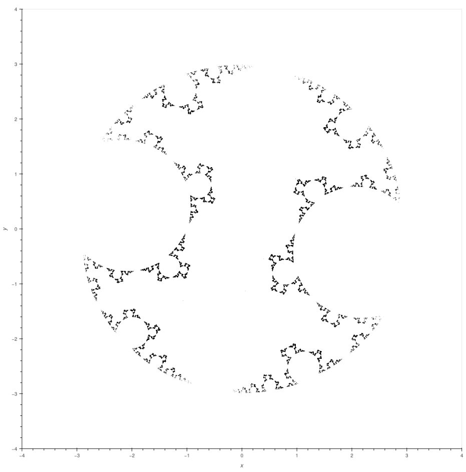
(c) $t_a = 2.44 + 0.9i$, $t_b = 2.19$



(d) $t_a = 1.96 + 0.9i$, $t_b = 2.19$



(e) $t_a = 1.96 + 0.9i$, $t_b = 2.19 + 0.46i$



(f) $t_a = 1.96 + 1.25i$, $t_b = 2.58 + 0.95i$

Figure 2: Limit sets for six of Grandma's groups.

(2.11) Example. The groups $G(t_a, t_b)$ of example (2.6) are quasi-Fuchsian if you start at $t_a = t_b = 3$ and take the parameter space to be maximal open set $\mathcal{QF}$ in $\mathbb{C}^2$ containing this point such that the domain of discontinuity $\Omega(G(t_a, t_b))$ does not split into more components or collapse to a single component. This is the **quasi-Fuchsian space of punctured tori**.

(2.12) Exercise. Classical Schottky groups are quasi-Fuchsian (of the second kind).

(2.13) Theorem (Alternative characterisations)**.** *Let G be finitely generated Kleinian. The following are equivalent.*

1. *G is quasi-Fuchsian (i.e. is a quasiconformal conjugate of a Fuchsian group);*
2. *there exist two disjoint quasidiscs Δ and Δ', both invariant under G, such that $\hat{\mathbb{C}} = \Delta \cup \Delta'$ and $\Lambda(G) \subset \partial\Delta = \partial\Delta'$;*
3. *there exist two disjoint topological discs Δ and Δ', both invariant under G, such that $\hat{\mathbb{C}} = \Delta \cup \Delta'$ and $\Lambda(G) \subset \partial\Delta = \partial\Delta'$.*

Proof. The only surprising thing is that we can lighten the condition on the domain from quasidisc to topological disc, which is a theorem of Bers [Ber70]. (It is this theorem which requires 'finitely generated'. As far as I am aware the result is not known for general Kleinian groups, and I would be surprised if it were true.)

(2.14) Example. One should carefully note that we need to specify that there are *two* quasidiscs preserved in $\Omega(G)$: we can deduce (as we could in the Fuchsian case) that if there is one quasidisc preserved then the limit set is contained in its boundary (and is of measure zero, but this is a very deep result [Mar16, Theorem 5.6.8]), but we can no longer expect its complement to be a quasidisc, or even non-empty(!). Indeed, there exist many groups G which have a single simply connected component of $\Omega(G)$ that is conformally a disc; these groups are called **degenerate groups** in the literature, c.f. [Mas87, Chapter IX], and cannot be obtained by quasiconformally deforming Fuchsian groups; they are much harder to understand, for instance no explicit constructions of these groups are known, only existence results and limiting processes. In some sense they are the obstruction to a complete classification of Kleinian groups.

Emily Dumas has produced (2007) images of limit sets of degenerate groups on the boundary of $\mathcal{QF}$ [Dum07], and we reproduce some of these in figure 3. The circular components which seem to appear inside the limit set are horodiscs corresponding to parabolic fixed points and will eventually be filled in by tendrils of the limit set. A similar image appears in Marden [Mar16, p. 313, fig. 5.8] for $\mathcal{QF}$; Marden also gives examples of groups obtained by gluing degenerate groups (i.e. degenerate groups on the boundary of function group space) [Mar16, §5.12]. The construction of such examples, at least those in Marden's book, is studied by Brock [Bro01b; Bro01a]; see also §3.7 of McMullen [McM96] who references preprints of Mumford, McMullen, and Wright which I have been unable to track down (though I have put in no effort to do so).

(2.15) Proposition. *Every parabolic in a finitely generated quasi-Fuchsian group $\tilde{G} = fGf^{-1}$ (where G is Fuchsian and f is a quasiconformal map as in the definition) represents a double cusp on the surfaces $f(\Delta)/\tilde{G}$ and $f(\hat{\mathbb{C}} \setminus \overline{\Delta})/\tilde{G}$.*

Remark. The image of a discrete, faithful representation $\rho : G \to \mathbb{M}$ where G is Fuchsian is said to have a **accidental parabolic** if there exists $g \in G$ that is not parabolic but such that $\rho(g)$ is parabolic. The proposition we have just stated is slightly stronger but essentially says that quasi-Fuchsian groups have no accidental parabolics *and* neither 'end' of the parabolic is pinched to nothing (which can occur e.g. in a degenerate group).

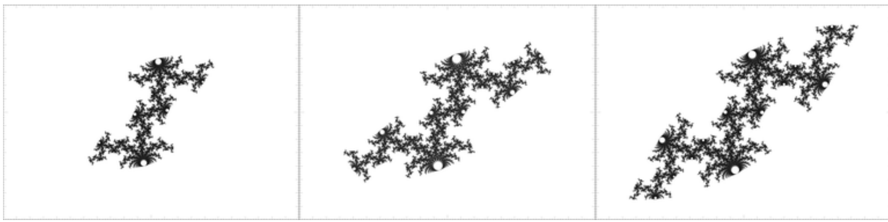


Figure 3: Dendrite limit sets of singly degenerate punctured torus groups drawn by E. Dumas.

Proof. There are no rank two parabolic groups in a Fuchsian group, so we may assume every parabolic in G generates a rank one parabolic group. A finitely generated Fuchsian group is geometrically finite [Mas87, V.G.14], hence every rank one parabolic group in G uniformises a double cusp [Mas87, VI.A.10]. Every parabolic $\tilde{G}$ is obtained from such a parabolic since the isomorphism $g \mapsto fgf^{-1}$ is type-preserving; and f is a homeomorphism so if $D_1 \cup D_2$ is the cusp region for $g \in G$ then $f(D_1) \cup f(D_2)$ is a cusp region for $fgf^{-1} \in \tilde{G}$.

§3. Embeddings of quasi-Fuchsian groups

Let (G, Δ) be Fuchsian. Then G acts on Δ as a group of hyperbolic isometries in the hyperbolic metric of Δ. However, if we view G as a Kleinian group this is not totally helpful: a Kleinian group in general acts only as a group of conformal maps on $\partial\mathbb{H}^3$, and the reason that we get hyperbolic isometries is essentially that when G is Fuchsian the quotient manifold $\mathbb{H}^3/G$ is just a product $(-1,1)\times S$ where S is a hyperbolic surface. There is a second disc on which G naturally acts, and this does have hyperbolic structure in $\mathbb{H}^3$: it is the dome $H = \text{h-conv}\,\partial\Delta \subset \mathbb{H}^3$ erected upon the boundary of the disc; the restriction of the metric of $\mathbb{H}^3$ to H gives it the structure of a hyperbolic plane. It is easy to see e.g. by Poincaré extension that if G preserves Δ then G preserves H, and in fact the action of G on $\mathbb{H}^3$ induces an action by G on H as a subgroup of the group of hyperbolic plane isometries of H. In particular, if Γ is some large ambient Kleinian group, and $G \leq \Gamma$ is a Fuchsian subgroup preserving Δ, then $G/\,\text{h-conv}(\partial\Delta)$ is a totally geodesic hyperbolic surface properly embedded into $\mathbb{H}^3/\Gamma$.

Suppose now that $G \leq \Gamma$ is quasi-Fuchsian. We may still construct $\text{h-conv}\,L$, where L is the Jordan curve preserved by G, but it is no longer a geodesic $\mathbb{H}^2$ inside $\mathbb{H}^3$: we still obtain a subsurface $(\text{h-conv}\,L)/G \subset \mathbb{H}^3/\Gamma$, but it is no longer a totally geodesic hyperbolic surface.

(3.1) Example. Consider the family of groups Γ_ρ generated by

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix}$$

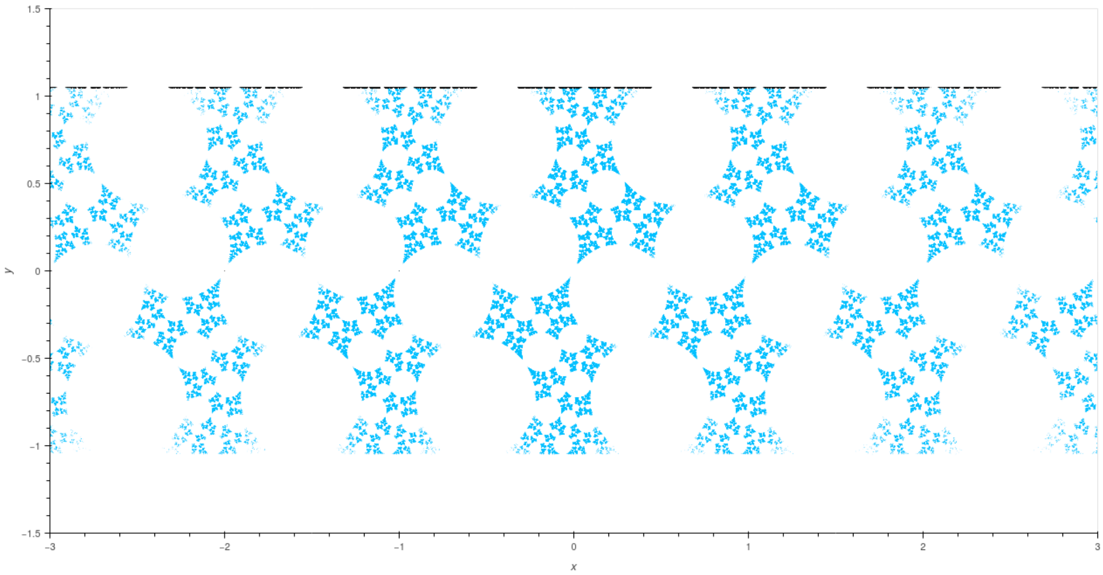
where $\rho \in \mathbb{C}$ is some parameter. We define four very special words in X and Y:-

$$\begin{aligned} U_1 &= yxYXYxyXY & V_1 &= X \\ U_2 &= yxYXYxyXYXyxYXYxyxYXyxyXY & V_2 &= yxYXYxyXYXyxyXY. \end{aligned}$$

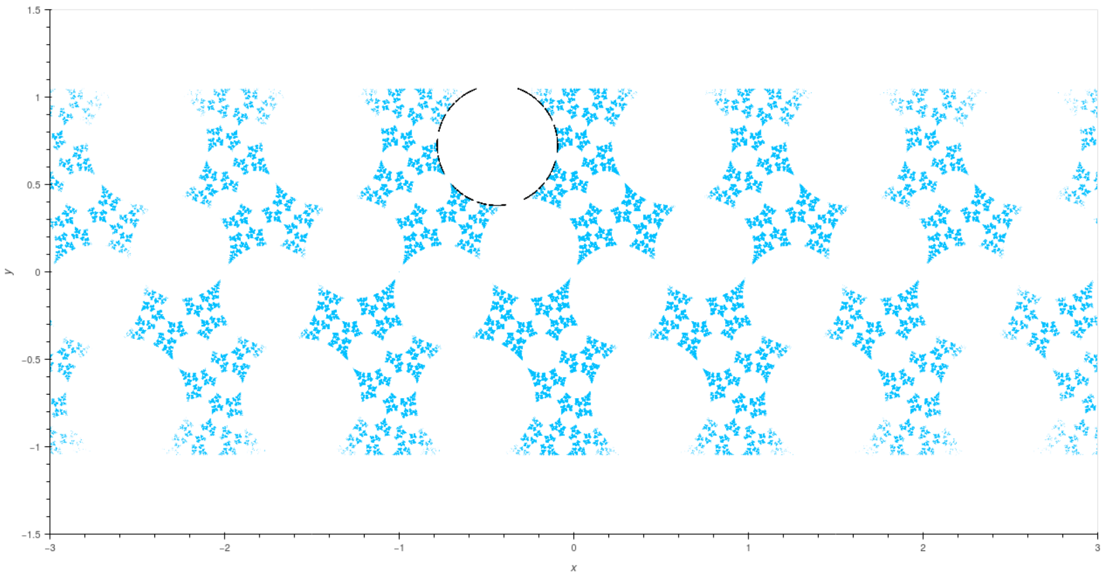
The reader may check that $U_1V_1 = U_2V_2$. We ask for $\rho \in \mathbb{C}$ such that all *five* traces

$$\begin{array}{ccc} \operatorname{tr} U_1 & & \operatorname{tr} V_1 \\ \operatorname{tr} U_1V_1 & = & \operatorname{tr} U_2V_2 \\ \operatorname{tr} U_2 & & \operatorname{tr} V_2 \end{array}$$

are real. It turns out that $\rho \approx -0.777994 + 1.47962i$ works. We plot the limit sets of $\langle X, Y\rangle$, $\langle U_1, V_1\rangle$, and $\langle U_2, V_2\rangle$ in figure 4. By exercise (2.4), both of the subgroups are Fuchsian. Small deformations

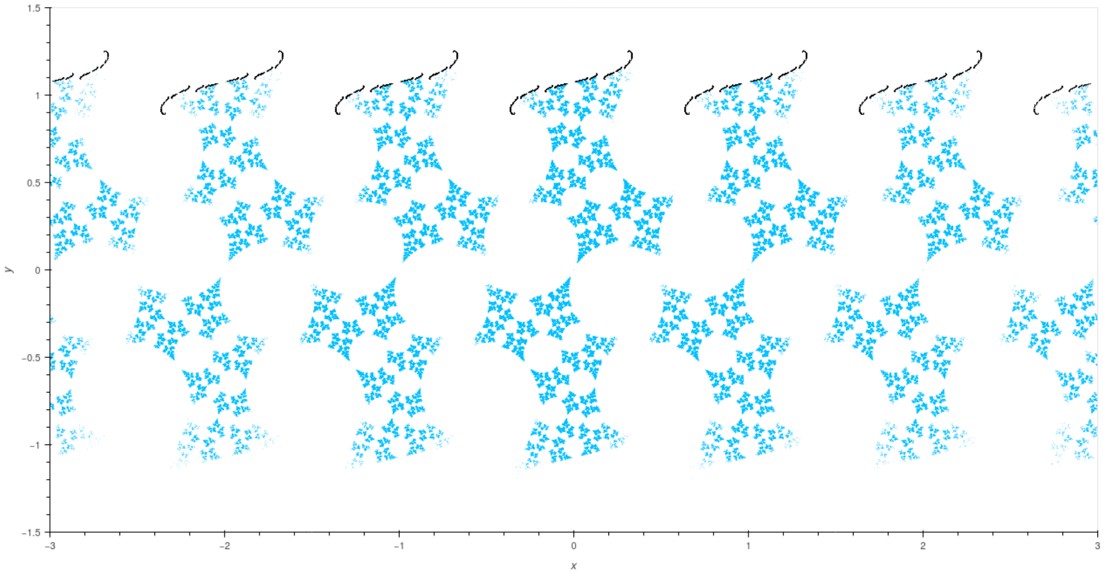


(a) Limit set of $\langle U_1, V_1 \rangle$.

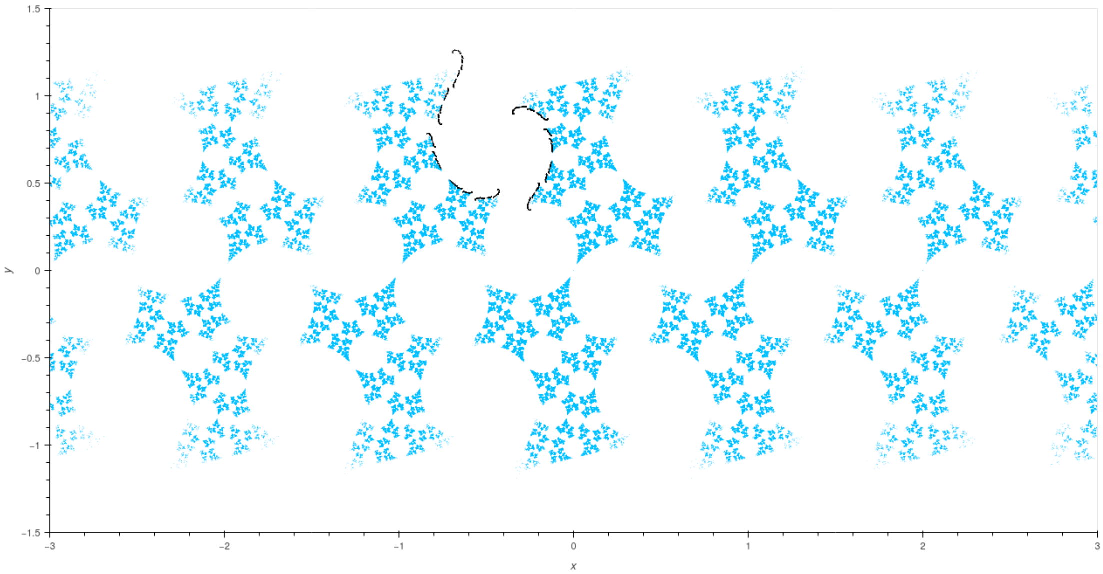


(b) Limit set of $\langle U_2, V_2 \rangle$.

Figure 4: Limit sets (black) of two Fuchsian subgroups of Γ_ρ (blue) for $\rho \approx -0.777994 + 1.47962i$.

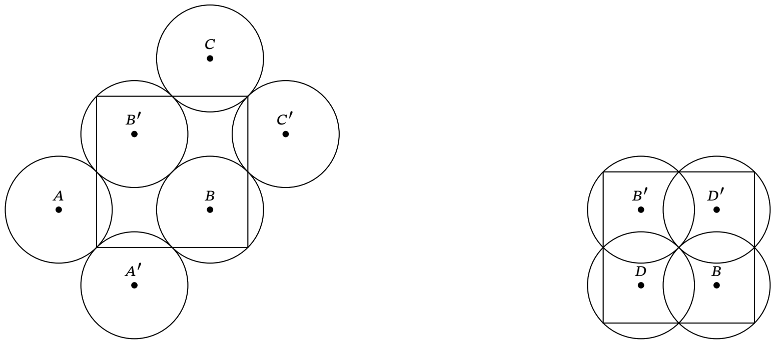


(a) Limit set of $\langle U_1, V_1 \rangle$.



(b) Limit set of $\langle U_2, V_2 \rangle$.

Figure 5: Limit sets (black) of two quasi-Fuchsian subgroups of Γ_ρ (blue) for $\rho \approx -0.721001 + 1.47962i$.



(a) Fundamental domain for G_1.

(b) Fundamental domain for G.

Figure 6: Fundamental domains for the groups in example (3.2).

of ρ will produce quasi-Fuchsian subgroups. This procedure, which is studied in detail in [EMS23] for this particular example, produces the groups shown in figure 5 where ρ has been moved to some $\rho + \varepsilon$ such that the global group is still discrete.

(3.2) Example. The following example is due to Wielenberg [Wie78, Example 3], see also [KAG86, Example 61]. We take a configuration of lines and circles on $\hat{\mathbb{C}}$ shown in figure 6a, and define the group

$$G_1 = \left\langle T = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ U = \begin{bmatrix} 1 & 2i \\ 0 & 1 \end{bmatrix},\ W = \begin{bmatrix} 1 & 0 \\ -1-i & 1 \end{bmatrix} \right\rangle;$$

then T and U pair the sides of the square in the figure, W sends B to B' (in fact these are the isometric circles of W), and A (resp. C) and A' (resp. C') are paired by $U^{-1}WT$ (resp. TWU^{-1}). Direct computation of angles (exercise) allows us to apply the Poincaré polyhedron theorem and find that the quotient manifold is a pair of thrice punctured spheres with a rank two cusp drilled out. We can also compute that the stabilisers of the thrice-punctured sphere components are

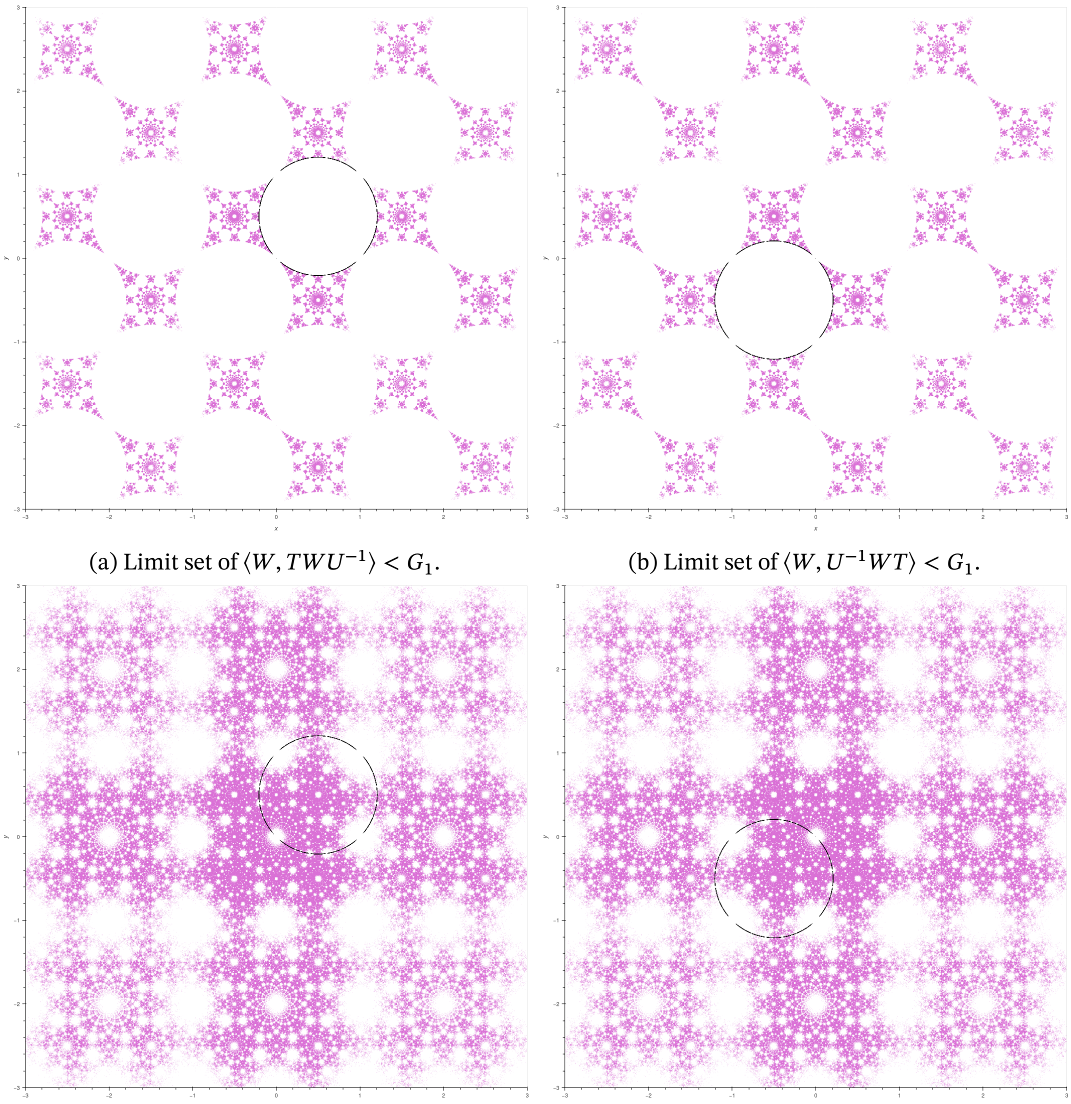
$$\langle W, U^{-1}WT \rangle \text{ and } \langle W, TWU^{-1} \rangle.$$

These two Fuchsian subgroups of G_1 are in fact peripheral subgroups, and are shown in figures 7a and 7b.

We now define a group extension $G = \langle G_1, V \rangle$, where

$$V = \begin{bmatrix} 1 & 0 \\ -1+i & 1 \end{bmatrix} \text{ satisfies } WV = VW, \text{ and } (U^{-1}WT)V = V(YWU^{-1});$$

in other words, V conjugates the two thrice-punctured spheres into each other and so $\mathbb{H}^3/G$ is obtained from $\mathbb{H}^3/G_1$ by gluing together the two boundary components. A fundamental domain for this new group is shown in figure 6b, and one can see that the resulting manifold is finite volume; from construction it is the complement of a link made up of four unknots cyclically chained together. This manifold has a totally geodesic embedded thrice-punctured sphere, uniformised by the (conjugate by V) Fuchsian subgroups whose limit sets are shown in figures 7c and 7d (of course, the limit set of G is dense in the plane, so the intricate patterns shown in this approximation are really reflecting the symmetric manner in which the Cayley graph of G in $\mathbb{H}^3$ is approaching every point on $\mathbb{S}^2$). This thrice-punctured sphere is a Seifert surface for the sublink of the three rank two cusps at the punctures, figure 8.



(a) Limit set of $\langle W, TWU^{-1}\rangle < G_1$.

(b) Limit set of $\langle W, U^{-1}WT\rangle < G_1$.

(c) Limit set of $\langle W, TWU^{-1}\rangle < G$.

(d) Limit set of $\langle W, U^{-1}WT\rangle < G$.

Figure 7: Limit sets for the Fuchsian subgroups in example (3.2).

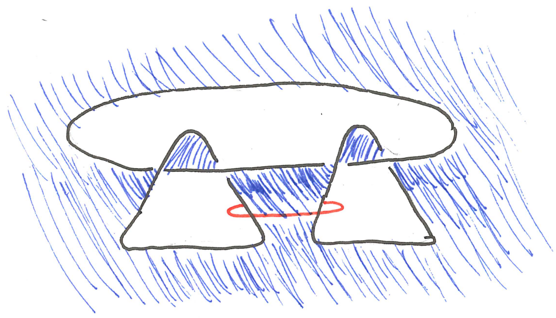
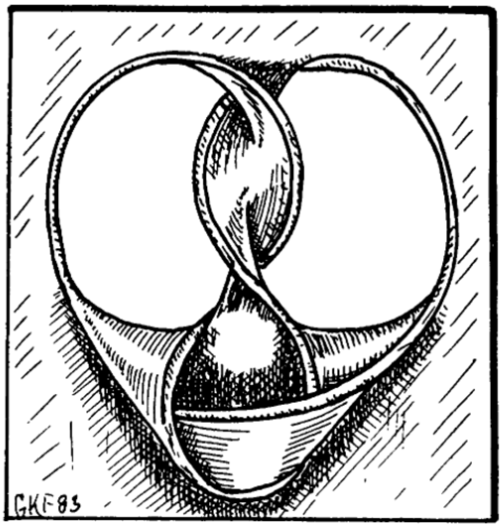
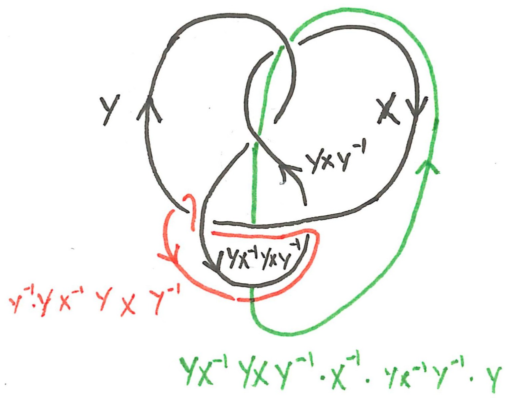


Figure 8: A Seifert surface for the alternating connect sum of two Hopf links and the additional rank 2 parabolic in the supergroup.



(a) Seifert surface for the figure eight knot, [Fra87, p. 156].



(b) Generators of the punctured torus.

Figure 9: Construction of a punctured torus subgroup of the figure eight knot group.

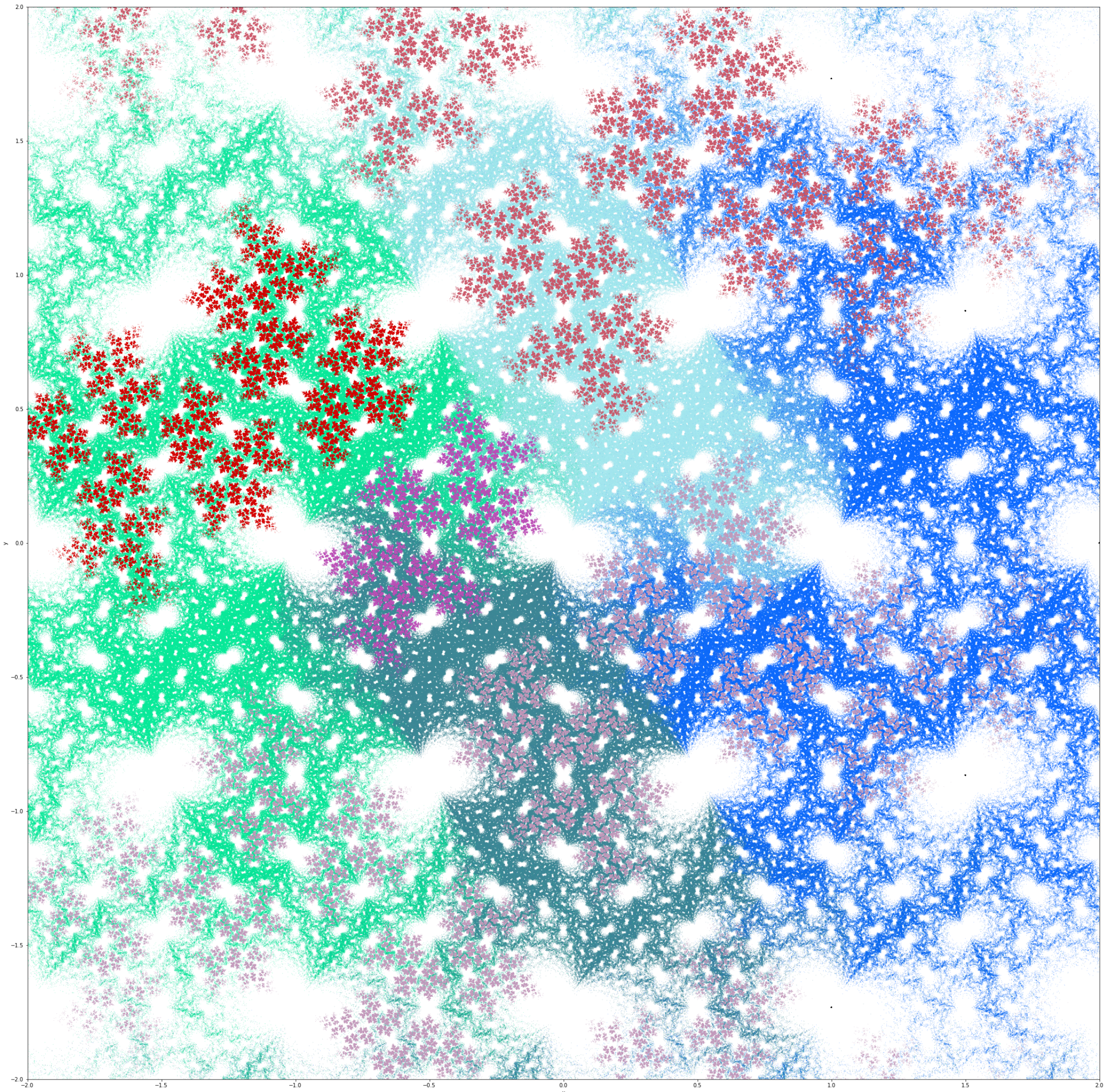


Figure 10: The limit set of a punctured torus subgroup (warm colours) in a figure eight knot group (cool colours).

(3.3) Exercise. Draw a convincing cartoon of the gluing process in example (3.2).

The following non-example shows that there exist subsurfaces of 3-manifolds which are not quasi-Fuchsian:

(3.4) Non-example. It is well-known [Ril75] that one pair of generators for the figure eight knot group (corresponding to the loops around the two bridge arcs) is

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 0 \\ -e^{2\pi i/3} & 1 \end{bmatrix}.$$

Let $\Gamma = \langle X, Y \rangle$. There exists a proper subsurface of $\mathbb{H}^3/G$ which is a once-holed torus Seifert surface of the figure eight knot, figure 9a. We can explicitly write down generators for the meridian and latitude of this torus, which are the red and green curves in figure 9b:

$$U = YxYXyxYx \text{ and } V = xYXy.$$

The limit set of $\langle U, V \rangle$ is laid over the limit set of $\langle X, Y \rangle$ in figure 10 (produced using Bella [Elz23], appendix B). In this picture it appears that the limit set of $\langle U, V \rangle$ is dense in $\hat{\mathbb{C}}$, and in the following lecture Emma will prove that this follows from the fact that the figure eight knot fibres over the circle with fibres exactly these Seifert surfaces. (The point essentially being that the group $\langle U, V \rangle$ does not have the correct ends—two surface ends—to be quasi-Fuchsian.) As a quick check of correctness the reader can observe that $[U, V]$ is parabolic.

§4. Knot complements

In the examples above, all the surfaces which we constructed were properly embedded but not closed. That this is always the case in knot complements is a conjecture of Menasco and Reid [MR92]. They prove that

(4.1) Theorem. *The complement of an alternating hyperbolic link or hyperbolic closed 3-braid in $\mathbb{S}^3$ cannot contain either a quasi-Fuchsian or a geometrically infinite closed embedded incompressible surface.*

Based on this, and various other results, they conjectured that

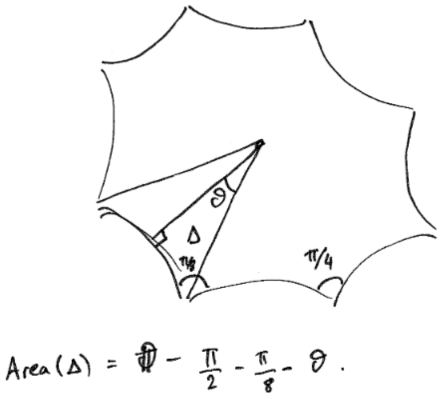
(4.2) Conjecture. The complement of a hyperbolic knot in $\mathbb{S}^3$ cannot contain an embedded totally geodesic closed incompressible surface.

They also give a complicated (eight component) example of a hyperbolic link with such a surface in its complement, so the restriction to the single component case is necessary.

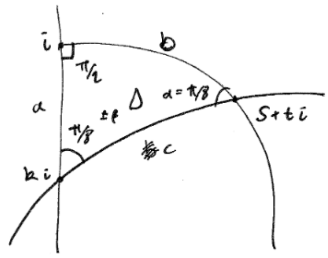
In a following section we will require the following result, which shows that the (possibly non-quasi-Fuchsian) groups corresponding to spanning surfaces of certain links do not contain accidental parabolics. This is fairly strong because it implies that the 3-manifolds of these groups are not degenerate with respect to algebraic limits.

(4.3) Theorem. *Let k be a link with a diagram that is connected, prime, and reduced alternating. Let Σ be a surface obtained via splitting crossings, gluing in discs to the components, and then adding back twist regions to glue the discs together so that the boundary of the result is a spanning surface (i.e. a* ***chessboard surface*** *for the link). Let F be a Fuchsian representative for Σ. Then the representation $F \to \pi_1(\Sigma) < \pi_1(\mathbb{S}^3 \setminus k)$ is type-preserving.*

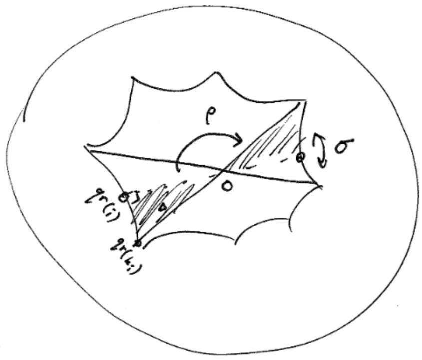
Sketch of proof. One first shows that if $\rho : F \to \pi_1(\Sigma)$ has an accidental parabolic γ then there is an essential annulus spanned by a component of k and the curve c on Σ represented by γ; this is the



(a) A (purported) $\pi/4$-regular hexagon in $\mathbb{H}^2$.



(b) The triangle Δ.



(c) The images of Δ under ρ and σ.

Figure 11: The trigonometry needed to produce the octagon O explicitly.

annulus along which c (which is a curve of nonzero length on Σ by assumption) can be moved to a parabolic fixed point. Now take a bounded polyhedral decomposition of $\mathbb{S}^3 \setminus k$ and move A to be in normal form. Show that the combinatorial area of A is zero. Carry out a study of the combinatorial possibilities for A based on this information and conclude that k is a $(2, q)$ torus link and Σ is the annulus spanning the two components; in particular c is boundary parallel on Σ which means it is isotopic to a parabolic point on Σ and hence $\rho^{-1}(\gamma)$ is parabolic (contradiction).

§A. Construction of a genus two Fuchsian group

We give a Fuchsian group F which glues each half-plane to a genus two surface; this group is obtained by writing down transformations in $\mathbb{M}$ which pair opposite sides of an octagon in $\mathbb{H}^2$ with equal side lengths and all interior angles equal to $\pi/4$.

Suppose such an octagon O exists. Then it glues to a genus two compact hyperbolic surface T_2. By the Gauss-Bonnet theorem we have

$$\text{Area}\, O = \text{Area}\, T_2 = -\int(-1)dA = -\int K(T_2)dA = -2\pi\chi(T_2) = 4\pi.$$

Cut O into right triangles (figure 11a). Such a triangle Δ has area $4\pi/16 = \pi/4$. By the hyperbolic area formula, the third angle of Δ (the one at the centre) is $\pi - \pi/2 - \pi/8 - \pi/4 = \pi/8$. We will now find an explicit coordinate realisation for this triangle in $\mathbb{H}^2$, and then rotate it around the $\pi/8$ angle to form the hexagon.

Without loss of generality, we can assume that the three vertices of Δ lie at i, ki, and $s + ti$ where $s^2 + t^2 = 1$, and that the right angle is at i (figure 11b). By the trigonometry of hyperbolic right-angled triangles [Bea83, §7.11], we can compute that

$$k = \cot\frac{\pi}{8} + \sqrt{\cot^2\frac{\pi}{8} - 1}, \quad s = \sqrt{1 - \tan^2\frac{\pi}{8}}, \text{ and } t = \tan\frac{\pi}{8}.$$

Now for simplicity we move from the upper half-plane model $\mathbb{H}^2$ to the disc model $\mathbb{B}^2$, conjugating $s + ti$ to 0. There is a standard map q which sends $\mathbb{H}^2 \mapsto \mathbb{B}^2$ with $i \mapsto 0$, given by

$$q(z) = \frac{2}{z+i} + i;$$

we therefore precompose q with a map sending $s + ti \mapsto i$, for instance

$$r(z) = \frac{z-s}{t}.$$

Hence in the ball model $\mathbb{B}^2$ our desired triangle is the image of Δ under $q \circ r$. Let $\rho : \Delta \to \Delta$ be the elliptic of order 8 which fixes 0, and let $\sigma : \Delta \to \Delta$ be the reflection across the geodesic joining 0 and $(qr)(i)$ in the disc (figure 11c). Then we wish to find an isometry f of $\mathbb{B}^2$ which sends the geodesic segment joining $(qr)(ki)$ to $(\sigma qr)(ki)$ to the geodesic segment joining $(\rho^4 qr)(ki)$ to $(\rho^4\sigma qr)(ki)$ (and the other side-pairing transformations which we want will be $\rho^n f \rho^{-n}$ for $n \in \{0, 1, 2, 3\}$). Of course, f should be a hyperbolic element of $\mathbb{M}$ which preserves Δ (so although ρ^4 does send one arc to the other it glues the endpoints in opposite order to that which we want). We can do this by writing down a Möbius transformation with real trace which fixes the two endpoints on the disc of the diameter through $(qr)(i)$ (i.e. $\pm(qr)(i)/|(qr)(i)|$), and then imposing the further condition that it sends $(qr)(i) \mapsto (\rho^4 qr)(i) = -(qr)(i)$. If the transformation is represented by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the desired conditions are

$$\begin{aligned} a\frac{\zeta}{|\zeta|} + b &= c\frac{\zeta^2}{|\zeta|^2} + d\frac{\zeta}{|\zeta|} \\ -a\frac{\zeta}{|\zeta|} + b &= c\frac{\zeta^2}{|\zeta|^2} - d\frac{\zeta}{|\zeta|} \\ a\zeta + b &= -c\zeta^2 - d\zeta \end{aligned}$$

where

$$\zeta = (qr)(i) = \frac{1 - t + is}{(1+t)i - s}.$$

Solving this system of equations, we find that one possibility is

$$A = \begin{bmatrix} 1+\sqrt{2} & \omega \\ \overline{\omega} & 1+\sqrt{2} \end{bmatrix} \text{ where } \omega = \sqrt{-6 + 2\sqrt{2} - 4i\sqrt{-2+2\sqrt{2}}}.$$

(We could take $\pm A$ or $\pm A^{-1}$ as well.) Now ρ is represented by the matrix

$$R = \begin{bmatrix} \exp(\pi i/8) & 0 \\ 0 & \exp(-\pi i/8) \end{bmatrix}$$

so the desired Fuchsian group uniformising T_2 is

$$F = \left\langle \begin{bmatrix} 1+\sqrt{2} & \omega \\ \overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (1+i)\omega \\ (1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & i\omega \\ -i\overline{\omega} & 1+\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1+\sqrt{2} & (-1+i)\omega \\ (-1-i)\overline{\omega} & 1+\sqrt{2} \end{bmatrix} \right\rangle.$$

§B. Computation of figure eight Seifert surface

```
""" Example: plotting the figure eight knot limit set along with
    the traces of elements in the group and the limit set of a
    Seifert surface subgroup.
"""

from bella import riley,cayley
from mpmath import mp
import holoviews as hv
hv.extension('matplotlib')

omega = mp.exp(2*mp.pi*1j/3)
G = riley.ClassicalRileyGroup(mp.inf, mp.inf, -omega)

traces = []
for word in G.free_cayley_graph_dfs(6):
  trace = cayley.simple_tr(G[word])
  traces.append(trace)
print(len(traces))

numpoints = 4*10**7
seed = G.fixed_points((0,1))[0]
limset = G.coloured_limit_set_fast(numpoints, seed=seed)
scatter = hv.Scatter(limset, kdims = ['x'], vdims = ['y','colour'])\
            .opts(marker = "d", s = .2,\
                  aspect=1, fig_size=1000,\
                  color = 'colour', cmap="glasbey_cool")\
              .redim(x=hv.Dimension('x', range=(-2, 2)),\
                     y=hv.Dimension('y', range=(-2, 2)))

P = G.subgroup([ G.string_to_word("YxYXyxYx"), G.string_to_word("xYXy") ])
limsetP = P.coloured_limit_set_fast(numpoints, (P.fixed_points((1,1)))[0])
scatterP = hv.Scatter(limsetP, kdims = ['x'], vdims = ['y','colour'])\
             .opts(marker = "d", s=.2, aspect=1, \
                   color = "colour", cmap="glasbey_warm")


scatter *= hv.Points([(float(z.real), float(z.imag)) for z in traces])\
             .opts(color='black', s=5) * scatterP

hv.save(scatter, 'fig8lattice_mpl2.png', fmt='png')
```

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