A problem session for weeks 3 and 4

31 July 2023

1 Braids

1.1 4-plats and 2-bridge knots

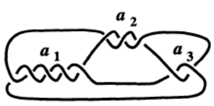
1. What does β determine in a torsion diagram? Hint: start at 0 and walk along the curve u according to its orientation. Where do you end up? Hence β is not the crossing number as in 'number of crossings', but in terms of 'number of *the* crossing'. The exercise is to check that this is actually what the homology number is measuring.

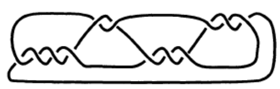
2. Show that the (α, β) torsion diagram with β *even* (so α is odd and the orientation of I_2 is reversed) is a diagram of the $(\alpha - \beta)/\alpha$ knot. Hence if the assumption 'β odd' is deleted then Schubert's theorem (Theorem 3.5) should be modified to read .... and $\beta^{\pm 1} = \pm\beta'$... in both cases. Hint: rotate I_2 by an isotopy of $\mathbb{R}^2$. The corresponding presentation is now on X and y^{-1} so to obtain the $(\alpha - \beta)/\alpha$ Riley word from the β/α one swap Y and y; instead of $VX = YV$ we also have $VX = yV$ so the new Farey word is $Vxvy$ up to inverses.

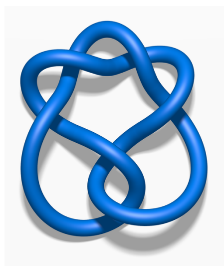
3. Classify the 2-bridge links with $\alpha \in \{0, 1\}$. Draw the corresponding 4-plats.

4. Draw the 3/4 torsion diagram and write down the corresponding word.

5. Give the rational number corresponding to the trefoil knot and the three knots below. Compute the corresponding Farey words.







6. Verify that the 5/7 and the 3/7 knots are the same. What are the correspoding Riley words? Conjecture a rule relating the p/q Riley word with the p^{-1}/q Riley word (inverses taken mod $2q$).

7. Compute the Riley word of an arbitrary 2-bridge link formally (i.e. prove Proposition 3.15). Hint: in the case of a 2-component link the Wirtinger presentation will give you two relations. These should correspond to the same relator, the Farey word.

8. On lens spaces.

   (a) $\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}$.

   (b) A homeomorphism $h : \partial U \to \partial U$ extends to an autohomeomorphism of U iff $h_*(\beta) = [\beta]^{\pm 1}$. (Here β is one of the loops in the standard basis, same notation as above.)

   (c) $L(1,0) = \mathbb{S}^3$ and $L(0,1) = \mathbb{S}^2 \times \mathbb{S}^1$. In fact, $L(1,q) = \mathbb{S}^3$ for all q.

   (d) $L(p,q) = L(p,q')$ if and only if $q \equiv \pm q' \pmod p$ or $q \equiv \pm q'^{-1} \pmod p$. Hint:- under these conditions there is a homeomorphism $h : L_{p,q} \to L_{p,q'}$ which preserves the two handebodies in the first case and swaps them in the second case.

   (e) Computer project. Draw pictures of lens spaces [Cou+22].

9. Use (4) of the previous exercise to prove that $\mathfrak{b}(\alpha,\beta)$ and $\mathfrak{b}(\alpha',\beta')$ are equivalent as unoriented links iff $\alpha = \alpha'$ and $\beta^{\pm 1} = \beta' \pmod \alpha$. Then prove that they are equivalent as oriented links iff $\alpha = \alpha'$ and $\beta^{\pm 1} = \beta' \pmod{2\alpha}$.

10. If an n-braid is chosen with permutation π, as in the definition, then there exists a link with μ components obtained by identifying the P_i with $Q_{\pi(i)}$. Give a formal definition of this link (the [closure!of a braid]closure of the braid). Prove (Alexander, 1928) that every link can be obtained as the closure of some braid [BZ03, §2D].

11. (Bankwitz-Schumann) All 2-bridge knots are alternating.

12. All 2-bridge knots are amphichiral.

13. Show that the two embeddings (Thurston's and Riley's) of the figure eight group into $\mathrm{PSL}(2,\mathbb{C})$ are conjugate subgroups:

$$\Pi_1 = \left\langle \phi_B = \frac{i}{\sqrt{\omega}}\begin{bmatrix}1 & 1\\ 1 & -\omega^2\end{bmatrix}, \phi_C = \begin{bmatrix}1 & \omega\\ 0 & 1\end{bmatrix}, \phi_D = \begin{bmatrix}2 & -1\\ 1 & 0\end{bmatrix}\right\rangle,$$
$$\Pi_2 = \left\langle \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \text{ and } Y \mapsto \begin{bmatrix}1 & 0\\ -\omega & 1\end{bmatrix}\right\rangle.$$

14. Computer project. Plot $\bigcup_{r\in\mathbb{Q}} \Lambda_r^{-1}(0)$.

1.2 Braids in general and mapping classes

1. Write down the relevant fibre bundle and prove the generalised Birman exact sequence using the same argument as the one-point version.

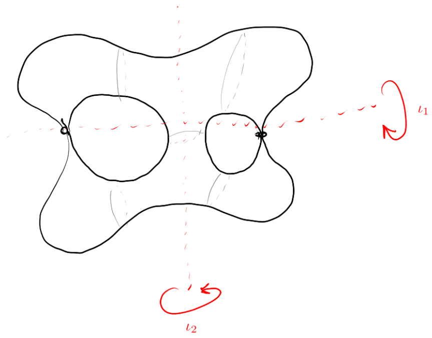
2. Supply a formal proof that $\pi_1(\mathrm{UConf}(\mathbb{B}^2,n)) \simeq \pi_1(\mathrm{UConf}(\mathbb{C},n))$.

3. Show that the (p,q) torus knot is the closure of the braid $(\sigma_1,\ldots,\sigma_{p-1})^q$ by embedding the latter braid on the torus.

4. (The four-times punctured sphere.) Let $S_{0,4}$ be the topological four-times punctured sphere.

   (a) Show that $\mathrm{Mod}(\mathbb{T}^2) = \mathrm{SL}(2,\mathbb{Z})$. [Hint: write $\mathbb{T}^2$ as the quotient of $\mathbb{C}$ by some lattice Λ, and show that $\mathrm{SL}(2,\mathbb{Z})$ is the maximal group which permutes all the different lattices that produce the same complex structure—see [IT87, §1.2] for details and pictures.]

(b) Observe that $\S_{0,4}$ and $\mathbb{T}^2$ are both produced by quotients of a quadrilateral in $\mathbb{C}$ and conclude that there is an induced surjective homomorphism $\mathrm{Mod}(S_{0,4}) \to \mathrm{SL}(2,\mathbb{Z})$ given by topological lifting; show that the kernel of this homomorphism is generated by two rotations by π of $S_{0,4}$:

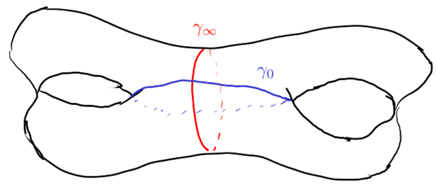


(c) Conclude that $\mathrm{Mod}(S_{0,4}) = \mathrm{SL}(2,\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.

(d) Describe the maps in the spherical Birman exact sequence, $1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathrm{UConf}(\mathbb{S}^2, 4)) \to \mathrm{Mod}(S_{0,4}) \to 1$.

(e) Recall that $\mathrm{SL}(2,\mathbb{Z})$ is generated by $R = (1, 1 \mid 0, 1)$ and $Q = (0, -1 \mid 1, 0)$. Write $L = (1, 0 \mid 1, 0)$. Show that $\mathrm{SL}(2,\mathbb{Z}) = \langle R, L\rangle \rtimes \langle Q\rangle$.

(f) Let $\Gamma_1 = \langle L, R\rangle$. Describe the action of Γ_1 as a subset of the mapping class group on the curves γ_0 and γ_∞ shown here:



(g) Compare with the discussion of the previous lecture on 2-bridge knots and links.

5. Prove that

$$B_n = \langle \sigma_1, \dots \sigma_{n-1} : \forall_{|i-j|>1} \sigma_i \sigma_j = \sigma_j \sigma_i, \\ \forall_i \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \rangle.$$

is a presentation of the braid group.

6. Recall that S_1^1 denotes the torus with a single boundary component. Prove that $\mathrm{Mod}(S_1^1) \simeq B_3$. (Hint: Take the quotient of S_1^1 by the hyperelliptic involution).

7. (The Birman exact sequence, revisited) Here we outline an alternative proof of the Birman exact sequence, using only hyperbolic geometry and Alexander's method, avoiding appealing to the long exact sequence in homotopy and deep results about the contractibility of spaces of homeomorphisms of surfaces.

Let S be a hyperbolic surface. Fix a point x in the interior of S.

(a) Prove that $\pi_1(S, x)$ has trivial center (Hint: Use the representation $\pi_1(S, x) \to \mathrm{Isom}^+(\mathbb{H}^2)$ and the classification of isometries of $\mathbb{H}^2$).

(b) Recall that if G is a group with $Z(G) = 1$, then, we have a short exact sequence,

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

(c) Show that the canonical homomorphism $\mathrm{Mod}(S, x) \to \mathrm{Aut}(\pi_1(S, x))$ is injective. (Hint: use Alexander's method [FM12, p. 59] applied to a suitable collection of curves based at the point x).

(d) Show that there is a natural, well-defined injection $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S, x))$ (use the fact that S has contractible universal cover to see injectivity). What about surjectivity? See the Dehn–Nielsen–Baer Theorem [FM12, Chapter 8].

(e) Consider the diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{Mod}(S,x) & \xrightarrow{\text{Forget}} & \mathrm{Mod}(S) & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(S,x) & \longrightarrow & \mathrm{Aut}(\pi_1(S,x)) & \longrightarrow & \mathrm{Out}(\pi_1(S,x)) & \longrightarrow & 1
\end{array}$$

Show that the image of $\pi_1(S, x)$ is contained in the image of $\mathrm{Mod}(S, x)$ as follows: Let α be a simple loop in S, based at x. Push α to the left a bit, to get α^+, and to the right a bit to get α^-. Show that the composition of Dehn twists $T_{\alpha^+}T_{\alpha^-}^{-1}$ acts as conjugation by α on $\pi_1(S, x)$. Conclude that the image of $\pi_1(S, x)$ lies within the kernel of Forget.

(f) Verify that the map $\pi_1(S, x) \to \mathrm{Mod}(S, x)$ is actually the push map, and complete the statement.

8. This exercise comes from a paper of Farb [Far22]. It provides a geometric explanation for the existence of an exceptional surjection.

(a) Let $n > m > 2$, and denote by Σ_n the symmetric group on n letters. Show that there exists an epimorphism $\Sigma_n \to \Sigma_m$ if and only if $(n, m) = (4, 3)$. (Hint: If $n \geq 5$, then A_n is simple).

(b) Find a lift of the homomorphism $\Sigma_4 \to \Sigma_3$ obtained above to $B_4 \to B_3$, where B_n denotes the braid group on n strands.

(c) Recall that $B_n = \pi_1(\mathrm{Poly}_n(\mathbb{C})) = \mathrm{Mod}(D_n)$, where $\mathrm{Poly}_n(\mathbb{C})$ denotes the space of monic, degree n, square free polynomials over $\mathbb{C}$. There is a map $\mathrm{Fer} : \mathrm{Poly}_4(\mathbb{C}) \to \mathrm{Poly}_3(\mathbb{C})$, called the resolving quartic map, induced via the following: send a configuration of 4 distinct points (q_1, q_2, q_3, q_4) to 3 *distinct* points (z_1, z_2, z_3) where,

$$\begin{aligned}
z_1 &= (q_1 - q_2 - q_3 + q_4)^2/4 \\
z_2 &= (q_1 - q_2 + q_3 - q_4)^2/4 \\
z_3 &= (q_1 + q_2 - q_3 - q_4)^2/4
\end{aligned}$$

Investigate the induced map $\mathrm{Fer}_* : B_4 \to B_3$.

9. (Capping and realizing B_3 as homeomorphisms) This exercise involves the braid group on 3 strands, and in particular, some consequences of viewing it as the mapping class group $\mathrm{Mod}(D_3)$. Is there a section to the projection $\mathrm{Homeo}^+(D_3, \partial D) \to B_3$? That is, can you realise the braid group on 3 strands as a group of homeomorphisms? The answer is *yes*, proven by Thurston (`https://mathoverflow.net/q/55555`).

The following discussion might be helpful regarding the above. We can cap the boundary of D_3 with a disk. If this disk is marked, then one has the following **capping exact sequence** [FM12, p. 82]:

$$1 \to \mathbb{Z} \to B_3 \to \mathrm{Mod}(S_{0,4}) \to 1$$

The homomorphism $\mathrm{Mod}(D_3) \to \mathrm{Mod}(S_{0,4})$ is simply given by extending as the identity, and the kernel is a Dehn twist about the boundary of D_3 (which, as you should check, generates the center of B_3). We show in another exercise that $\mathrm{Mod}(S_{0,4}) \simeq \mathrm{PSL}(2,\mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

Switching gears a bit, suppose we cap the boundary component of D_3 with just a disk (no marked point). It can be shown (see for example [FM12, p. 104]) that one has an exact sequence,

$$\cdots \to \pi_1(\mathrm{Diff}^+(S^2)) \to \pi_1(UTS^2) \to B_3 \to \mathrm{Mod}(S_{0,3}) \to 1$$

where UTS^2 is the unit tangent bundle of S^2. Recall also that $\mathrm{Diff}^+(S^2)$ has the homotopy type of SO(3). Now, the unit tangent bundle of the 2-sphere UTS^2 can be identified with $\mathbb{R}\P^3$, so, in particular, $\pi_1(UTS^2) = \mathbb{Z}/2\mathbb{Z}$. One can try and place this exact sequence into the context of the Birman exact sequence for the 3-stranded spherical braid group and obtain a similar picture to the four-punctured sphere case.

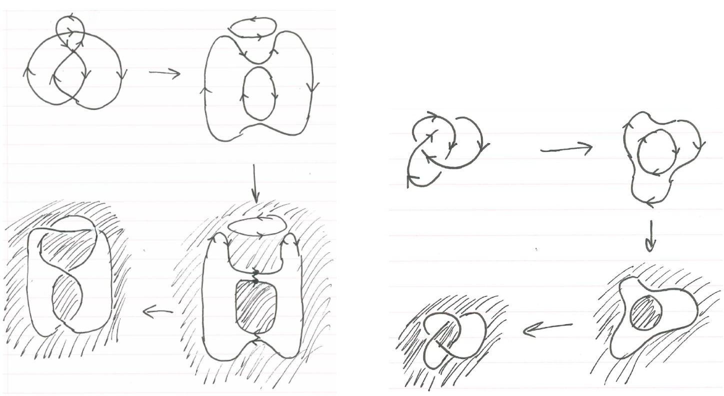
2 Knot polynomials

2.1 The Alexander–Conway polynomial

1. Draw a Seifert surface for the $(3,4,3)$ pretzel knot.

2. Show that the skein relations are invariant under Reidemeister moves.

3. Give the genera for the following two Seifert surfaces:



4. Let L be the link consisting of two parallel trefoil knot complements. Construct a surface spanned by L by taking a narrow rectangular strip of paper and tying it up in a trefoil knot with the two short ends suitably identified. Show that this surface is non-orientable. Draw a Seifert surface for L.

5. Show that for any oriented link L, $\Delta_L(t) = \Delta_L(t^{-1})$; and for any oriented knot k, $\Delta_k(1) = \pm 1$.

6. In this long exercise, we will compute the Alexander–Conway polynomials (or 'A–C polynomials' for short) for all 2-bridge knots, following [BZ03, § 12C].

   Define the **Fibonacci polynomials** $\mathsf{fib}_n(z)$ by

$$\mathsf{fib}_0(z) = 0, \qquad \mathsf{fib}_1(z) = 1,$$
$$\mathsf{fib}_{n+1}(z) = z\,\mathsf{fib}_n(z) + \mathsf{fib}_{n-1}(z),$$
$$\mathsf{fib}_{-n}(z) = (-1)^{n+1}\,\mathsf{fib}_n(z) \text{ for } n \geq 0.$$

   (a) Show that the Fibonacci polynomials for $n \geq 0$ are of the form

$$f_{2n-1} = 1 + a_1 z^2 + a_2 z^4 + \cdots + a_{n-1} z^{2n-2}$$
$$f_{2n} = z(b_0 + b_1 z^2 + b_2 z^4 + \cdots + b_{n-1} z^{2n-2})$$

   for some $a_i, b_i \in \mathbb{Z}$, with $a_{n-1} = b_{n-1} = 1$.

   Let $k = \mathfrak{b}(\alpha, \beta)$ be a two-bridge *knot*—so $\alpha \equiv \beta \equiv 1 \pmod 2$. Represent this knot by the braid

$$\sigma_2^{a_1}\sigma_1^{-2b_1}\cdots\sigma_2^{a_k}$$

   where $k = (m+1)/2$. (There is always a unique generalised Euclidean algorithm of this form, [BZ03, Proposition 12.7].)

   (b) Using the skein relations, show that the A–C polynomial of the 4-plat defined by σ_2^a for $a > 0$ is $\Delta_a(z) = (-1)^{a+1}\,\mathsf{fib}_a(z)$.

   (c) Show that $\Delta_{-a} = (-1)^{a+1}\Delta_a$.

   (d) Assume that $a > 0$, $b > 0$, and $c \neq -1$. Show that the A–C polynomial of the 4-plat defined by $\sigma_2^a\sigma_1^{-2b}\sigma_2^c$ is

$$\Delta_{abc}(z) = \Delta_{a-1}(z)\Delta_c(z) + \Delta_a(z)\Delta_{c+1}(z) - bz\Delta_a(z)\Delta_c(z).$$

   Hint: use the skein relations on the double points of σ_2^a from top to bottom.

   (e) Use (1) to show that if $c > 0$,

$$\deg \Delta_{abc} = a + c - 1 \text{ and } |\mathrm{LC}(\Delta_{abc})| = |b+1|$$

   and if $c < 0$ then

$$\deg \Delta_{abc} = a - c - 1 \text{ and } |\mathrm{LC}(\Delta_{abc})| = |b|.$$

   One can also show that if $a < 0$, $b < 0$, and $c \neq 1$ then

$$\Delta_{abc} = \Delta_{a+1}\Delta_c + \Delta_a\Delta_{c-1} - bc\Delta_a\Delta_c$$

   and so $\deg \Delta_{abc} = |a| + |c| - 1$, $C(\Delta_{abc}) = |\beta| + 1 - \eta$ where $\eta = 1$ or 0 according to whether $c > 0$ or $c < 0$. But the proof is boring so just take it for granted. We continue.

   (f) Suppose that $\beta = \sigma_2^{a_1}\sigma_1^{-2b_1}\beta'$ and $\beta' = \sigma_2^{a_2}\sigma_3^{-2b_2}\cdots$ where $a_1 > 0$ and $a_2 > 0$. Show that the A–C polynomial of β is

$$\Delta_\beta = \Delta_{a_1}\Delta_{\sigma_2\beta'} + \Delta_{a_1-1}\Delta_{\beta'} - b_1 z\Delta_{a_1}\Delta_{\beta'}$$

   and $\deg \Delta_\beta = \deg \Delta_{a_1}\Delta_{\sigma_1\beta'}$.

(g) Conclude by induction that

$$\deg \Delta_k = |a_1| - 1 + \sum_{j>1} |a_j| \text{ and } |\mathrm{LC}(\Delta_k)| = \prod_{j=1}^{k-1} (|b_j| + 1 - \eta_j).$$

(h) It is a classical theorem that the genus of an alternating knot is $(d+1)/2$ where d is the degree of its A–C polynomial. Compute the genus of every 2-bridge knot. Conclude also that there are infinitely many knots of positive genus.

2.2 Quantum invariants

1. Show that if $c \in \mathrm{Aut}(V \otimes V)$ is an R-matrix then so are λc, c^{-1}, and $\tau \circ c \circ \tau$ where τ is the flipping map and λ is a scalar.

2. Show that the map $c \in \mathrm{Aut}(V \otimes V)$ defined by

$$c(e_i \otimes e_i) := q e_i \otimes e_i$$
$$c(e_i \otimes e_j) := \begin{cases} r_{ji} e_j \otimes e_i & \text{if } i < j; \\ r_{ji} e_j \otimes e_i + (q - pq^{-1}) e_i \otimes e_j & \text{if } i > j. \end{cases}$$

is an R-polynomial and verify that it satisfies the quadratic polynomial

$$c^2 - (q - pq^{-1})c - p\, \mathrm{id}_{V \otimes V} = 0.$$

3. (a) Show that the dual vector space of a coalgebra C is an algebra: consider the map $\overline{\lambda} : C^\vee \otimes C^\vee \to (C \otimes C)^\vee$ defined by $(f \otimes g)(u \otimes v) := g(v) \otimes f(u)$ and define $A = C^\vee$, $\mu = \Delta^\vee \circ \overline{\lambda}$ and $\eta = \varepsilon^\vee$.
   (b) Show that the algebra of functions structure defined on $k\{X\}^\vee$ is indeed the natural one arising from the coalgebra structure on $k\{X\}$ via the construction of (a) above.
   (c) Show that the dual vector space of a finite dimensional algebra is a coalgebra. Hint: in the finite dimensional setting, $\overline{\lambda}$ is an isomorphism.

4. Define **cocommutativity** of a bialgebra A. Show that the flip $\tau_{V,W} : V \otimes W \to W \otimes V$ is an isomorphism of A-modules when A is cocommutative. Show that addition in $\mathbb{A}^1$ is cocommutative in $k[x]$.

5. (Fun for 334 students.) Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Define a **convolution** operation on $\mathrm{Hom}(H, H)$ by the composition

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

An endomorphism $S \in \mathrm{Hom}(H, H)$ is a **antipode** for the bialgebra H if $S * \mathrm{id}_H = \mathrm{id}_H * S = \eta \circ \varepsilon$. A **Hopf algebra** is a bialgebra with an antipode. Define the commutative algebras

$$M(2) = k[a, b, c, d]$$
$$GL(2) = M(2)[t]/((ad - bc)t - 1)$$
$$SL(2) = GL(2)/(t - 1) = M(2)/(ad - bc - 1).$$

(a) Show that for any commutative algebra A there are bijections $\mathrm{Hom}(GL(2), A) \simeq \mathrm{GL}_2(A)$ and $\mathrm{Hom}(SL(2), A) \simeq \mathrm{SL}_2(A)$, where GL_2 and SL_2 are the classical matrix algebras over A.

(b) Define $\Delta : M(2) \to M(2) \otimes M(2) \simeq k[a', a'', b', b'', c', c'', d', d'']$ by

$$\Delta(a) = a'a'' + b'c'', \qquad \Delta(b) = a'b'' + b'd''$$
$$\Delta(c) = c'a'' + d'c'', \qquad \Delta(d) = c'b'' + d'd''.$$

Show that for any commutative algebra A, Δ corresponds to usual matrix multiplication in $M_2(A)$.

(c) Show that $\Delta(ad - bc) = (a''d'' - b''c'')(a''d'' - b''c'')$.

(d) Observe that Δ induces maps $GL(2) \to GL(2) \otimes GL(2)$ and $SL(2) \to SL(2) \otimes SL(2)$.

(e) Define suitable morphisms $GL(2) \to k$ and $SL(2) \to k$ corresponding to units, and suitable automorphisms of $GL(2)$ and $SL(2)$ corresponding to inversions. Check that you now have a Hopf algebra structure on $GL(2)$ and $SL(2)$.

6. (Fun for 334 students who also like quantum field theory.) The affine plane is the algebra generated freely by x and y modulo the relation $yx = xy$. The **quantum commutation relation** is the relation $yx = qxy$, where $q \in k^*$. Let I_q be the two-sided ideal of the free algebra $k\langle x, y\rangle$ generated by $yx - qxy$, and let the **quantum plane** be the quotient $k_q[x, y] := k\langle x, y\rangle/I_q$.

(a) Let R be an algebra without zero divisors. If α is an algebra endomorphism of R, then an α-**derivation** of R is a linear map $\delta : R \to R$ such that for all $a, b \in R$, $\delta(ab) = \alpha(a)\delta(b) + \delta(a)\alpha(b)$. Given an injective algebra endomorphism $\alpha : R \to R$ and an α-derivation δ of R there exists a unique algebra structure on the free module of polynomials $R[t]$ such that the natural inclusion $R \to R[t]$ is an algebra morphism and $ta = \alpha(a)t + \delta(a)$. This algebra structure is called the **Ore extension** $R[t, \alpha, \delta]$. (A proof of existence and uniqueness is [Kas95, Theorem I.7.1].) If R is (left) Noetherian, then so is the Ore extension [Kas95, Theorem I.8.3].

Show that if α is the automorphism of $k[x]$ determined by $\alpha(x) = qx$, then $k_q[x, y]$ is isomorphic to the Ore extension $k[x][y, \alpha, 0]$. Conclude that $k_q[x, y]$ is Noetherian with no zero divisors and has basis $\{x^iy^j\}_{i,j\geq 0}$.

(b) Show also that for any pair (i, j) of nonnegative integers, $y^ix^j = q^{ij}x^jy^i$ and for any k-algebra R there is a natural bijection between $\mathrm{Hom}(k_q[x, y], R)$ and $\{(X, Y) \in R \times R : YX = qXY\}$. These pairs are [points of the quantum plane]R-points of the quantum plane.

(c) Let A be the algebra of smooth complex functions on $\mathbb{C} \setminus \{0\}$. Let $q \in \mathbb{C} \setminus \{0, 1\}$. Consider the elemenets of $R = \mathrm{End}_{\mathrm{lin.}}(A)$ given by

$$\tau_q(f)(x) = f(qx) \text{ and } \delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x}.$$

Show that (τ_q, δ_q) is an R-point of $k_q(x, y)$ and justify the equation $\lim_{q\to 1} \delta_q = d/dx$.

7. (a) Compute the HOMFLY polynomial of the trefoil knot and the Hopf link.

(b) Show that if L is a link and L' is its mirror image then $P_{L'}(x, y) = P_L(x^{-1}, y^{-1})$. Conclude that the trefoil knot is not amphichiral.

8. Show that the HOMFLY polynomial is invariant under mutation, hence does not distinguish between the Kinoshita–Terasaka and Conway knots.

9. Let (L_+, L_-, L_0) be a Conway triple. Show that there exist tangles L_1, L_2, L_3, L_4 such that

$$L_+ = L_1 \circ (L_2 \otimes X_+ \otimes L_3) \circ L_4$$
$$L_- = L_1 \circ (L_2 \otimes X_- \otimes L_3) \circ L_4$$
$$L_0 = L_1 \circ (L_2 \otimes X_0 \otimes L_3) \circ L_4.$$

10. On representations of B_n, [Kas95, § X.6.2]. Let V be a vector space, $c \in \mathrm{Aut}(V \otimes V)$, and $n > 1$ an integer. For $1 \le i \le n-1$ define $c_i \in \mathrm{Aut}(V^{\otimes n})$ by

$$c_i = \begin{cases} c \otimes \mathrm{id}_{V^{\otimes n-2}} & \text{if } i = 1 \\ \mathrm{id}_{V^{\otimes n-1}} \otimes c \otimes \mathrm{id}_{V^{\otimes n-i-1}} & \text{if } 1 < i < n-1 \\ \mathrm{id}_{V^{\otimes n-2}} & \text{if } i = n-1. \end{cases}$$

   (a) Show that if $|i - j| > 1$ then $c_i c_j = c_j c_i$.
   (b) Show that $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$ for all i if and only if c is an R-matrix (i.e. a solution of the Yang-Baxter equation).
   (c) Let $c \in \mathrm{Aut}(V \otimes V)$ be an R-matrix. Show that for any $n > 0$ there exists a unique group morphism $\rho_n^c : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $1 \le i \le n-1$. (In other words, R-matrices manufacture representations from B_n onto $V^{\otimes n}$ for all $n \ge 2$.)

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