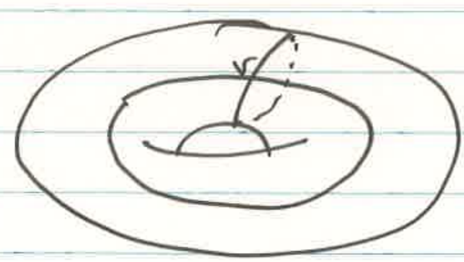
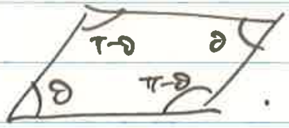
Example (primary school)

Consider $\mathbb{T}$:

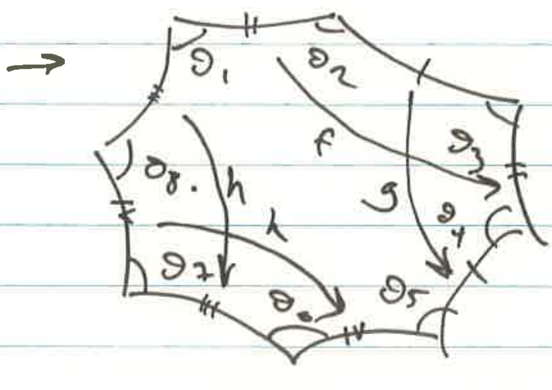


By lemma 9.2 of Lee, "Intro. to Riemannian Manifolds", Ed. 1, the ~~total~~ sum of angles at the vertex v must be 2π & hence we can cut to a parallelogram:



Example (intermediate school):

$$\sum_{i=1}^{8} \theta_i = 2\pi.$$



Walking ~~along~~ around v, we obtain the relation

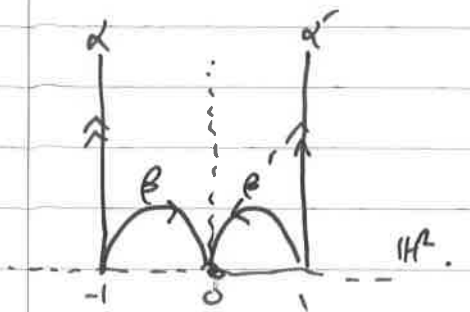
$$(fgf^{-1}g^{-1})(h^{-1}k^{-1}kh) = 1$$

which is the standard relation for S_2.

Given any octagonal tiling of $\mathbb{H}^2$, where the tiles are positioned & edge-to-edge, we obtain an S_2 by gluing:

Example (the simplest non-identity gp and some relations).

Consider an (∞,∞,∞) triangle & reflections in its sides.
To make the result orientation-preserving, we double it:



~~An the isometry sending α to α' is of the form $z \mapsto z+2$, followed by the~~

~~An isometry sends~~

We take the isometry $X = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, which sends $\alpha \to \alpha'$;
we compute all isometries sending ~~the~~ $\beta \to \beta'$:

$Y = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ st $Y(-1) = 1$
and $Y(0) = 0$

then: $\dfrac{-a+b}{-c+d} = +1 \rightsquigarrow$ ~~so~~ $b-a = d-c$

$b/d = 0 \rightsquigarrow b = 0$ so $a = c-d$.

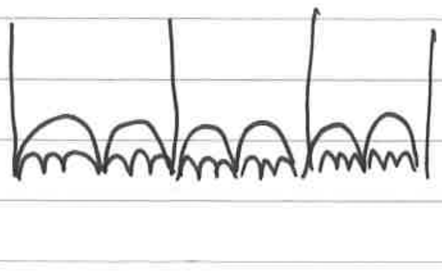
But $\det Y = ad - bc = ad = 1$ so $d = a^{-1}$

$\rightsquigarrow c = a+d = a+a^{-1}$ &

$Y = \begin{pmatrix} a & 0 \\ a+a^{-1} & a^{-1} \end{pmatrix}$ is the general isometry.

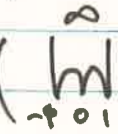
Observe this is parabolic iff ~~$a+a^{-1} = \pm 2$~~
$(a+a^{-1})^2 = 4$, i.e. $a \in \{\pm 1\}$.

Parabolic case:



$\leftarrow$ nice tiling.

Suppose $a=2$, so $\quad Y = \begin{pmatrix} 2 & 0 \\ 5/2 & 1/2 \end{pmatrix}$.

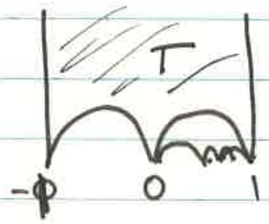
We compute $Y\Big($  $\Big)$:

$$Y(\infty) = \frac{2\cdot\infty}{\frac{5}{2}\cdot\infty + 1/2} = \frac{2}{5/2} = 4/5.$$

$$Y(-1) = \frac{-2}{-5/2 + 1/2} = {-2}/{-4/2} = 1.$$

$$Y(0) = 0$$

$$Y(1) = \frac{2}{\frac{5}{2}+\frac{1}{2}} = \frac{2}{6/2} = \frac{4}{6} = \frac{2}{3}.$$



Rmk. $\mathrm{Fix}(Y) = \{0, \frac{3}{5}\}$.

Conclude that $\langle X, Y\rangle \cdot T$ does not fill $\mathbb{H}^2$.

(This shows that there exist discrete groups which do not tile the full plane.)

~~Example (Indiscrete gp that does not tile.)~~

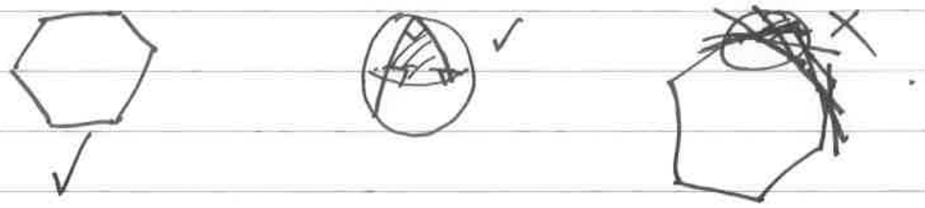
~~Ex~~

~~We observe that~~

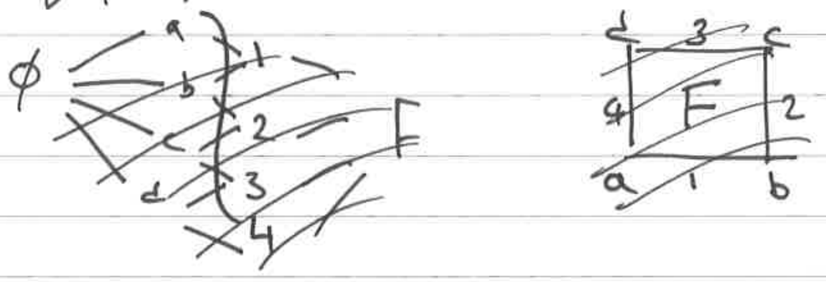
~~$\Rightarrow$ if a tiling of~~

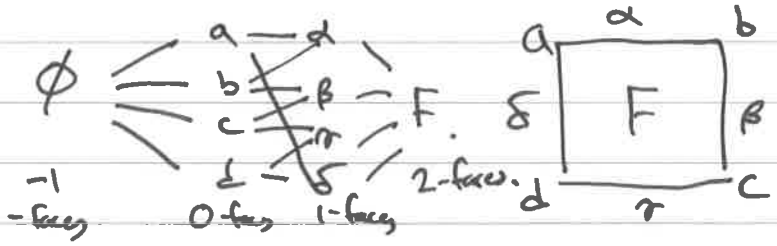
Fix X to be one of S^n, $\mathbb{H}^n$, $\mathbb{E}^n$ for $n \geq 2$.

Definition: A polyhedron in X is an intersection of countably many half-spaces such that only finitely many defining hyperplanes meet any cpct subset of X.



Given any polyhedron we have a natural lattice:





Fact. ~~Defn.~~ A topological group is discrete if there is a sequn of elents approximating the identity.

Lemma. (Exercise.) If G acts by homeo. ~~by isometry~~ on a topological ~~metric~~ space such that the images $G \cdot U$ of some ~~open~~ set with non-empty interior are disjoint, then G is discrete. $\square$.

Slogan: "Tilings induce discreteness."

we consider polyhedra P which come equipped with a side-pairing: for every side (codim. 1 face) s there exists a side s' and an isometry $g_s \in \mathrm{Isom}^+ X$ such that

$$g_s(s) = s', \quad g_{s'} = g_s^{-1}, \quad \text{and } (s')' = s.$$

This induces an equivalence relation on $\overline{P}$: the smallest equiv. rel. containing $x \sim y$ if $\exists$ a side-pairing sending x to y, for $x, y \in \partial P$, and the trivial eq. rel. on the interior.

Consider the quotient $\overline{P}/\sim$. For there to be any hope of manifold structure, the sum of angles around each codim two face must be 2π.

We may also project the metric d_X to $\overline{P}/\sim$: if $\pi : \overline{P} \to \overline{P}/\sim$ is the projection, set

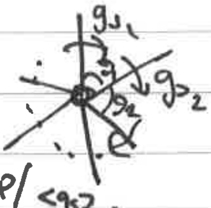
$$\overline{d}_X(p, q) = \inf_{\substack{\tilde{p} \in \pi^{-1}(p), \\ \tilde{q} \in \pi^{-1}(q)}} d_X(\tilde{p}, \tilde{q}).$$

<u>Theorem</u> (Poincaré). Let P be a polyhedron with a side pairing structure. Assume also that:

iii) $\forall$ sides s, $g_s(P) \cap P = \emptyset$;

iv) the projection $\pi : \overline{P} \to \overline{P}/\sim$ is finite-to-one;

vi) if you walk around an edge, the product $g_{s_1} \cdots g_{s_n} = 1$ and $\sum \theta_i = 2\pi$.



Then a) $\overline{P}/\sim = \langle g_s \rangle P / \langle g_s \rangle$. b) the only relations in $\langle g_s \rangle$ are from (vi). c) $\langle g_s \rangle$ is discrete.

<u>Rmk</u> Usually one requires completeness of the development of $\overline{P}/\sim$ but this is not strictly necessary. Proof: carefully read IV.F & IV.E of [M].

§ (G,X)-structures.

RECALL. A <u>manifold</u> of dimension n is a second-ctble Hausdorff topological space which admits an open cover $\{U_\alpha\}$ and a family of open maps ~~maps~~ $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that

~~$\phi_\beta^{-1}\phi_\alpha : U_\alpha \cap U_\beta \to$~~

$\phi_\beta \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is a homeomorphism for all α, β.

<u>Definition</u>. Let X be a ~~simply~~ connected manifold and let G be a gp acting on X by homeomorphisms.
A (G,X)-structure on a manifold Y is an open cover $\{U_\alpha\}$ and a family of open maps $\phi_\alpha : U_\alpha \to Y$ such that
$\phi_\beta \phi_\alpha^{-1}$ is a restriction of an elmt of G for all α, β.

<u>Observe</u> that if we construct a manifold $M = \cancel{X}/G$ ~~where~~ ~~$X \in \{\mathbb{H}^n, \mathbb{E}^n, S^n\}$ and~~ $\bar{P}/\sim$ as in prev. sec. ~~G is a gp of isometries~~,
then M admits an $(X, \mathrm{Isom}(X))$-structure: around each pt draw a small ball; either it is in interior of P (trivial to find a local chart) or ~~it's~~ on an edge (deform P slightly).

Other examples.

- A Riemann surface is a 2-manifold with a $(\mathbb{C}, \mathrm{Holo}(\mathbb{C}))$ structure. — a conformal structure

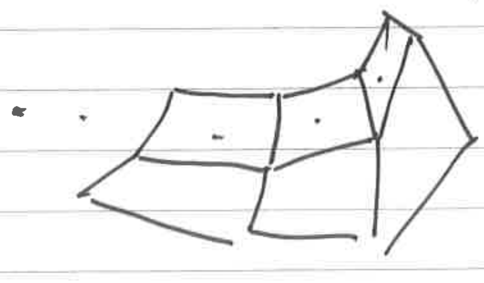
- Given a Riemann surface we can ask for more restrictive structures: e.g.

$(\hat{\mathbb{C}}, PSL(2,\mathbb{C}))$ — a projective or Kleinian structure.

$(\mathbb{H}^2, PSL(2,\mathbb{R}))$ — a hyperbolic or Fuchsian structure.

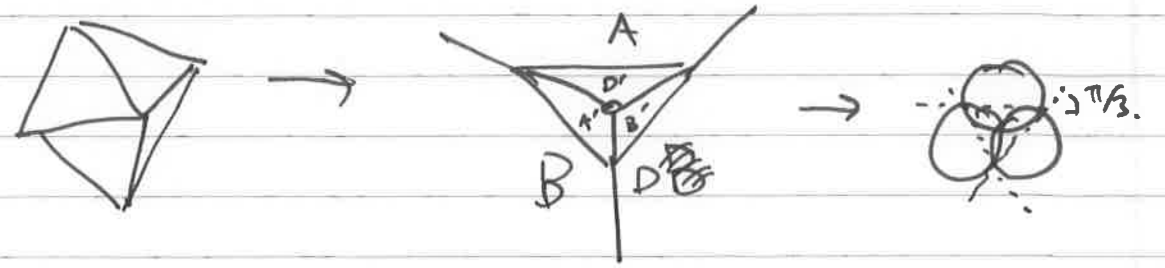
One conformal structure — many projective structures — unique complete hyperbolic structure.

- $(\mathbb{R}^n, \mathrm{Aff}(\mathbb{R}^n))$, e.g.

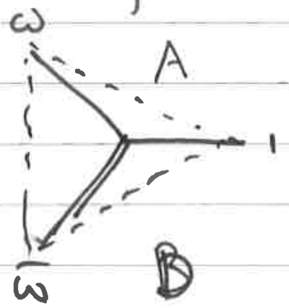


§ Examples.

Recall e.g. $(5,3)$:

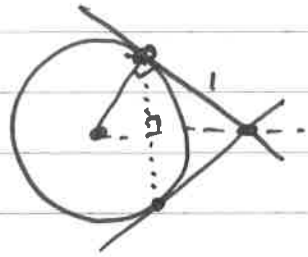


Pick your favorite equilateral triangle:



$\omega = e^{2\pi i/3}$.

Consider the circle tangent to the lines $[\omega, 0]$ & $[\bar{\omega}, 0]$:



$$\frac{\omega + \bar{\omega}}{2} = \cos\frac{2\pi}{3} = \frac{1}{2}.$$

Anyway: A sends $(\infty, \omega, 1)$ to $(\bar{\omega}, \omega, 0)$.
B sends $(\infty, \bar{\omega}, \omega)$ to $(1, \bar{\omega}, 0)$
D sends $(\infty, \bar{\omega}, 1)$ to $(1, \omega, 0)$.

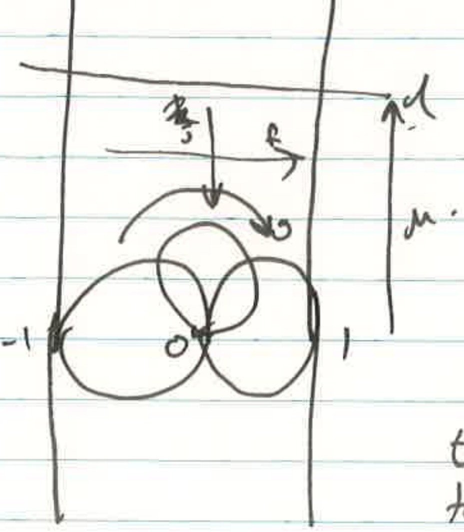
A: $a = -b$, $a = c\bar{\omega}$, $a\omega + b = c\omega^2 + d\omega$

$$\begin{aligned}
0 &= c\omega^2 + (d - c\bar{\omega})\omega + c\bar{\omega} \\
&= \cancel{c\omega^2 + d\omega + c\bar{\omega}}. \quad \cancel{d = -c.} \\
&= c\omega^2 + d\omega - c + c\bar{\omega}. \\
&= 2c\bar{\omega} + d\omega - c. \\
&= (2\bar{\omega} - 1)c + d\omega.
\end{aligned}$$

$$d = \frac{1 - 2\bar{\omega}}{\omega} c = (\bar{\omega} - 2\omega)c.$$

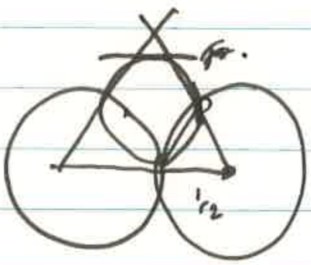
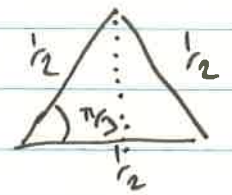
$\rightsquigarrow$ Choose $c = i/\sqrt{3}$: $A = \frac{1}{2}\begin{pmatrix} 1 - i/\sqrt{3} & -1 + i/\sqrt{3} \\ 2i/\sqrt{3} & 3 + i/\sqrt{3} \end{pmatrix}$. etc.

Example 2.



$$f = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$
$$g = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

take any circle through O tangt to $\mathbb{R}$, ~~say of radius 1: then it meets the two circles of g at:~~ of radius $1/2$ for simplicity; let μ be the height of some horizontal line.

$$\frac{1}{2}\cdot \sin\frac{\pi}{3} = \frac{\sqrt{3}}{4}.$$

Wish to send $\left(0, \frac{1}{4}+\frac{\sqrt{3}}{4}i, -\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)$

to $(\infty, 1+\mu i, -1+\mu i)$:

so: $d=0$; ~~$b \ne c$, so $ad-bc = -c^2$.~~

$\omega = \frac{1}{4}(1+\sqrt{3}i)$; $\sigma = 1+\mu i$.

Now $a\omega + b = c\omega\sigma$

and $-a\omega + b = c\omega\sigma$

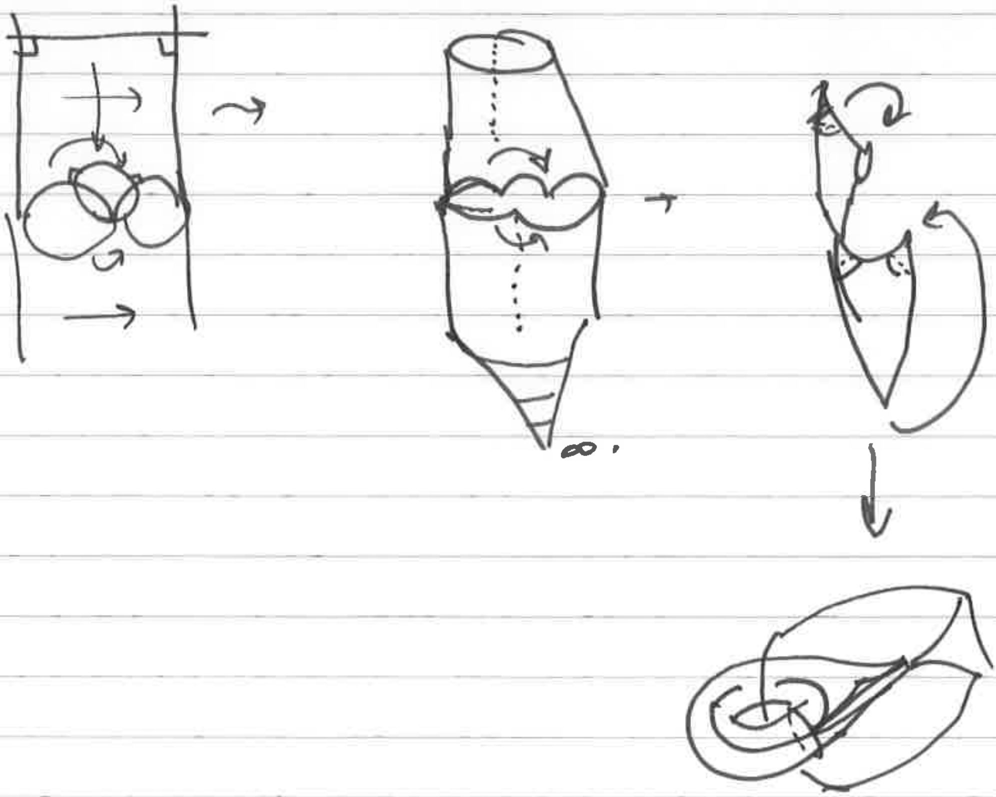
$2a\omega = 0 \rightsquigarrow a=0.$

$b = c\omega\sigma$: $\quad ad-bc=1$, so $-bc=1$, $b=-1/c$.

$-\frac{1}{c} = c\omega\sigma$

$-\frac{1}{\omega\sigma} = c^2. \rightsquigarrow c = \dfrac{2}{\sqrt{(\sqrt{3}-i)(\mu-i)}}.$

Quotient:



Example 3. $\mathrm{Isom}(S^3) = SO(4) \cong \frac{S^3 \times S^3}{\mathbb{Z}/2\mathbb{Z}}$

(where each S^3 is acting like ~~$SO(3)$~~ unit quaternions).

Lens group is a cyclic gp gen by $\rho : (z, w) \mapsto (e^{2\pi i/p} z, e^{2\pi i/q} w)$.
We consider the ~~action~~ on ~~[illegible]~~ $\mathbb{P}^3\mathbb{R}$:

$$\rho \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} =$$