A problem session for weeks 1 and 2

17 July 2023

1 Classical knot theory

1.1 Basic definitions and first examples

1. Show that the figure eight knot is amphichiral.
2. Show that if k is any knot and $\pi : k \to \mathbb{R}^2$ is some projection which induces a diagram then there exists an alternating knot k' with the same projection as a subset of $\mathbb{R}^2$ (Tait, late 1800s).
3. Define the **writhe** of a diagram δ of a knot k to be

$$w(\delta) = \sum_{v \in V(\delta)} \varepsilon(v)$$

(compare the definition of linking number, where the sum is only over intersections of two different components). Show that w is invariant under the second and third Reidemeister moves, but not the first: in fact adding a single 'loop' (either over or under) to a knot diagram adds 1 to the writhe. In fact the writhe is a topological invariant of the knot k *together with* a choice of section of the unit normal bundle to k, or (equivalently) a 'ribbon' thickening of k. This additional structure on k is called a **framed knot** (and has an obvious generalisation to links).

4. Show that the only knot of crossing number 0 is the unknot; that there are no knots of crossing number 1 or 2; that the only knot of crossing number 3 is the trefoil knot; that the only knot of crossing number 4 is the figure eight knot. Conclude that the figure eight and trefoil knots are distinct.
5. Show that the figure eight knot is amphichiral.

1.2 The fundamental group

1. Show that the Fox knot (fig. 1) is not the unknot. (See e.g. [Fox49] for one method)
2. Show that the fundamental groups of two separated rings and two linked rings (the **Hopf link**) are not isomorphic. The Hopf link is so named because every pair of circles in the Hopf fibration form a Hopf link in $\mathbb{S}^3$.
3. Exhibit 3-bridge presentations for the Kinoshita–Terasaka and Conway knots.
4. Find all presentations of the fundamental group of the trefoil knot onto A_5.

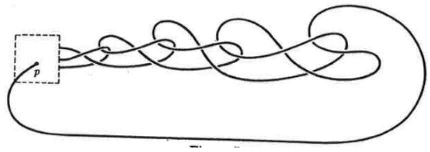

Figure 1: A wild knot, the **Fox knot** [CF08, p. 6]. Observe that this can somehow be unravelled, but it is not isotopic to the unknot [Fox49]! See also [Kau87, p. 52].

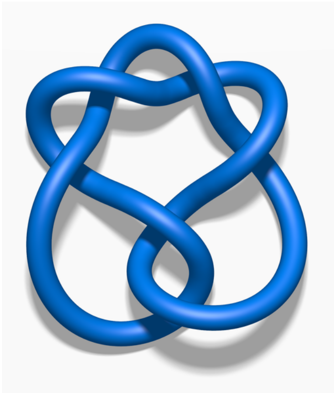

Figure 2: The stevedore's knot. Image by Jim.belk, released to public domain (see `http://commons.wikimedia.org/wiki/File:Blue_Stevedore_Knot.png`)

5. Show that the fundamental group of the Klein bottle is $\pi_1(K) = \langle x, y : y = xyx \rangle$. Show that no knot group admits a surjective representation onto $\pi_1(K)$.

6. Show that tricolourability of k is equivalent to the existence of a *surjective* homomorphism $\pi_1(k) \to S_3$.

7. Show that the trefoil knot is the $(2, 3)$ torus knot. Show that the (p, q) and (q, p) torus knots are equivalent.

8. Show that the stevedore's[1] knot (fig. 2) is 2-bridge and give a two generator presentation for its group.

9. Show that $\langle x, y : yxy = xyx \rangle \simeq \langle a, b : a^2 = b^3 \rangle$. Hint: $b = xy$ and $a = xyx$. Observe that this is a presentation for $\mathrm{PSL}(2, \mathbb{Z})$ [Ser02, Example 1.5.2 of Chapter I].

10. (Brauner's theorem, [Mil69, p. 4]) The (p, q)-torus knot is cut out by intersecting a sufficiently small 3-sphere in $\mathbb{C}^2$ with the algebraic curve $\mathbf{V}(z^p + w^q)$, i.e. it is an algebraic knot.

[1]"A workman employed either as overseer or labourer in loading and unloading the cargoes of merchant vessels." (OED)

2 Geometric knot theory

2.1 Geometric structures on knot complements

1. If you know Chapter VII of Maskit [Mas87]: write the figure eight group in terms of the amalgamated products and HNN extensions of the cyclic groups generated by

$$\begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

where $\omega = e^{2\pi/3}$.

2. Describe in SU(2), in terms of the group structures (where α denotes the upper-left-hand element of a generic element, eq. (2.11)),

   (a) the latitudes: the set of all $U \in \mathrm{SU}(2)$ such that $\mathrm{Re}\,\alpha$ is some fixed value (hint: this was already done for $x = \pm 1$);

   (b) the longitudes: the set of $U \in \mathrm{SU}(2)$ cut out by any hyperplane ($\mathbb{R}^3$) in $\mathbb{C}^2$ which passes through $\pm I$.

3. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus.

   (a) Show that the linear automorphism of $\mathbb{R}^2$ represented by $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ descends to T. The resulting map on the torus is the **Arnold's cat map** α.

   (b) Draw the mapping torus of α, $M_\alpha := (T \times [0,1])/((x,1) \sim (\alpha(x), 0))$. This manifold is a Sol-manifold.

4. [Thu97, Exercise 4.7.1] If ϕ is an isometry of $\mathbb{S}^2$ then the mapping torus M_ϕ is an $(\mathbb{S}^2 \times \mathbb{E}^1)$-manifold. In fact it is the quotient of $\mathbb{S}^2 \times \mathbb{E}^1$ by the discrete group generated by the transformation $(v,t) \mapsto (\phi v, t+1)$ where $v \in \mathbb{S}^2$. The manifold is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ when ϕ is orientation-preserving and is non-orientable otherwise. What other manifolds admit $\mathbb{S}^2 \times \mathbb{E}^1$ structures?

   (a) Any discrete subgroup of isometries of $\mathbb{S}^2 \times \mathbb{E}^1$ acts discretely (but not necessarily freely or effectively) on $\mathbb{E}^1$.

   (b) An infinite discrete group of isometries of $\mathbb{E}^1$ is isomorphic to $\mathbb{Z}$ or $C_2 * C_2$.

   (c) There are only three closed 3-manifolds, up to diffeomorphism, that admit $(\mathbb{S}^2 \times \mathbb{E}^1)$-structures. Two are orientable and one is not.

2.2 Hyperbolic invariants and computation

1. [Mar16, Exercise 3-46] (Halpern's inequality) Suppose G is a torsion-free Fuchsian group (i.e. discrete subgroup of $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Isom}^+(\mathbb{H}^2)$) acting on the upper half-plane $\mathbb{H}^2 = \{x + ti \in \mathbb{C} : t > 0\}$. Assume G has a parabolic fixed point at ∞ and that the parabolic subgroup is generated by $T : z \mapsto z + 1$. Prove that for every $A \in G$ that does not fix ∞, $|c| \geq 2$ where c is the lower-left-hand entry of A. (Hint: compute $\mathrm{tr}\, TATA^{-1}$.)

2. [Mar16, Exercise 3-5] A **Dirichlet region** for a Kleinian group G with centre $z \in \mathbb{H}^3$ is the closed convex hyperbolic polyhedron

$$\bigcap_{g \in G} H_g$$

where H_g is the relatively closed half-space which is bounded by the perpendicular bisector of $[z, gz]$ containing z. This is a fundamental polyhedron for G [Mas87, §IV.G].

Find a Dirichlet region for the rank two parabolic group generated by $z \mapsto z+1$ and $z \mapsto \tau$ for $\tau \in i\mathbb{R}_{>0}$. Show that generically it has six edges, but sometimes only four. Compute the hyperbolic volume of the part of the polyhedron lying above a general horosphere based at ∞. Show that the quotient $\mathbb{H}^3 \cup \mathbb{C}/G$ (this is OK since $\Omega(G) = \hat{C} \setminus \{\infty\}$!) is homeomorphic to $\{0 < |z| \leq 1 : z \in \mathbb{C}\} \times \S^1$, i.e. the complement of the core circle is the solid torus. This is the prototype of the local structure about a hyperbolic knot, the parabolic fixed point is 'stretched' onto the knot.

3. [Mar16, Exercise 6-1] Let Γ be the group of isometries of $\mathbb{E}^3$ which is generated by $(x, y, t) \mapsto (x+1, y, t)$ and $(x, y, t) \mapsto (-x, y+1, -t)$. Let $\Gamma_0 = \langle (x, y, t) \mapsto (x+1, y, t), (x, y, t) \mapsto (x, y+1, t) \rangle$
   (a) The group Γ preserves $\mathbb{C}$ and $\mathbb{C}/\Gamma$ is the Klein bottle.
   (b) The interior of $\mathbb{T}^2 \times [0, 1]$ obtained by thickening the torus $\mathbb{T}^2$ is almost hyperbolic except for the existence of hyperbolic essential cylinders with one boundary component on $\mathbb{T}^2 \times \{0\}$. The interior is $\mathbb{E}^3/\Gamma_0$. (This is the only manifold whose boundary components are tori whose interior does not have a complete finite-volume hyperbolic structure, by a theorem from the lecture.)
   (c) The torus $\mathbb{C}/\Gamma_0$ is the two-sheeted orientable cover of $\mathbb{C}/\Gamma$ and the cover transformation is 'flipping'. The corresponding 3-manifold $\mathbb{E}^3/\Gamma$ is called the **twisted I-bundle** over the Klein botthe and is the only homotopically atoroidal manifold whose interior does not have a hyperbolic structure.

References

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