Knot Knotes

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January 6, 2024

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Introduction

These are the notes for a eight-lecture minicourse given at the University of Auckland in July 2023. The course follows a fairly traditional set of topics—fundamental groups, invariants from geometric topology, braids, and knot polynomials—but given the particular interests of the audience we will go a bit more deeply into some aspects which do not normally end up in textbooks, in particular the role of group representations (both finite and infinite). We will also give full technical proofs of many of the results we use instead of just taking them for granted. In general we will expect the audience to have a good understanding of basic topology, group theory, and hyperbolic geometry. References on the background for each section may be found in the individual introductions.

> *"Oh, there we are back to those parallel lines," answered Whatif, "I admit that you can prove that if the alternating angles are equal then those lines must be parallel, but nobody could prove the converse. This is why Euclid put the converse (or what is equivalent to it), as his famous fifth postulate for the Euclidean plane. But now we are in a..."*
>
> *"diabolic plane?" asked Alice.* [65, p. 64]

Chapter 1

Classical knot theory

In this first week, we will look at classical knot theory—by this, we mean knot theory pre-Thurston (so up until the 1970s). A lengthy description of the history of knotting, including the mathematics, may be found in the delightful anthology [71]. We will emphasise the algebraic aspects, in particular the representation theory of knot complement groups (following R. Riley).

For these notes, we follow in particular the textbooks of Crowell–Fox [19], Kauffman [40], and Lickorish [45]; but since these books do not go deeply into a lot of what we want to do (Riley's work). The prerequisite topology and group theory may be found in the book by Stillwell [64].

1.1 Basic definitions and first examples

Let $\mathbb{S}^n$ be the n-sphere; usually we identify $\mathbb{S}^n := \mathbb{R}^n \cup \{\infty\}$. A **knot** is an embedding $k : \mathbb{S}^1 \to \mathbb{S}^3$. A **link** is an embedding $k : \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1 \to \mathbb{S}^3$. A **component** of a link is just a topological component of the image. The actual parameterisation k is not important, we usually identify the knot or link with the image. Often we will say 'knot' when we mean 'knot or link', hopefully in the places that it matters we remember to say so.

1.1 Example. The **unknot** is the image of the map $[0, 2\pi] \to \mathbb{R} \times \mathbb{C} = \mathbb{R}^3$ given by $t \mapsto (0, \exp(it))$. The **figure eight knot** and the **trefoil knot** (also called the **cloverleaf knot**) are shown along with the unknot in Fig. 1.1.

Knots are defined up to ambient isotopy in $\mathbb{S}^3$: two knots k, l are equivalent if there exists a continuous map $F : \mathbb{S}^3 \times [0, 1] \to \mathbb{S}^3$ such that $F(\cdot, 0)$ is the identity map, $F(\cdot, t) : \mathbb{S}^3 \to \mathbb{S}^3$ is an isotopy for all $t \in [0, 1]$, and $F(k(\cdot), 1) = l(\cdot)$.

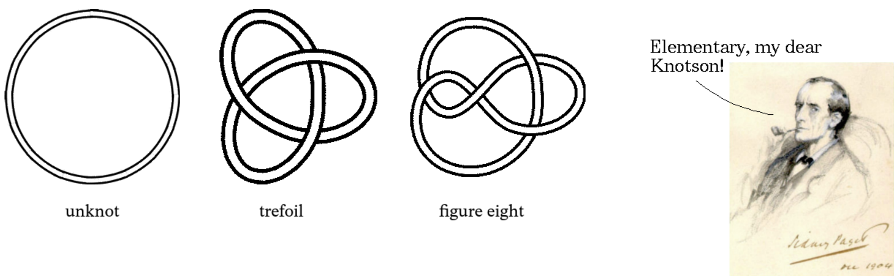

Figure 1.1: Three elementary knots.

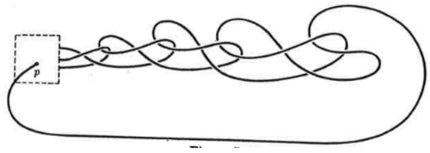

Figure 1.2: A wild knot, the **Fox knot** [19, p. 6]. Observe that this can somehow be unravelled, but it is not isotopic to the unknot [26]! See also [40, p. 52].

Remark. Some people say that knots are defined up to homeomorphism of $\mathbb{S}^3$, i.e. if there exists a homeomorphism $f : \mathbb{S}^3 \to \mathbb{S}^3$ which sends one knot onto the other. Clearly if two knots are equivalent up to ambient isotopy then they are equivalent up to ambient homeomorphism. The converse is almost true. If two knots are equivalent under orientation preserving homeomorphism then they are equivalent up to ambient isotopy [19, p. 10]. Two knots which are equivalent up to orientation reversing homeomorphism are said to form a **chiral pair**, and a knot equivalent up to ambient isotopy with its chiral twin (mirror image) is called **amphichiral**.

Finally we say that a knot is **polygonal** if it is piecewise linear except for finitely many vertices, and a knot is **tame** if it is equivalent to a polygonal knot. In Fig. 1.2 we show an example of a **wild** (i.e. non-tame) knot. We shall from this point assume that every knot is tame unless otherwise stated.

Usually we will work with planar projections of knots. We will give a formal definition but in reality the technicalities get in the way so we will hardly ever phrase anything in terms of the function ob which we are about to define.

1.2 Definition. A **knot diagram** of a link k is a planar[1] 4-valent graph δ together with a function $\mathrm{ob} : V(\delta) \to 2^{E(\delta)}$ which assigns to every vertex v an unordered pair $\mathrm{ob}(v) = \{e, f\}$ $(e \neq f)$ of edges incident with v such that in the planar embedding $k \hookrightarrow \mathbb{R}^2$ the edges e and f are on opposite sides of v, i.e. any arc from the midpoint of e to the midpoint of f crosses an edge originating from v that is neither e nor f.

Almost all projections $\mathbb{R}^3 \to \mathbb{R}^2$ of a 3-plane containing a link to a 2-plane not incident with the knot induce a knot diagram: the projection induces a 4-valent graph, and the function ob sends v to the pair of edges of the diagram which are the projection of the piece of the knot furthest away from the plane of projection (Fig. 1.3). Conversely a knot diagram clearly induces a knot (by simply separating the two strands into the third dimension at each vertex).

One can play around with 'bad' projections and produce some amusing results: `https://youtu.be/SqpzP81Z0BA`.

We will describe here a couple of other things we need from knot diagrams. Suppose k is an oriented knot, that is take an orientation of $\mathbb{S}^1$ and push it forward onto the image $k(\mathbb{S}^1)$; let δ be a diagram of k. Then every edge $e \in E(\delta)$ inherits an orientation, and the 'divalence' (number of in edges minus number of out edges) of every vertex $v \in V(\delta)$ is zero.

1. We can assign a **sign** $\epsilon(v)$ to each vertex $v \in V(\delta)$ according to the convention Fig. 1.4.

2. Define an equivalence relation $\leftrightsquigarrow$ on the set of edges $E(\delta)$ by $e \leftrightsquigarrow f$ iff there exists a vertex v such that $\{e, f\} = \mathrm{ob}(v)$. This sets up a partition $V(\delta)/\leftrightsquigarrow$ of the set of vertices, and the parts of this partition are the **arcs** of the diagram. We will write $\mathrm{arcs}(\delta)$ for this set of arcs. Note also that ob sets up a map $V(\delta) \to \mathrm{arcs}(\delta)$ which we also denote by ob; it is this function which is really what we are trying to formalise (a crossing is a place where an arc crosses over another arc). In Fig. 1.5 we show a diagram of the figure eight knot with four arcs (the connected

[1] i.e. comes equipped with a given fixed embedding into $\mathbb{R}^2$

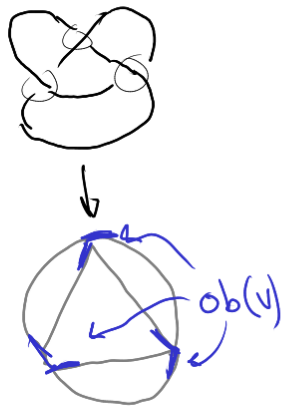

Figure 1.3: Formally encoding the data of a crossing via the function ob.

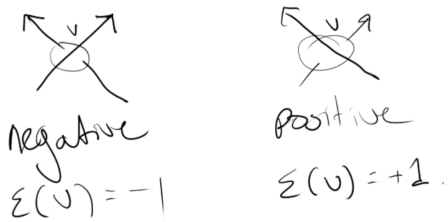

Figure 1.4: Sign of a vertex v.

components of the left-hand image). The **arc graph** of δ is the graph with vertex set arcs(δ) and an edge between arcs α and β iff there is a vertex of δ at which α and β meet (right hand image of Fig. 1.5)

On the subject of arcs, let k be a knot in $\mathbb{S}^3$ which meets a plane $\mathbb{R}^2$ in $2m$ points such that the arcs (in the usual topological sense) of k contained in each halfspace cut out by $\mathbb{R}^2$ are orthogonally projected onto arcs on $\mathbb{R}^2$ which are simple and mutually disjoint from the other arcs from the same halfspace. The minimal m for which this is possible is called the **bridge number** of k. It is very hard to compute this number in general. The only 1-bridge knot is the trivial knot; we will classify 2-bridge knots later on; and m-bridge knots for $m > 2$ do not admit a known classification.

It is an important (but hard to prove) theorem of Reidemeister that the topological definition of knot equivalency can be reinterpreted in terms of the combinatorics of knot diagrams:

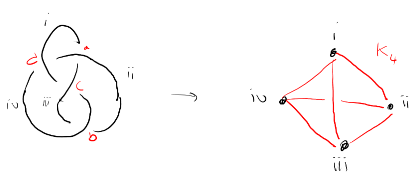

Figure 1.5: The arc graph of the figure eight knot.

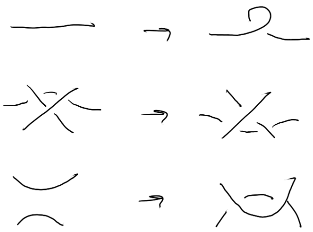

Figure 1.6: The three Reidemeister moves.

1.3 Theorem (Reidemeister)**.** *Two tame links k, k' are equivalent if and only if there exists a finite sequence of* ***Reidemeister moves*** *(Fig. 1.6) changing a diagram of k into a diagram of k'.*

In practice this theorem is useful when trying to define so-called *knot invariants*.

1.4 Definition. Let k be a link with diagram δ and arc graph $\mathrm{arcs}(\delta)$. Then $\mathrm{arcs}(\delta)$ is said to be **tricolourable** if it admits a (possibly non-proper) vertex colouring on 3 colours such that (i) at least two colours are used, (ii) any lollipop is coloured with either exactly one or exactly two colours, and (iii) any 3-cycle is coloured with either exactly one or exactly three colours.

1.5 Lemma. *Let k be a link. If there exists a diagram δ of k which has tricolourable arc graph, then every diagram of k has tricolourable arc graph. Hence the function $t : \mathsf{Link} \to \{0, 1\}$ which assigns to each knot the value 1 if it is tricolourable and 0 otherwise is well–defined, i.e. it does not depend on the diagram chosen.*

Proof. Reidemeister moves preserve tricolourability.

This is the first example of a **knot invariant**, a function $\mathsf{Link} \to S$ where S is a known set. It is not a very good one, but at least we get the following:

1.6 Corollary. *The figure eight knot is nontrivial (i.e. is not equivalent to the unknot).*

Proof. The incidence graph of the figure eight knot is K_4 (Fig. 1.5), but the unknot is tricolourable (its arc graph is a single vertex with no edges).

We have distinguished at least two knots, but we need better invariants—for instance we still cannot prove that the trefoil (Fig. 1.3) is knotted.

There is a simple to define invariant which does distinguish the three knots of Fig. 1.1 (though it is in general hard to compute). Define the **crossing number** of a link to be the minimal number of crossings of any regular diagram. It is intuitively obvious that the figure eight knot has crossing number 4 and the trefoil knot has crossing number 3: one can prove this via enumerating all knot diagrams(!). First, show that the only diagrams on 0, 1, or 2 crossings represent the unknot; we can exhibit a diagram of the trefoil with 3 crossings, and one can enumerate all diagrams with 3 crossings and show that they are all unknots or trefoils; and the figure eight knot admits a diagram with 4 crossings so this must be the minimal number. (We will give an alternative proof that these knots are distinct in Example 1.25.)

Proceeding in this way one can enumerate (in principle) all knots, and indeed most knot tables like the famous Rolfsen table (helpfully placed online in a useful form as part of Dror Bar-Natan and Scott Morrison's *Knot atlas*, `http://katlas.org/wiki/The_Rolfsen_Knot_Table`) use crossing number as the first-order measure of knot complexity. However while one can enumerate all knots algorithmically in this way it is not easy to check that they are all distinct, and indeed knots 10_{161}

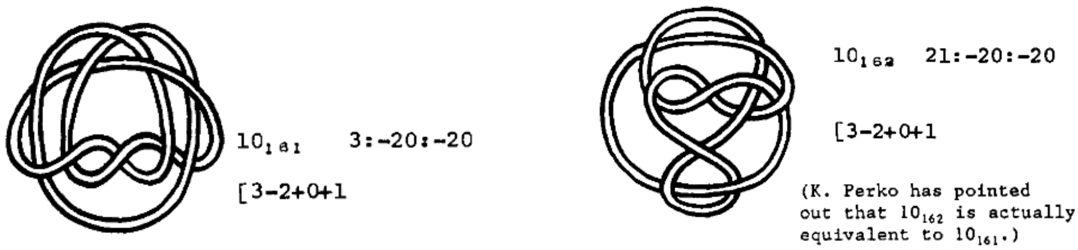

Figure 1.7: An excerpt from the table of Rolfsen [59]. For at least 73 years the Perko Pair was listed as two distinct knots in tables. But they are the same!

and 10_{162} in the Rolfsen table (the 10 refers to the crossing number) shown in Fig. 1.7 are in fact the same knot; they are known as the **Perko pair** [41]. The error originates in the Tait–Little table of 1900, and the pair gives a counterexample to a 'theorem' of Tait (that the **writhe** of a knot, defined in the exercises, is a knot invariant).

To show how hard the computation of crossing number is in general, we give a hard theorem now and an open problem in a bit. A knot diagram is **alternating** if, when walking along the knot, one encounters over- and under-crossings alternately. A diagram in the plane P is **reducible** if there is a round circle in P that intersects the knot diagram transversely in exactly one crossing, called a **nugatory crossing**, and a diagram is **reduced** if it is not reducible.

1.7 Theorem (First Tait conjecture)**.** *Any reduced diagram of an alternating link has the fewest possible crossings.*

One can prove the first Tait conjecture using the machinery of the Jones polynomial.

We can define slightly finer invariants almost immediately for links.

1.8 Lemma. *Let $L = l \sqcup k$ be a link of two oriented components. Let $l \cap k$ be the set of crossings in some diagram. Then the **linking number***

$$\mathrm{lk}(l, k) = \frac{1}{2} \sum_{p \in l \cap k} \varepsilon(p)$$

where $\varepsilon(p)$ is the sign of the crossing (i.e. depending on the orientation) is independent of the diagram and hence is an invariant of the link.

Proof. Reidemeister moves preserve lk.

1.9 Corollary. *There exists a nontrivial link (i.e. a link which is not equivalent to two unknots that lie in disjoint 3-balls in $\mathbb{S}^3$).*

Again this invariant is not good enough for simple examples like the Borromean rings.

In the next lecture we will derive a function $\pi_1 : \mathsf{Knot} \to \mathsf{Group}$ which provides a better invariant (and which is algorithmically computable), and the definition of even better (faster, easier to compute, more geometrically meaningful: pick any two) knot invariants is a theme of the next few weeks. But for the rest of today we will pause to have a look at some fun tricks and constructions to pick up some intuition that will be very useful.

1.10 Construction. The **connected sum** of two *oriented* knots k, k', denoted $k \oplus k'$, is defined by cutting tiny arcs out of k and k' and gluing the ends in an orientation-compatible way. Clearly if 1 is the unknot then $k \oplus 1 = k = 1 \oplus k$.

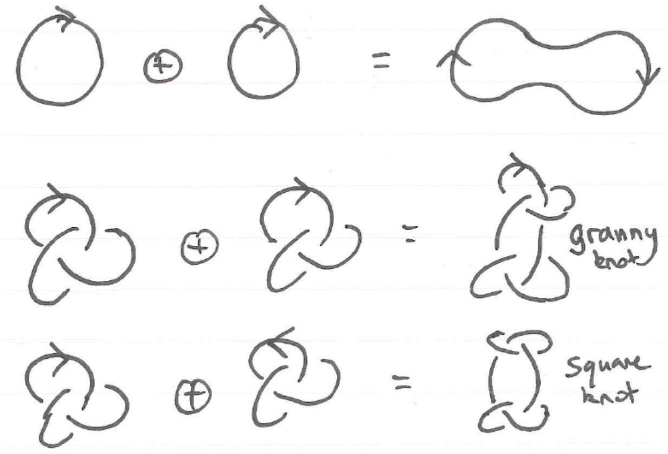

Figure 1.8: Three connected sums.

1.11 Lemma. *Connected sum is associative and commutative (up to knot equivalence).*

1.12 Example. We exhibit the **granny knot** and the **square knot** as connected sums of trefoil knots in Fig. 1.8. Observe that the sum depends on orientation!

A knot is called **prime** if whenever $k = k' \oplus k''$ then either $k = k'$ or $k = k''$. The following trick shows that the unknot is prime.

1.13 Trick (The Eilenberg-Mazur swindle)**.** Knots don't cancel: i.e. given two knots k, k', if $k \oplus k'$ is unknotted then k and k' are unknotted. We follow the proof indicated in [40, Theorem 4.6, p.55]. Suppose $k \oplus k'$ is unknotted; then form the wild knot $k \oplus k' \oplus k \oplus k' \oplus \cdots$ (c.f. Fig. 1.2). But

$$k \oplus k' \oplus k \oplus k' \oplus \cdots = (k \oplus k') \oplus (k \oplus k') \oplus \cdots = 1 \oplus 1 \oplus \cdots = 1.$$

On the other hand,

$$k \oplus k' \oplus k \oplus k' \oplus \cdots = k \oplus (k' \oplus k) \oplus (k' \oplus k) \oplus \cdots = k \oplus 1 \oplus 1 \oplus \cdots = k.$$

Thus $k = 1$.

Remark. Compare the proof of Conway, `https://youtu.be/lwWeRMmXIoU`, where he basically uses the idea that we use to prove associativity of connected sum and so (really) it is the same proof.

Now for the open problem on crossing numbers which we promised earlier:

1.14 Problem. Is crossing number additive under connected sum?

This is true for alternating knots, and for certain other specialised classes of knots (e.g. torus knots).

The connected sum is obtained by taking two 3-balls which each contain an arc with endpoints on the boundary, and gluing them in some way so as to identify those endpoints. One can do a similar thing for 3-balls containing two arcs:

1.15 Construction (Mutation)**.** The **Kinoshita–Terasaka knot** [42] and the **Conway knot** [17, ???] of Fig. 1.9 are distinct (we will prove next time). They are related by the process of **mutation**: if $k \subseteq \mathbb{S}^3$ is a knot and $B \subseteq \mathbb{S}^3$ is a 3-ball with $|\partial B \cap k| = 4$ then cut B out of $\mathbb{S}^3$ and glue it back in after a rotation by π so that the four bits of the knot are matched up.

K-T

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Figure 1.9: The Kinoshita–Terasaka knot (L) and the Conway knot (R) [45, Figure 3.3].

We end with a final remarkable construction of knots due e.g. to Brauner, though we follow the excellent exposition of Milnor [50].

1.16 Construction. Let $V \subseteq \mathbb{C}^2$ be an affine algebraic curve cut out by a square-free polynomial $f(w, z)$. Let r be the number of local analytic branches of V passing through $(0, 0)$. Since $(0, 0)$ is either a simple point or an isolated singularity, there exists $\varepsilon > 0$ such that the intersection $S_\varepsilon \cap V$ of a 3-sphere of radius ε with V is a smooth compact 1-manifold with r components, i.e. it is a link of r components. Such a link is called an **algebraic link**.

1.17 Exercises.

1. Show that the figure eight knot is amphichiral.
2. Show that if k is any knot and $\pi : k \to \mathbb{R}^2$ is some projection which induces a diagram then there exists an alternating knot k' with the same projection as a subset of $\mathbb{R}^2$ (Tait, late 1800s).
3. Define the **writhe** of a diagram δ of a knot k to be

$$w(\delta) = \sum_{v \in V(\delta)} \varepsilon(v)$$

(compare Lemma 1.8, where the sum is only over intersections of two different components). Show that w is invariant under the second and third Reidemeister moves, but not the first: in fact adding a single 'loop' (either over or under) to a knot diagram adds 1 to the writhe. In fact the writhe is a topological invariant of the knot k *together with* a choice of section of the unit normal bundle to k, or (equivalently) a 'ribbon' thickening of k. This additional structure on k is called a **framed knot** (and has an obvious generalisation to links).

4. Show that the only knot of crossing number 0 is the unknot; that there are no knots of crossing number 1 or 2; that the only knot of crossing number 3 is the trefoil knot; that the only knot of crossing number 4 is the figure eight knot. Conclude that the figure eight and trefoil knots are distinct.

1.2 The fundamental group

Recall that a knot is **prime** if it does not decompose under connected sum, i.e. k is prime iff whenever $k = k' \oplus k''$ one of k' or k'' is the unknot, and a knot is **tame** if it is isometric to a knot which is made up of finitely many straight line segments. We write $\mathbb{S}^3 \setminus k$ for the complement 3-manifold of k, and $\pi_1(k) \coloneqq \pi_1(\mathbb{S}^3 \setminus k)$.

1.18 Theorem (Gordon–Luecke, 1989 [31])**.**

1. *Fundamental groups are knot invariants: If $(\mathbb{S}^3 \setminus k) \simeq_{homeo.} (\mathbb{S}^3 \setminus k')$, then $k \sim k'$.*

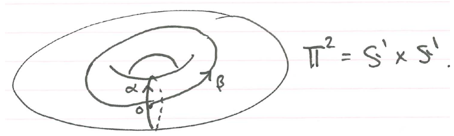


Figure 1.10: Median and latitude of a torus.

2. *The converse is true for prime knots: If k and k' are prime, and $\pi_1(k) \simeq \pi_1(k')$, then $k \sim k'$.*

The Gordon–Luecke theorem *does not hold* for links [59, §9.H].

Usually when we compute the fundamental group we will obtain it in terms of generators and relations. Having a group in terms of generators and relations is not to really *know* the group! Hence this invariant, while 'easy' to compute (we will see an algorithm in a bit), is not in practice so useful on its own.

We recall first some basic algebraic topology which we will use throughout the remainder of the lecture.

1.19 Definition. Let H_1 and H_2 be groups, and L be a third group equipped with maps $\Phi_1 : L \to H_1$ and $\Phi_2 : L \to H_2$. Then the **amalgamated free product** $H_1 *_L H_2$ is a group equipped with maps $f_1 : H_1 \to H_1 *_L H_2$ and $f_2 : H_2 \to H_1 *_L H_2$ such that $f_1 \circ \Phi_1 = f_2 \circ \Phi_2$ satisfying the universal property "if G is a group equipped with maps $g_1 : H_1 \to G$ and $g_2 : H_2 \to G$ such that $g_1 \circ \Phi_1 = g_2 \circ \Phi_2$, then there exists a unique map $\Psi : H_1 *_L H_2 \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
& H_1 \arrow[dr, "f_1"] \arrow[drr, bend left, "g_1"] & & \\
L \arrow[ur, "\Phi_1"] \arrow[dr, "\Phi_2"'] & & H_1 *_L H_2 \arrow[r, dotted, "\Psi"] & G.\text{''} \\
& H_2 \arrow[ur, "f_2"'] \arrow[urr, bend right, "g_2"'] & &
\end{tikzcd}$$

This group is $(H_1 * H_2)/K$, where K is the normal closure of the subgroup of $H_1 * H_2$ generated by the words $\Phi_1(l)\Phi_2(l)^{-1}$ for all $l \in L$.

1.20 Theorem (Seifert–Van Kampen [12, Theorem III.9.4]). *Let $X = U \cup V$ with each of U, V, $U \cap V$ open, non-empty, and path connected. Fix a common base point $x_0 \in U \cup V$. Then the canonical maps of the fundamental groups of U, V, and $U \cap V$ into that of X induce an isomorphism*

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \simeq \pi_1(X).$$

The following technical lemma is the fount of all places that coprime pairs will appear.

1.21 Lemma. *Coordinatise $\mathbb{S}^1 \subseteq \mathbb{C}$ in the usual way via* exp, *and let the 'standard torus' be $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$. The fundamental group $\pi_1(\mathbb{T}^2, (1,1))$ is a free Abelian group with standard basis given by the images of $\alpha, \beta : (I, \partial I) \to (\mathbb{T}^2, (1,1))$ defined by*

$$\alpha(t) = (e^{2\pi i t}, 1), \qquad \beta(t) = (1, e^{2\pi i t}).$$

(So far so good.) An element of $\pi_1(\mathbb{T}^2)$ is represented by a simple loop iff it has homotopy class $[\alpha]^p[\beta]^q$ with $(p,q) = 1$.

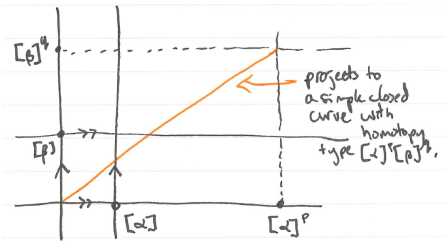


Figure 1.11: Every simple closed curve on the torus is a projection of a line of rational slope.

Proof. "$\Leftarrow$": if it has given homotopy class then it is parametrised by $t \mapsto (e^{2\pi pit}, e^{2\pi qit})$ which is simple (Fig. 1.11). "$\Rightarrow$": suppose $\omega(t)$ parameterises a simple curve, and cut along it, opening the torus into an annulus. Since α also has this property there is a homeomorphism $h : \mathbb{T}^2 \to \mathbb{T}^2$ with $h\alpha = \omega$ (cut via omega and reglue via alpha to define h). Define $p, q, r, s \in \mathbb{Z}$ by $h_*(\alpha) = [\omega] = [\alpha]^p[\beta]^q$ and $h_*(\beta) = [\alpha]^r[\beta]^s$. Since h_* is an automorphism of $\pi_1 \simeq [\alpha] \times [\beta]$, we have $\begin{vmatrix} p & r \\ q & s \end{vmatrix} = \pm 1$. ⛵

1.22 Example (Torus knots)**.** Let $p, q \in \mathbb{Z}$ be coprime. Fix a basis $[\alpha], [\beta]$ for $\pi_1(\mathbb{T}^2)$. Then there exists a unique up to homotopy curve on the torus $\mathbb{T}^2$ with homotopy class $[\alpha]^p[\beta]^q$ (Lemma 1.21). Embed $\mathbb{T}^2$ into $\mathbb{S}^3$ in an unknotted way. The resulting curve is the (p, q) **torus knot** $k_{p,q}$. We can apply Theorem 1.20 to compute $\pi_1(k_{p,q})$. Let T be the torus, U be a slight open thickening of the portion of $\mathbb{S}^3 \setminus k$ not exterior to T, and V a slight open thickening of the portion of $\mathbb{S}^3 \setminus k$ not interior to T. Both U and V are solid torii, so $\pi_1(U) = \langle x \rangle$ and $\pi_1(V) = \langle y \rangle$. Now observe that from the perspective of U, $U \cap V$ is a thickened annulus winding p times around, and from the perspective of V $U \cap V$ winds q times. We therefore have $\pi_1(U \cap V) = \langle x^p \rangle \subseteq \pi_1(U)$ and $\pi_1(U \cap V) = \langle y^q \rangle \subseteq \pi_1(V)$. Thus $\pi_1(U \cup V) = \langle x, y : x^p = y^q \rangle$.

In general we can give an algorithm for the computation of the fundamental group, first described by Wirtinger circa. 1905 (according to the historical notes to [15, Chapter 3]).

1.23 Algorithm (Wirtinger presentation)**.** Let δ be a diagram of an oriented link k.

1. Enumerate the arcs of δ, so $\mathrm{arcs}(\delta) = \{x_1, \dots, x_n\}$.

2. For every vertex v of δ, let i, j, k be the indices of the three arcs at v in such a way that $\mathrm{ob}(v) = x_k$ and such that x_i is walked before x_j when travelling in the orientation direction. If $\varepsilon(v) = +1$ then let $W_v = x_k x_i x_k^{-1} x_j^{-1}$, otherwise set $W_v = x_k^{-1} x_i x_k x_j^{-1}$. Let $\mathrm{words}(\delta) = \{W_v : v \in V(\delta)\}$.

3. Then $\langle \mathrm{arcs}(\delta) : \mathrm{words}(\delta) \rangle$ is a presentation for $\pi_1(k)$, the **Wirtinger presentation**.

Observe that, if we write the Wirtinger presentation of a link, generators which are loops around the same component are all conjugate. In particular we see that the Abelianisation of $\pi_1(k) = \mathbb{Z}^n$ where n is the number of components of k.

1.24 Example. The unknot has fundamental group $\mathbb{Z}$.

1.25 Example. We can compute the group of the trefoil knot k as follows. Label the three arcs of k by x, y, z as in Fig. 1.12. Then by applying the vertex rules we get the following relations for each vertex:

1. $z = yxy^{-1}$,

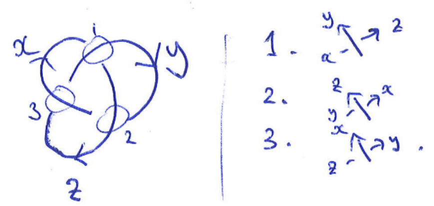


Figure 1.12: Generators and relations for the Wirtinger presentation of the trefoil knot.

2. $x = zyz^{-1}$,
3. $y = xzx^{-1}$.

Hence

$$\pi_1(k) = \langle x, y, z \,:\, yxy^{-1}z^{-1} = zyz^{-1}x^{-1} = xzx^{-1}y^{-1} = 1\rangle.$$

But by relation (1) we can eliminate the generator z; this also eliminates one of the other two generators (one becomes the inverse of the other) and in total we have

$$\pi_1(k) = \langle x, y \,:\, yxy = xyx\rangle.$$

We now observe that there is a surjective map $\pi_1(k) \to S_3$: the symmetric group is generated by $(1\,2)$ and $(2\,3)$, so define the map $\phi \,:\, \pi_1(k) \to S_3$ by $x \mapsto (1\ 2)$ and $y \mapsto (2\ 3)$; this is possible since

$$(2\ 3)(1\ 2)(2\ 3) = (1\ 3) = (1\ 2)(2\ 3)(1\ 2).$$

By another application of the algorithm, we get that the fundamental group of the figure eight knot l is

$$\pi_1(l) = \langle x, y \,:\, yxy^{-1}xy = xyx^{-1}yx\rangle.$$

We claim that there is no surjective map $\pi_1(l) \to S_3$, and prove this by contradiction. First note that x and y are conjugate so their images in S_3 must be distinct (as S_3 is not cyclic) conjugate elements. Further since the map is surjective their images cannot be cycles of length 3, since two cycles of length 3 generate a proper subgroup of S_3. We therefore see that x and y must be mapped to two transpositions, without loss of generality $x \mapsto (1\ 2)$ and $y \mapsto (2\ 3)$. But one can easily check that the relation

$$(2\ 3)(1\ 2)(3\ 2)(1\ 2)(2\ 3) = (1\ 2)(2\ 3)(2\ 1)(2\ 3)(1\ 2)$$

does not hold—the left hand side is $(1\ 2)$ and the right hand side is $(1\ 3)$, so the only possible map $\{x, y\} \to \mathbb{S}^3$ cannot extend to a homomorphism, giving the desired contradition.

By Theorem 1.18 we therefore see that since $\pi_1(k) \not\cong \pi_1(l)$, $k \not\equiv l$.

Note that for the trefoil and figure eight knots we could reduce the number of generators down to 2 from the *a priori* number 3.

1.26 Lemma. *The minimal number of generators of a Wirtinger presentation is exactly the bridge number of the knot.*

Proof. Let b be the bridge number of k and let m be the minimal number of generators of a Wirtinger presentation. Since the number of generators in the Wirtinger presentation coming from a b-bridge presentation is b, we have $m \leq b$. On the other hand the bridge number is bounded above by the number of arcs in any given diagram, and each of these gives a presentation, so $b \leq m$.

We shall now turn to the proof of correctness of Algorithm 1.23 which is fairly standard; we steal pictures from the version given in §10.2 of Armstrong [6] since they are particularly clearly drawn. Observe without loss of generality we may assume our link is embedded in $\mathbb{R}^3$.

Proof of correctness of Algorithm 1.23. Let k be a link, let $P = \{(x, y, z) : z = 0\}$ be the plane disjoint from k which induces a diagram δ via orthogonal projection π; the claim is that a presentation for $\pi_1(k)$ is given by $\langle \text{arcs}(\delta) : \text{words}(\delta)\rangle$. Let S be a bounded closed disc in P which includes in its interior the diagram δ, for every crossing $v \in V(\delta)$ let R_v be a closed subset of k which is projected onto a small closed neighbourhood of v of the undercrossing at v chosen in such a way that all the R_v are mutually disjoint (except for if R_v and R_v' are adjacent). In Fig. 1.13 the arcs R_v are the lighter coloured arcs. It should now be clear that we can move the knot via an isotopy such that the sets R_v are all disjoint subsets of P and the remainder of the knot lies entirely in one of the open half-spaces bounded by P, say $P_+ = \{(x, y, z) : z > 0\}$—see Fig. 1.14. We can identify the connected components of $P_+ \cap k, \alpha_1, \dots, \alpha_r$ with the elements of the set arcs(δ) and without loss of generality we can assume that the orthogonal projections $\pi(\alpha_i)$ of these components to P are disjoint.

Pick a basepoint in P_+ that is far away from P and the knot, say $x_0 = (0, 0, z_0)$ where $z_0 \gg 0$ and let $\overline{P_+} = \{(x, y, z) : z \geq 0\}$ be the closed half-space. For each arc α_i let x_i be a loop based at z_0 which goes around α_i according to the right-hand rule and comes straight back up, as in Fig. 1.15.

Claim: $\pi_1(\overline{P_+} \setminus k, z_0)$ is the free group generated by the x_i. *Proof of claim:* For each arc α_i let B_i be a thickening (i.e. small open neighbourhood) of the set $\bigcup_{x\in\alpha_i}[x, \pi(x)]$; the latter set looks like a wall under α_i (Fig. 1.16). Delete all these neighbourhoods and start adding them (minus the knot) back in one at a time inductively—the set $P_* \setminus \bigcup B_i$ is simply connected, each $B_i \setminus k$ has cyclic fundamental group generated by x_i; to be fully rigorous we need to adjoin to B_i a long thin open 'noodle' which goes up to z_0 and doesn't intersect any other B_i except in a tiny ball around z_0, then these intersections have trivial fundamental group and so by Theorem 1.20 the fundamental group of $(P_+ \setminus \bigcup B_i) \cup (B_1 \setminus k) \cup \cdots \cup (B_r \setminus k)$ is exactly the free product $\langle x_1\rangle * \langle x_2\rangle * \cdots \langle x_r\rangle$ as desired. *This ends the proof of the claim.*

We now need to add in the lower half-space $\overline{P_-} \setminus k$. Suppose we look at the local picture at some vertex with incident arcs indexed i, j, k and with the lower arc going from i to j as you look along k (the other orientation is the same argument), depicted in Fig. 1.17. Suppose for the sake of labelling that this is vertex v. Draw a small box D_v made up of the square cylinder in P_- capped with a square ∂D_v on P surrounding the underpass R_v. Topologically, we can thicken D_v slightly into P_+. The fundamental group of the thickened D_v is still trivial but the intersection of this thickening with $P_+ \setminus k$ is an annulus, namely it is a thickening of the square ∂D_v minus the central arc R_v (see Fig. 1.18). The fundamental group of this intersection is generated by the loop indicated in Fig. 1.17. Observe that this loop is homotopic in P_+ to the loop $x_i x_k x_j^{-1} x_k^{-1}$. By Theorem 1.20 we therefore have that $\pi_1(P_+ \setminus k \cup D_v) = \pi_1(P_+ \setminus k) *_N \pi_1(D_v)$, where N is the (normal closure of the) group generated by $x_i x_k x_j^{-1} x_k^{-1}$. This is exactly the element of words(δ) coming from the vertex v . By induction, since all the D_v for different groups are disjoint, we get

$$\pi_1((P_+ \setminus k) \cup \bigcup D_v) = \langle x_1, \dots, x_r : \text{words}(\delta)\rangle.$$

Finally observe that the remaining part of P_i is simply connected and has simply connected intersection with the set whose fundamental group was just computed, so by a final application of Theorem 1.20 we get

$$\pi_1(\mathbb{R}^3 \setminus k) = \langle x_1, \dots, x_r : \text{words}(\delta)\rangle$$

as desired. ⅅ

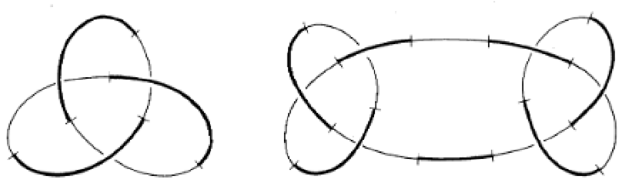


Figure 1.13: Undercrossings (light) and arcs (dark). Figure from [6, Fig. 10.6].

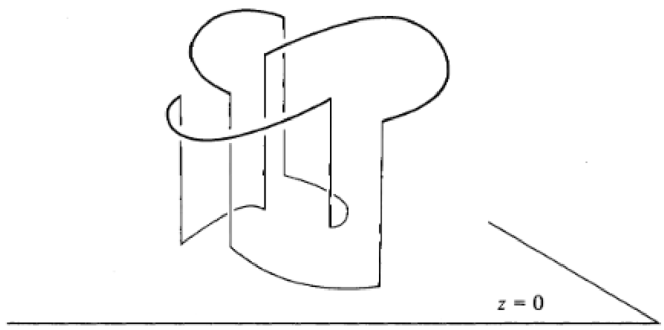


Figure 1.14: A knot isotoped to lie entirely in the plane $z = 0$ except for the finitely many arcs of some diagram. Figure from [6, Fig. 10.7].

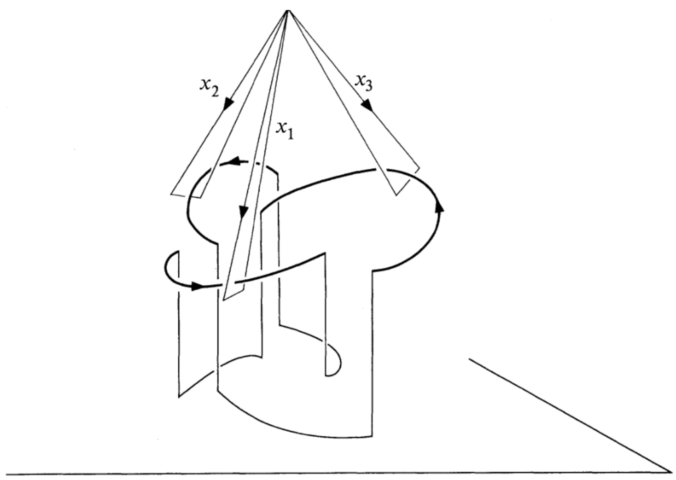


Figure 1.15: The generators of the Wirtinger presentation. Figure modified from [6, Fig. 10.8].

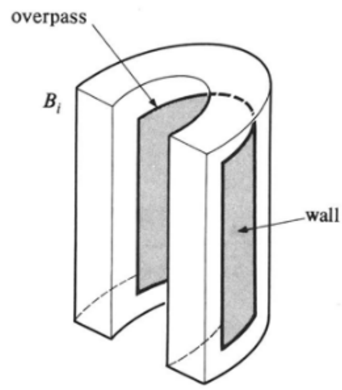


Figure 1.16: The open set B_i is a thickening of the 'wall' set $\bigcup_{x\in\alpha_i}[x,\pi(x)]$. Figure modified from [6, Fig. 10.9].

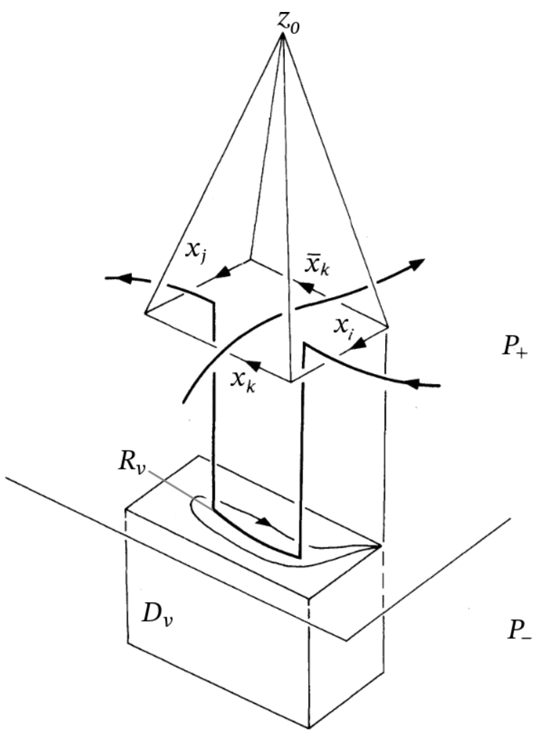


Figure 1.17: The local picture of the fundamental group around the vertex with incident arcs indexed i, j, k (and with the lower arc going from i to j as you look along k). Figure modified from [6, Fig. 10.10].

Figure 1.18: Square doughnuts. Image from `https://www.bakingbusiness.com/articles/54433-square-is-the-new-round-at-udf`.

We already mentioned that just knowing presentations of groups is not good enough to distinguish them. The most classical way of dealing with this is to study representations onto simpler groups; we now exhibit some results of Riley for the case of representations onto finite groups.

Remark. The computation of the Alexander modules, which we will do much later on, is a similar kind of idea: instead of studying representations onto finite groups, one studies representations onto infinite cyclic groups and the associated group algebras and homology groups. Thus the reader can postpone caring about the following discussion until then.

Following Riley [57] we will outline a scheme for computing the representations $\pi_1(k) \to L_p \coloneqq \mathrm{PSL}(2, p)$ for k a knot (of one component). Suppose we have a Wirtinger representation for $\pi_1(k)$ of minimal number of generators (or alternatively we take an arbitrary Wirtinger representation and reduce it by substituting relations, so we keep the property that every relator is of them form $x_i = W^{-1}x_jW$ for some $W \in \pi_1(k)$); say to fix notation that $\pi_1(k) = \langle x_1, \dots, x_n : r_1, \dots, r_{n-1}\rangle$ where n is the bridge number of k. Suppose also for simplicity that $n = 2$ or $n = 3$. Given some $\theta : \pi_1(k) \to G$ for any finite group G, since all the generators x_i are conjugate, their images have the same order; we say that this is the **order** of the representation θ and it is defined up to equivalence of representations. Let p be an odd prime: we will classify the representations of order p from $\pi_1(k)$ to L_p. To do this we need to study the elements of order p in L_p since these are the possible images of the x_i.

1.27 Lemma (Structure of L_p)**.** *We recall some properties of L_p from [16, Chapter XIV] (n.b., Burnside's H is our L_p).*

1. *$|L_p| = p(p^2 - 1)/2$ (§221).*
2. *There are two conjugacy classes of elements of order p in L_p and each class contains $(p^2 - 1)/2$ elements (§227).*
3. *Fix α an element of order p. Given a second element β of order p, either (i) $\beta = \alpha^n$ for some n, or (ii) $L_p = \langle\alpha, \beta\rangle$.*
4. *In case (i), the element $\beta = \alpha^n$ is conjugate to α iff n is a square mod p (there are $(p-1)/2$ squares mod p).*
5. *The elements of order p which are not powers of α lie in $(p-1)/2$ orbits of p elements each under conjugacy by α.*
6. *The automorphism group of L_p acts transitively on elements of order p, and the stabiliser of α is the group generated by α-conjugation.*

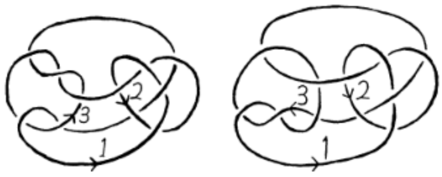


Figure 1.19: Generators for the Wirtinger presentations of the Kinoshita–Terasaka and Conway knots [57, Figure 2].

	$\theta_1 x_1$	$\theta_1 x_2$	$\theta_1 x_3$	$\theta_2 x_1$	$\theta_2 x_2$	$\theta_2 x_3$
KT	α	(1675243)	(1452736)	α	α	(1675243)
C	α	(1675243)	(1723654)	α	α	(1264735)

Table 1.1: Representations onto PSL(2, 7) of the Kinoshita–Terasaka and Conway knots. Table an exerpt from p.615 of [57].

1.28 Proposition ([57, §3]). *Fix a surjective representation $\theta : \pi_1(k) \to L_p$ of order p. The equivalence class of θ can be found by performing $E(n)$ experiments—n the bridge number—where*

$$E(n) = \left(\frac{p-1}{2}\right)^{n-1} \frac{(p+1)^{n-1}-1}{p}.$$

Proof. We have $\pi_1(k) = \langle x_1, \dots, x_n : r_1, \dots, r_{n-1} \rangle$. Fix $\alpha \in L_p$ of order p, without loss of generality we can take $\alpha = (1, 1|0, 1)$. We can assume up to automorphism that $\theta(x_1) = \alpha$. The image $\theta(x_2)$ is of order p and is conjugate to α, hence is either a power of α and there are $(p-1)/2$ choices by (4) in the lemma, or lies in one of the $(p-1)/2$ orbits mentioned in (5). Choose a representative $\alpha_1, \dots, \alpha_{(p-1)/2}$ for each of these; then $\theta(x_2) = \alpha^s$ for some s or $\theta(x_2) = \alpha_j$ for some j.

If $n = 2$, then by surjectivity we must have $\theta(x_2) = \alpha_j$ hence $E(2) = (p-1)/2$.

If $n = 3$ then either $\theta(x_2) = \alpha_j$ for some j in which case $\langle \theta x_1, \theta x_2 \rangle = L_p$ and the isomorphism class is determined by j and θx_3, or $\theta x_2 = \alpha^s$ so we swap x_2 and x_3 since in this case again by surjectivity we must have $\theta x_3 = \alpha_j$ for some j; hence the number of choices is $\frac{p-1}{2}\frac{p^2-1}{2}$ (case I) plus $\frac{p-1}{2}\frac{p-1}{2}$ (case II) and one can check that this is OK.

By similar arguments for $n > 3$ one gets the claimed formula.

1.29 Example. One can use this to check that the Kinoshita–Terasaka knot and the Conway knot are distinct (c.f. Construction 1.15). The point will be the consideration of representations $\pi_1 \to L_7$. One can check that KT and C have presentations on three generators and two relations, by taking the Wirtinger presentation and then eliminating all but the three generators shown in Fig. 1.19. Then there are exactly two representations for each knot, and they are distinct [57, p. 615]. One can choose a permutation representation for L_7 where $\alpha = (1234567)$ (warning: we write and multiply permutations from left-to-right) and the $(p-1)/2$ orbits of other elements of order p are represented respectively by the elements $\alpha_1 = (1675243)$, $\alpha_2 = \alpha_1^2$, and $\alpha_3 = \alpha_1^4$. We end up with table Table 1.1.

When we have a permutation representation $\rho : \pi_1(k) \to G$ where G is a finite group acting transitively on some finite set $\{1, \dots, s\}$; let H be the subgroup of G defined by pulling back the stabiliser of 1 through ρ. The covering space $\mathcal{U}$ of $\mathbb{S}^3 \setminus k$ defined by H is a non-compact s-sheeted cover, and the integral homology $H_1(\mathcal{U}, \mathbb{Z})$ is a knot invariant: it depends up to group isomorphism only on $\pi_1(k)$, the representation ρ, and the number s.

	θ_1	θ_2
KT	$\mathbb{Z} \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(28)$	$\mathbb{Z} \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(238)$
C	$\mathbb{Z} \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(12)$	$\mathbb{Z} \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(2)\mathbb{Z}/(8) \oplus \mathbb{Z}/(40)$

Table 1.2: Integral homology of the 7-sheeted covers of $\mathbb{S}^3 \setminus KT$ and $\mathbb{S}^3 \setminus C$ defined via the representations θ_1 and θ_2 of Example 1.29. Table an exerpt from p.615 of [57].

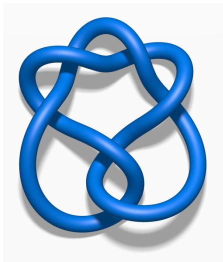


Figure 1.20: The stevedore's knot. Image by Jim.belk, released to public domain (see `http://commons.wikimedia.org/wiki/File:Blue_Stevedore_Knot.png`)

1.30 Example. The Kinoshita–Terasaka knot and the Conway knot have integral homologies coming from the two representations we just described with image in S_7. The respective homology groups are listed in Table 1.2.

Remark. Many other amazing results on knot groups and $\mathrm{PSL}(2,\mathbb{F}_p)$ are known; for instance, $\pi_1(KT)$ has quotient groups isomorphic to $\mathrm{PSL}(2,\mathbb{F}_p)$ for infinitely many p [46, Theorem 2] and there are some knots which admit homomorphisms onto $\mathrm{PSL}(2,\mathbb{F}_p)$ for all p [57, pp. 609–610], this is particularly remarkable since the groups $\mathrm{PSL}(2,\mathbb{F}_p)$ are incredibly varied, see e.g. [60, pp. 224–227]

1.31 Exercises.

1. Read the paper of Fox [26] showing that the Fox knot (Fig. 1.2) is not the unknot. (Now you have read and fully understood an *Annals* paper!) One can also try to see that it is not unknottable by looking at the diagram—the point is that the isotopy 'pull from the right' is not defined in the neighbourhood of p in the knot complement.

2. Show that the fundamental groups of two separated rings and two linked rings (the **Hopf link**) are not isomorphic. The Hopf link is so named because every pair of circles in the Hopf fibration (Example 2.10) form a Hopf link in $\mathbb{S}^3$.

3. Exhibit 3-bridge presentations for the Kinoshita–Terasaka and Conway knots.

4. Find all presentations of the fundamental group of the trefoil knot onto A_5.

5. Show that the fundamental group of the Klein bottle is $\pi_1(K) = \langle x, y \,:\, y = xyx\rangle$. Show that no knot group admits a surjective representation onto $\pi_1(K)$.

6. Show that tricolourability of k is equivalent to the existence of a *surjective* homomorphism $\pi_1(k) \to S_3$.

7. Show that the trefoil knot is the $(2,3)$ torus knot. Show that the (p,q) and (q,p) torus knots are equivalent.

8. Show that the stevedore's[2] knot (Fig. 1.20) is 2-bridge and give a two generator presentation for its group.

9. Show that $\langle x, y : yxy = xyx \rangle \simeq \langle a, b : a^2 = b^3 \rangle$. Hint: $b = xy$ and $a = xyx$. Observe that this is a presentation for $\mathrm{PSL}(2, \mathbb{Z})$ [63, Example 1.5.2 of Chapter I]. This will be explained in Example 2.14.

10. (Brauner's theorem, [50, p. 4]) The (p, q)-torus knot is cut out by intersecting a sufficiently small 3-sphere in $\mathbb{C}^2$ with the algebraic curve $\mathbf{V}(z^p + w^q)$, i.e. it is an algebraic knot (Construction 1.16).

[2]"A workman employed either as overseer or labourer in loading and unloading the cargoes of merchant vessels." (OED)

Chapter 2

Geometric knot theory

In this week we will study the hyperbolic geometry of knot complements. A very nice historical overview of the contributions of Thurston may be found in his article [69]. We will begin by reviewing briefly hyperbolic geometry; we will then give the historical motivation for, and some examples of, the Riley–Thurston theorem ("most knot complements are hyperbolic"). In the second lecture we will compute some geometric invariants and explain how they can be mechanicised following the work of Jeff Weeks.

There are a plethora of nice books on this area, but we will mainly follow Thurston [70, 68], Purcell [53], and Benedetti–Petronio [8]. We also found several sets of notes very useful in the preparation of this chapter: [10, 61]. Basic hyperbolic geometry may be found in [65, Chapter 4] and [7, Chapter 7].

2.1 Geometric structures on knot complements

We saw in the previous section that the study of representations $\pi_1(k) \to \mathrm{PSL}(2, p)$ is a fruitful one when trying to define knot invariants. This representation space has a big disadvantage: the groups $\mathrm{PSL}(2, p)$ are all finite, and so do not carry a lot of information about $\pi_1(k)$. In addition, these groups do not have obvious geometric interpretations in terms of the knot k. Recall from covering space theory that $G = \pi_1(k)$ can be viewed as a group of homeomorphisms of a simply-connected topological 3-manifold M, the universal cover of $\mathbb{S}^3 \setminus k$, in such a way that M/G is homeomorphic to $\mathbb{S}^3 \setminus k$. The manifold M can be viewed as the 'unrolling' of $\mathbb{S}^3 \setminus k$ via the action of $\pi_1(k)$. The geometric study of knots comes from the observation that it might be possible to keep geometric as well as topological information: that is, it might be possible to find a nice Riemannian manifold M and a faithful representation $\rho : \pi_1(k) \to \mathrm{Isom}^+(M)$ such that if $G = \rho(\pi_1(k))$ then M/G is a Riemann manifold homeomorphic to $\mathbb{S}^3 \setminus k$. In fact in most cases this is possible, and even better there is a *unique* such geometric structure (i.e. a unique representation ρ) such that this all works!

2.1 Definition. A **model geometry** (X, G) is a manifold X together with a Lie group G of diffeomorphisms of X such that:

1. X is connected and simply connected;

2. G acts transitively on X with compact point stabilisers;

3. G is not contained in any larger group of diffeomorphisms of X with compact point stabilisers; and

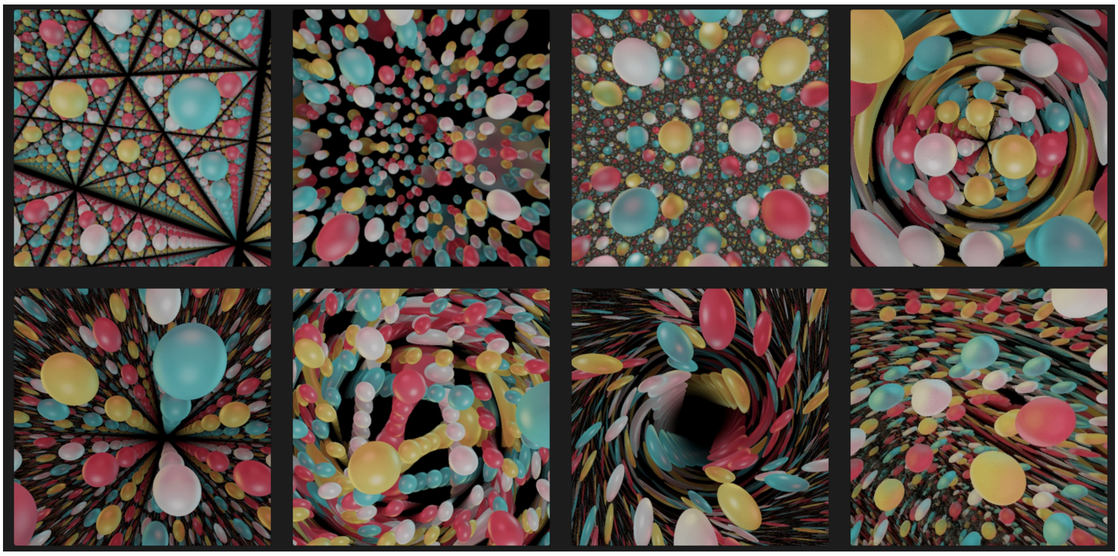


Figure 2.1: The eight Thurston geometries, from `https://www.3-dimensional.space/`. From top left: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, Nil, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and Sol.

4. there exists at least one compact manifold modelled on (G, X).

To clarify the last point, if X is a metric space and G is a group of diffeomorphisms of X then a manifold M is **modelled on** (G, X) (or we say M is a (G, X)-**manifold**, or **locally** X if $G \leq \mathrm{Isom}(X)$) if it admits an atlas of charts $\phi_i : U_i \to X$ such that $\phi_i \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is the restriction of an element of G for all i, j such that $U_i \cap U_j \neq \varnothing$.

2.2 Theorem (Thurston, c.1980)**.** *There are exactly eight three-dimensional model geometries* (G, X)*, called the* ***Thurston geometries****:*

1. *If the point stabilisers are three-dimensional, then X is either $\mathbb{S}^3$, $\mathbb{E}^3$, or $\mathbb{H}^3$.*
2. *If the point stabiliers are one-dimensional, then X fibres over one of $\mathbb{S}^2$, $\mathbb{E}^2$, or H^2 in such a way that is G-invariant and there is a G-invariant Riemann metric on X such that the connection orthogonal to the fibres has curvature* 0 *or* 1*:*
   (a) *Curvature* 0*: X is $\mathbb{S}^2 \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^1$.*
   (b) *Curvature* 1*: X is* Nil *(fibreing over $\mathbb{E}^2$) or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (fibreing over $\mathbb{H}^2$).*
3. *If the point stabilisers are zero-dimensional, then X is* Sol.

Remark. The point stabiliser dimensions 3, 1, and 0 come from the fact that the identity component of the point stabiliser must be SO(3), SO(2), or the trivial group [70, p. 181].

For the sake of completeness we show pictures of the eight geometries in Fig. 2.1, but we will introduce the ones we need as we go. If we ever want to distinguish the Euclidean structure on $\mathbb{R}^n$ as opposed to the algebraic structure, we write $\mathbb{E}^n$.

Hyperbolic 3-space $\mathbb{H}^3$ is the unique simply connected Riemannian manifold of constant sectional curvature -1. The upper half-space model is given by the topological manifold

$$\mathbb{H}^3 := \{(z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$$

equipped with the Riemann metric
$$ds^2 = \frac{dz^2 + dt^2}{t};$$
the geodesic lines in this metric are the half-circles which are orthogonal to the sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A 3-manifold is **hyperbolic** if it is locally modelled on $\mathbb{H}^3$. A knot is **hyperbolic** if the complement $\mathbb{S}^3 \setminus k$ admits a Riemannian metric that turns it into a hyperbolic 3-manifold.

There is a natural isomorphism between the group $\mathrm{Isom}^+(\mathbb{H}^3)$ of orientation preserving isometries of $\mathbb{H}^3$ and the group of conformal maps of the sphere $\hat{\mathbb{C}}$ which is identified with the group of Möbius transformations $\mathbb{M}$ given via extension of the action on geodesics to their endpoints; we identify $\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathbb{M} \simeq \mathrm{PSL}(2, \mathbb{C})$. A discrete subgroup of $\mathbb{M}$ is called **Kleinian**.

2.3 Theorem. *Given any Kleinian group G, the quotient $\mathbb{H}^3/G$ is a complete hyperbolic manifold with holonomy group G. Conversely, given any complete hyperbolic manifold M with holonomy group G, G is a Kleinian group with $\mathbb{H}^3/G \simeq_{\mathrm{isom.}} M$.*

By standard algebraic topology, since $\mathbb{H}^3$ is simply connected there is a natural identification between the discrete group G and $\pi_1(\mathbb{H}^3/G)$. To see this concretely, given a nontrivial $g \in G$ there are four possibilities for its action on $\mathbb{H}^3$:

Elliptic: there is a hyperbolic geodesic λ which is fixed pointwise by g, and g acts as a finite-order rotation around λ;

Hyperbolic: there is a hyperbolic geodesic λ which is left invariant by g, and g acts as a translation along λ;

Loxodromic: g is a composition of an elliptic and an hyperbolic with the same axis;[1]

Parabolic: there is exactly one family of horospheres in $\mathbb{H}^3$ (that is, a Euclidean sphere in the upper half-plane model of $\mathbb{H}^3$ tangent to $\hat{\mathbb{C}}$; locally they are E) which are preserved by g.

We will always assume in these notes that Kleinian groups are torsion-free (so we exclude elliptics, but all three other types are possible). Take a loxodromic element with axis λ; the quotient of λ by $\langle g \rangle$ is a circle of circumference the translation length of g, and the projection of λ to $M = \mathbb{H}^3/G$ is a homotopically nontrivial loop in M of minimal length in its homotopy class. (There is also some twisting going on because of the rotational component of g but this is not relevant to the homotopy theory.) On the other hand, given a parabolic element g fix a horosphere Σ. One can always pick a horocircle σ on Σ which is preserved by g, and this projects down to a homotopically nontrivial loop in M. However one may always pick a smaller horosphere Σ' and obtain a shorter loop which is homotopically equivalent; thus g represents a homotopy class of nontrivial curves in M with lengths tending to zero. One should think of loxodromic elements of $G \simeq \pi_1(M)$ as representing hyperbolic geodesics in M of definite length that wrap around large homotopy obstructions (for instance a crossing in a knot complement), while parabolic elements represent infinitesimal obstructions at infinity known as **cusps** (e.g. a single arc of the knot). In a hyperbolic knot complement, the group should be generated by loops around just the arcs, i.e. a representation into $\mathbb{M}$ should send meridians to parabolics. In general there are many representations $\pi_1(\mathbb{S}^3 \setminus k) \to \mathbb{M}$: in the next lecture we will explain how to distinguish the 'correct' one.

2.4 Problem. Give, for each hyperbolic n-bridge knot group, a faithful representation into $\mathbb{M}$ with n parabolic generators.

[1] The OED cites *Penny Cyclopaedia* XIV. 183/1: "Loxodromic spiral, the curve on which a ship sails when her course is always on one point of the compass. It is called in English works Rhumb Line."

Figure 2.2: Thurston's hexahedral face pairing. Figure taken from [27, Fig. 2 of Chapter 8].

Riley [58] does this for 2-bridge and torus knots.

We will show that the figure eight knot complement is hyperbolic. One way to do this is to exhibit a polyhedron $P \subseteq \mathbb{H}^3$ and an edge-pairing structure on P in the sense of the Poincaré polyhedron theorem, and this is how the result was proved by Thurston [68, §3.1]—but it does not give an explicit holonomy group. In the next section we will give the original proof of Riley [56]. The history surrounding this discovery is very interesting; various accounts beyond [69] include [55] and the accompanying commentary [13], and the additional references given in the historical notes to Section 10.3 on p.504 of [54].

2.5 Theorem. *The figure eight knot k is hyperbolic.*

Proof. Consider an ideal hyperbolic triangular bipyramid: that such a vegetable exists can be seen by gluing a pair of regular tetrahedra, and we can take these two tetrahedra to have vertex sets $\{0, 1, \omega, \infty\}$ and $\{1, \omega, \omega + 1, \infty\}$ where $\omega = e^{2\pi i/3}$. Two of the tetrahedron faces are already equal (the convex hull of $\{1, \omega, \infty\}$), and we pair the remaining six faces as in the labelling of Fig. 2.2. This pairing satisfies the hypotheses of the Poincaré polyhedron theorem if all the angles are $\pi/3$.

We can write down the corresponding group in terms of matrices:

$$\pi_1(k) = \left\langle \phi_B = \frac{i}{\sqrt{\omega}} \begin{bmatrix} 1 & 1 \\ 1 & -\omega^2 \end{bmatrix}, \phi_C = \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \phi_D = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle. \tag{2.6}$$

(In one of the exercises you are invited to struggle to show that ϕ_B is redundant and hence we have a parabolic representation.)

It remains to convince ourselves that the result is indeed the figure eight knot... this can be done via the deformations shown in Fig. 2.3.

We would like to give criteria for a given knot complement to have a geometric (and more specifically hyperbolic) structure (i.e. to admit a Riemannian metric which is locally X for one of the model geometries X). Such a criteria comes as a consequence [53, Theorem 8.17] of a very deep theorem of Thurston, the geometrisation theorem for Haken manifolds [69, 66] whose detailed proof occupies the monograph of Kapovich [37]. The specific case for knot complements requires a couple of definitions:

2.7 Definition. A knot k is a **satellite** if its complement contains an incompressible torus which is not boundary-parallel (a picture like Fig. 2.4 makes this clearer). A knot is a **torus knot** if it can be embedded (without crossings) onto the boundary of a torus. (We already classified all such knots, Example 1.22.)

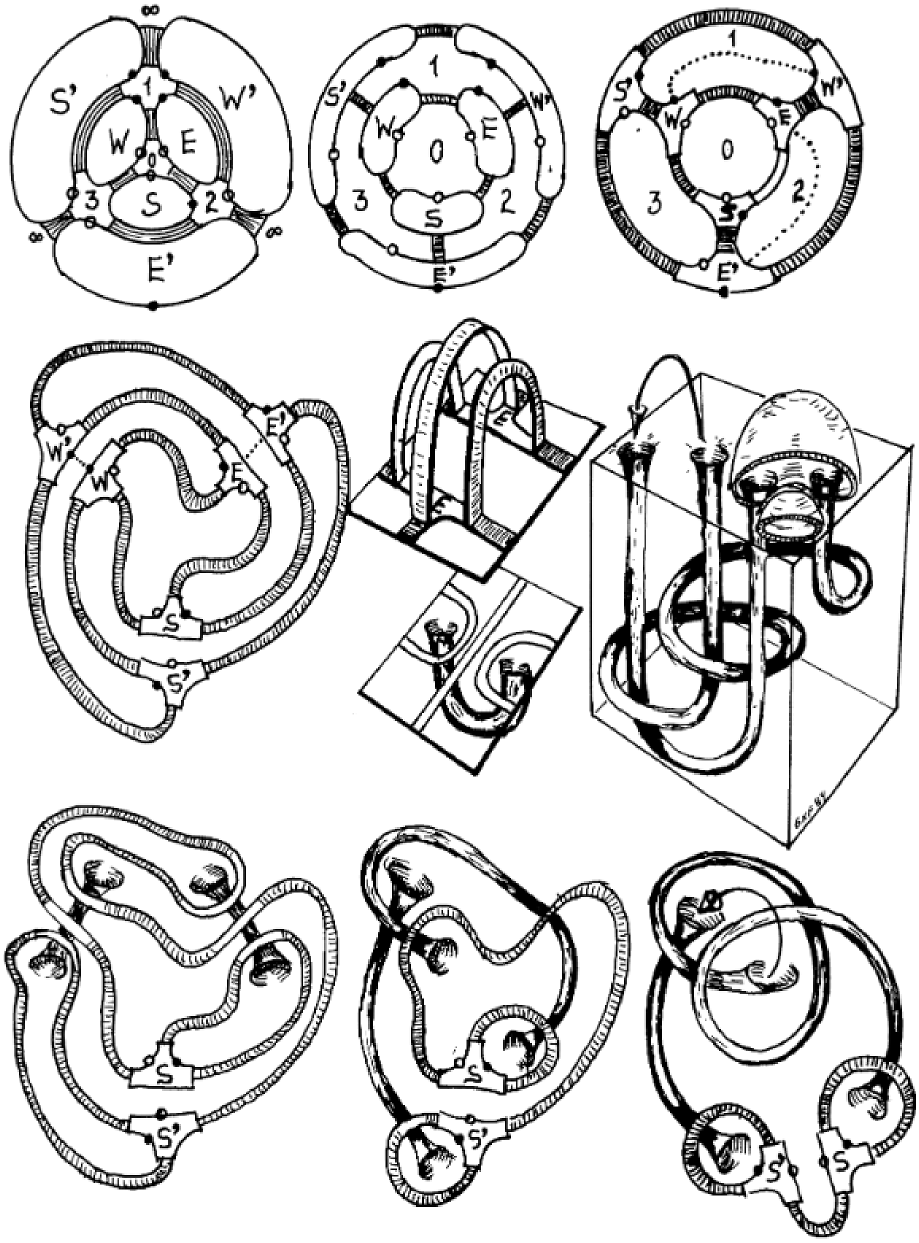


Figure 2.3: Proof that the face-pairing of Fig. 2.2 does indeed give the figure eight knot complement. Figure taken from [27, Fig. 3 of Chapter 8].

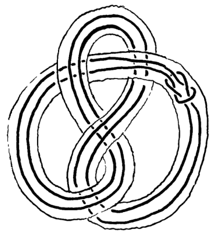


Figure 2.4: A satellite of the trefoil knot. Figure from [59, §9.J.10].

G	$\mathfrak{g}$	$\dim/F$	$B(X,Y)$
$\mathrm{GL}(n,F)$	$\mathfrak{gl}(n,F) = \mathrm{Mat}(n,F)$	n^2	$2n\,\mathrm{tr}\,XY - 2\,\mathrm{tr}\,X\,\mathrm{tr}\,Y$
$\mathrm{SL}(n,F)$	$\mathfrak{sl}(n,F) = \{A \in \mathfrak{gl}(n,F) : \mathrm{tr}\,A = 0\}$	$n^2 - 1$	$2n\,\mathrm{tr}\,XY$
$\mathrm{SO}(n)$	$\mathfrak{so}(n) = \{A \in \mathfrak{gl}(n,\mathbb{R}) : \mathrm{tr}\,A = 0 \text{ and } A + A' = 0\}$	$n(n-1)/2$	$(n-2)\,\mathrm{tr}\,XY$
$\mathrm{SU}(n)$	$\mathfrak{su}(n) = \{A \in \mathfrak{gl}(n,\mathbb{C}) : \mathrm{tr}\,A = 0 \text{ and } A + A^* = 0\}$	$n(n-1)/2$	$2n\,\mathrm{tr}\,XY$

Table 2.1: Little list of Lie groups (always $n \geq 2$).

We will now state the Riley–Thurston theorem:

2.8 Theorem (Riley–Thurston (c.1982), [69, Corollary 2.5])**.** *Let $k \subseteq S^3$ be a knot. Then k has a geometric structure if and only if k is not a satellite knot, and k has a hyperbolic structure iff it is neither a satellite nor a torus knot.*

The remainder of this section will be spent on the non-hyperbolic knots.

2.9 Definition (Some Lie groups)**.** We recommend Fulton and Harris [30] for full detail, but we only need a brief precis of the land all of which one may have seen in 725. A **Lie group** is a smooth manifold which also admits a group action such that multiplication and inversion are smooth. Examples of Lie groups are $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{Mat}(n,\mathbb{C})$; also the universal cover of a Lie group is a Lie group, the most important example for us is $\widetilde{\mathrm{SL}}(2,\mathbb{C})$ which does not admit a faithful matrix representation. Fix a Lie group G. Then G acts on itself by conjugation, say $\phi_g : G \to G$ is conjugation by g. Let T_eG be the tangent space to G at the identity. Then $d\phi_g : T_eG \to T_eG$ induces a map $\mathrm{Ad} : G \times T_eG \to T_eG$, this is the **adjoint action** of G on T_eG and has kernel $Z(G)$ if G is connected. We can further take the differential of this map with respect to the first argument, obtaining a map $\mathrm{ad} : T_eG \times T_eG \to T_eG$. For matrix Lie groups, i.e. $G \leq \mathrm{Mat}(n,F)$ for a field F, $\mathrm{Ad}(g) : T_eG \to T_eG$ is defined by $\mathrm{Ad}(g)(X) = gXg^{-1}$ and $\mathrm{ad}(X) : T_eG \to T_eG$ is given by $\mathrm{ad}(X)(Y) = [X,Y] = XY - YX$. More generally a **Lie algebra** is an algebra admitting a skew-symmetric bilinear may $[\cdot,\cdot]$ which admits the Jacobi identity $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$, and if G is a Lie group then the canonical Lie algebra T_eG is denoted $\mathfrak{g}$. A Lie algebra is equipped with a second natural bilinear form, the **Killing form** $B : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ defined by

$$B(X,Y) \coloneqq \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

For convenience, we include a table of Lie groups, Table 2.1.

2.10 Example (The Hopf fibration)**.** First, observe that $\mathrm{SU}(2)$ is a 3-sphere. More precisely, write the generic element of $\mathrm{SU}(2)$ as

$$U = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \tag{2.11}$$

where $\det U = \|\alpha\|^2 + \|\beta\|^2 = 1$; this exibits SU(2) as the usual 3-sphere in $\mathbb{C}^2$ with $\pm I$ being the north and south poles $(\pm 1, 0, 0, 0)$.

Putting topology aside for a moment, we now define a continuous representation $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, or equivalently an isometric action by SU(2) on $\mathbb{S}^2$. Observe that $\mathfrak{su}(2)$ is isomorphic (as a vector space) to $\mathbb{R}^3$ via the following basis:

$$u_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, u_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \text{ and } u_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

With this basis, the matrix of the Killing form B is $\mathrm{diag}(-2, -2, -2)$, hence $-\frac{1}{2}B = I$ is the usual Euclidean quadratic form on $\mathbb{R}^3$. The adjoint action $\mathrm{Ad} : \mathrm{SU}(2) \times \mathfrak{su}(2) \to \mathfrak{su}(2)$ preserves B (exercise) hence preserves $-\frac{1}{2}B$. In particular we have a morphism $\varphi : \mathrm{SU}(2) \to \mathrm{SO}(3)$, where SU(2) is acting on the level-sets of $-\frac{1}{2}B$ as an element of SO(3).

Fix the particular level set $-\frac{1}{2}B = 1$. The unit vector u_1 lies in this level-set; we compute the stabiliser of this element. Let $U = U(\alpha, \beta)$ be the generic element of SU(2) as in Eq. (2.11). We have $\mathrm{Ad}(U)(u_1) = u_1$ iff

$$u_1 = U u_1 U^{-1} = \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} \overline{\alpha} & -\beta \\ \overline{\beta} & \alpha \end{bmatrix} = \begin{bmatrix} i(\|\alpha\|^2 - \|\beta\|^2) & -2i\alpha\beta \\ -2i\overline{\alpha\beta} & -i(|\beta|^2 + |\alpha|^2) \end{bmatrix}$$

this equality implies $\|\alpha\|^2 - \|\beta\|^2 = 1$ which together with $|\beta|^2 + |\alpha|^2$ shows $\|\alpha\|^2 = 1$ and $\|\beta\|^2 = 0$; since the converse is clearly true we have $\mathrm{Ad}(U)(u_1) = u_1$ iff $\|\alpha\|^2 = 1$. Hence the stabiliser of u_1 is a $\mathbb{S}^1$.

In summary, then, we have an action of $\mathrm{SU}(2) \simeq \mathbb{S}^3$ on $\mathbb{S}^2$ with stabilisers $\mathbb{S}^1$; this exhibits SU(2) as a $\mathbb{S}^1$-bundle over $\mathbb{S}^2$. This structure is called the **Hopf fibration**.

Remark. Another way of constructing the Hopf fibration is as follows. Let $\mathbf{H}$ be the Hamiltonian quaternion algebra, that is the algebra on $\mathbb{R}^4 = \mathbb{R}\{1, i, j, k\}$ with multiplication given by the Fano relations $-1 = i^2 = j^2 = k^2 = ijk$. With the usual norm on $\mathbb{R}^4$, the subgroup of unit norm quaternions is isomorphic to SU(2) via the map $U(\alpha, \beta) \to \alpha + \beta j$ (notation as in Eq. (2.11)), see e.g. [7, §2.4]. Identify $\mathbb{R}^3$ with the subspace of $\mathbf{H}$ spanned by $\{i, j, k\}$. Then SU(2) acts on $\mathbb{R}^3$ via quaternion multiplication as a group of rotations. With a little bit of algebra one can show that this is actually the same action we just described via the adjoint action!

We note that a similar kind of thing occurs for $\mathbb{H}^3$: one can define an action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{H}^3$ by identifying $\mathbb{H}^3$ with the set $\alpha + \beta j \in \mathbf{H}$ where $\alpha \in \mathbb{C}$ is arbitrary but $\beta \in \mathbb{R}_{>0}$ and then setting

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .(\alpha + \beta j) \coloneqq (a(\alpha + \beta j) + b)\,(c(\alpha + \beta j) + d)^{-1}.$$

See the elementary proof in §4.1 of [7].

All of this is due to coincidences that occur in low-rank Lie groups: many of them are strongly related to low-dimension Clifford algebras which can be constructed with quaternions and octonions in natural ways (see for instance the rather remarkable Chapter 21 of [52]).

As a quick application of the previous example, we can derive the famous belt trick.

2.12 Trick. Since $\mathrm{SU}(2) \simeq \mathbb{S}^3$ is connected, the kernel of this map is $Z(\mathrm{SU}(2)) = \{\pm I\}$ and so φ induces an injective continuous map $\overline{\varphi} : \mathrm{SU}(2)/\{\pm I\} \to \mathrm{SO}(3)$; by invariance of domain,[2] $\overline{\varphi}$ is open;

[2]**Theorem.** *If M and N are topological n-manifolds and $f : M \to N$ is continuous and injective, then f is open.*
Proof. [12, Corollary IV.19.9]. ♞⥯

but domain and codomain are connected compact manifolds so this implies that $\overline{\varphi}$ is onto. Viewing SU(2) as the 3-sphere so $-I$ acts to send a point to its diametric opposite, it should be clear that $\mathbb{RP}^3 \simeq \mathrm{SU}(2)/\{\pm I\} \simeq \mathrm{SO}(3)$, hence $\pi_1(\mathrm{SO}(3)) = \pi_1(\mathbb{RP}^3) = \mathbb{Z}/2\mathbb{Z}$.

Consider now the embedded 1-sphere $\overline{\gamma}$ in SO(3) given by taking the projection via $\overline{\varphi}$ of the path γ from $\gamma(0) = I$ to $\gamma(1) = -I$ given by

$$[0, \pi] \ni t \mapsto \begin{bmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{bmatrix} \in \mathrm{SU}(2).$$

This loop cannot be contracted since the path above cannot be contracted to a point. Thus it represents the nontrivial element of $\pi_1(\mathrm{SO}(3))$.

One now observes that $\overline{\gamma}(t)$ represents (as an element of SO(3)) rotation by an angle $2\pi t$. In particular since $\overline{\gamma} * \overline{\gamma}$ (as an element of $\pi_1(\mathrm{SO}(3))$) is trivial, this means that the map $\lambda : t \mapsto$ rotation by $4\pi t$ represents a homotopically trivial loop in SO(3). That is, there is a homotopy $F : [0,1]^2 \to \mathrm{SO}(3)$ with

$$\begin{array}{ll} F(s,0) = \mathrm{id} & F(s,1) = \lambda(s) \\ F(0,t) = \mathrm{id} & F(1,t) = \mathrm{id}. \end{array}$$

With this defined, consider the map $\Phi : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$ given by

$$\Phi(x,t) = \begin{cases} F(|x| - 1, 1 - t)x & \text{for } 1 \le |x| \le 2 \\ x & \text{otherwise.} \end{cases}$$

This sets up the following physical experiment (following Bredon). Suspend a hollow ball (of radius 1 centred at the origin) in an infinite bath of ideal jelly; rotate the ball twice around some axis; fix the ball from any further movement and let go. Then the jelly can return to its original (unwound) state via the isotopy F which leaves the ball fixed and which also leaves the jelly far away from the origin fixed. This is known as the **belt trick**.

2.13 Definition. We will endow $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with a geometric structure, following [18, §10]. We have already seen that SU(2) acts on 2-spheres foliating a copy of 3-space $\mathbb{R}^3$ which gives it a fibre bundle structure over $\mathbb{S}^2$ (Example 2.10), and similarly we will construct a decomposition of $\mathrm{SL}(2,\mathbb{R})$ as a fibre bundle over $\mathbb{H}^2$. Consider the adjoint action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathfrak{sl}(2)$ (2×2 real matrices with zero trace). The Killing form on $\mathfrak{sl}(2)$ is given by

$$B(X,Y) = 4\,\mathrm{tr}(XY)$$

and has signature $(2,1)$. Level sets of this form are $\mathbb{H}^2$'s and are the orbits of the $\mathrm{SL}(2,\mathbb{R})$ action; the point-stabilisers are topologically $\mathbb{S}^1$'s (they are isomorphic to $O(2)$) and hence we have $\mathrm{SL}(2,\mathbb{R}) \simeq \mathbb{H}^2 \times \mathbb{S}^1$; the universal cover $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is a line bundle over $\mathbb{H}^2$ of curvature 1, also called $\mathbb{H}^2 \tilde{\times} \mathbb{E}^1$. For some visualisations, see [51].

Remark. We won't use it, but here are some properties of the isometry groups of the geometries fibreing over $\mathbb{H}^2$ [70, §4.7]. Fix X such a geometry, i.e. X is $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Let $G = \mathrm{Isom}(X)$. There is a natural projection $p : G \to \mathrm{Isom}(\mathbb{H}^2) = \mathrm{PSL}(2,\mathbb{R})$. If $\Gamma \le G$ is discrete, then $p(\Gamma)$ is either discrete (i.e. a Fuchsian group) or is virtually Abelian. Even better, Γ is finite covolume iff $p(\Gamma)$ is discrete, finite covolume, and $\ker p$ is infinite.

2.14 Example (The trefoil knot)**.** Recall that the trefoil knot is a torus knot. By the Riley–Thurston theorem, it has a geometric but not hyperbolic structure. We claim that it has $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ structure (which is a special case of Example 2.17 below) and in fact we can exhibit it explicitly as

$$\widetilde{\mathrm{SL}(2,\mathbb{R})} \Big/ \widetilde{\mathrm{SL}(2,\mathbb{Z})} \tag{2.15}$$

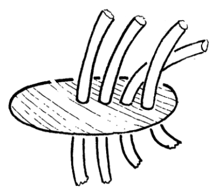


Figure 2.5: A 'slice' of a fibred solid torus. Modified from [59, §10.K.1].

We give a proof of this fact which was written up by Milnor [49, p. 84], though he attributes it to D. Quillen, and which ties together all the remarkable views of this manifold enumerated in [63, Example 1.5.2 of Chapter I].

Observe first that we can reduce the problem to the study of

$$M = \mathrm{SL}(2,\mathbb{R})\big/\mathrm{SL}(2,\mathbb{Z})$$

since by definition $\widetilde{\mathrm{SL}(2,\mathbb{Z})}$ is the inverse image of $\mathrm{SL}(2,\mathbb{Z})$ in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The manifold M is naturally identified with the space of unit-area lattices in $\mathbb{C}$.

Consider the space of *all* lattices in $\mathbb{C}$, call it $\hat{M}$. Given any lattice L there is a **Weierstrass function** $\wp_L$ which is meromorphic on $\mathbb{C}$, doubly periodic with respect to L, and with poles exactly at the lattice points $\lambda \in L$ of the form

$$\wp_L(z+\lambda) = z^{-2} + \sum_{n=1}^{\infty} a_{2n} z^{2n}.$$

The Weierstrass function satisfies the differential equation [3, §7.3.3]

$$\left(\frac{\mathrm{d}\wp_L}{\mathrm{d}z}\right)^2 = 4\wp_L^3 - g_2\wp_L - g_3$$

where g_2 and g_3 are defined respectively as

$$g_2 = 60 \sum_{\lambda\in L^*} \lambda^{-4}, \qquad g_3 = 140 \sum_{\lambda\in L^*} \lambda^{-6}.$$

Further, the pair (g_2, g_3) determine $\wp_L$ and L uniquely. Conversely a pair (g_2, g_3) determines a lattice iff the three roots of the polynomial $f(z) = 4z^3 - g_2 z - g_3$ are all distinct [3, §7.3.4], and hence the manifold $\hat{M}$ is diffeomorphic to the complement of the variety cut out by the discriminant of f, i.e.

$$\hat{M} \simeq \mathbb{C}^2 \setminus \mathbf{V}(27g_3^2 - g_2^3).$$

We have already seen that the trefoil knot is the $(2,3)$-torus knot and that the $(2,3)$-torus knot is the algebraic knot corresponding to the point $(0,0)$ on $\mathbf{V}(w^2 - z^3)$ (for both statements see Exercises 1.31). Hence (modulo scaling one coordinate, which is a diffeomorphism) we see that the trefoil knot complement in $\mathbb{S}^3$ is $\hat{M} \cap S_\varepsilon$ for some small $\varepsilon > 0$. But for every element of M (i.e. every unit-area lattice) there is a unique lattice on the sphere S_ε of $\hat{M}$ obtained by scaling; this scaling is a smooth map and hence we have a diffeomorphism $M \simeq \hat{M} \cap S_\varepsilon$ as desired.

2.16 Definition. A **trivial fibred solid torus** is the solid torus $\mathbb{S}^1 \times \mathbb{B}^2$ with the product foliation of circles, $(\mathbb{S}^1 \times \{x\})_{x\in\mathbb{B}^2}$ (Fig. 2.5). A **fibred solid torus** is a solid torus together with a foliation by circles that is finitely covered by a trivial solid torus; these fibred tori can all be obtained by cutting

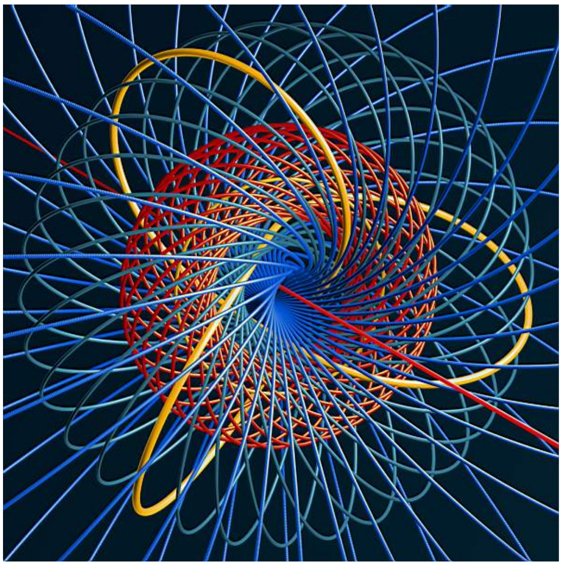


Figure 2.6: A Seifert fibration of $\mathbb{S}^3$ with generic fibre the $(1, 1)$-torus knot. Image by Ian Agol, `https://mathoverflow.net/a/248120/150082`.

a trivial fibred solid torus along one of the discs, rotating by q/p (q, p coprime), and regluing. (The induced foliation on the boundary is the (p, q) curve on the torus.)

A **Seifert fibration** on a 3-manifold M is a decomposition of M into disjoint simple closed curves (**fibres**) such that every fibre has a neighbourhood U that is diffeomorphic to a fibred solid torus in a fibre-preserving way.

Warning. A Seifert fibration of a knot complement is *not* to be confused with a fibration by Seifert surfaces: the figure eight knot complement admits the latter structure [27, pp. 159–160] but not the former (as it is hyperbolic).

2.17 Example. Torus knot complements admit Seifert fibrations. First observe that if γ is a curve on the boundary of the solid torus, then the solid torus admits a Seifert fibration which restricts to a foliation parallel to γ on the boundary (Fig. 2.6). Now given the p/q-torus knot, cut along the embedded torus so obtaining one solid torus glued to another along the boundary with a p/q-curve on one glued to a median on the other. Fibre each torus separately, and then the two surface foliations on the torus agree giving a fibration of the whole thing. On the other hand, if $\mathbb{S}^3 \setminus k$ admits a Seifert fibration where k is a knot, then k is a torus knot. Proof: the fibration extends to a Seifert fibration of $\mathbb{S}^3$ by adding in k, let U be a neighbourhood of k which is a fibred torus and clearly a fibre on the boundary of this torus is isotopic to k.

This classifies topologically the only class of non-hyperbolic geometric knots. (The complement manifolds of the third class of knots, the satellite knots, can be decomposed by cutting along a compact surface such that each piece is either hyperbolic or admits a Seifert fibration. This is called the **characteristic torus decomposition** [10, Theorem 3.4].)

Remark. There are many more Seifert fibred *links*: see [14] for some characterisations (e.g. it is equivalent to the link group having a nontrivial center).

2.18 Theorem. *If the complement of a knot k admits a Seifert fibration, then it admits a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometry and a $\mathbb{H}^2 \times \mathbb{E}^1$ geometry (and these geometries are not rigid).*

Sketch of proof. Suppose M is the knot complement, so we have a (p, q)-torus knot. Define an orbifold X to be the quotient space of M by the relation 'points become equal if they lie on the same circular fibre': so X is an orbifold surface. There is an exact sequence $1 \to K \to \pi_1(M) \to \pi_1(X) \to 1$ where K is the infinite cyclic group generated by a regular fibre of X. One can show that $\chi(X) < 0$ (it is a disc with one p-cone point and one q-cone point, so $\chi(X) = \chi(\text{disc}) - (1-1/q) - (1-1/p) = 1-2+1/q+1/p$ by [68, §13.3]), i.e. it is hyperbolic and so $X = \mathbb{H}^2/\pi_1(X)$ where $\pi_1(X)$ is a Fuchsian group. Choose natural generators for $\pi_1(X)$ and lifts of these generators to $\pi_1(M)$; we can choose these generators such that if X has genus g with n cone points of orders $\alpha_1, \dots, \alpha_n$ we have a presentation

$$\pi_1(X) = \left\langle a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_n \,:\, \forall_r (x_r^{\alpha_r} = 1), \prod_{i=1}^{g} [a_i, b_i] x_1 \cdots x_n = 1 \right\rangle.$$

The point now is to lift this action of $\pi_1(X)$ on $\mathbb{H}^2$; there are two ways of doing this, one in a twisted way giving a embedding of $\pi_1(M)$ into $\mathrm{Isom}(\widetilde{\mathrm{SL}(2,\mathbb{R})})$ and one in a non-twisted way giving an embedding into $\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{E}^1)$. See [61, Theorem 5.3(ii)] for details, which are a little too complicated for us.

2.19 Exercises. 1. If you know Chapter VII of Maskit [48]: write the figure eight group in terms of the amalgamated products and HNN extensions of the cyclic groups generated by

$$\begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

where $\omega = e^{2\pi/3}$.

2. Describe in SU(2), in terms of the group structures (where α denotes the upper-left-hand element of a generic element, Eq. (2.11)),
   (a) the latitudes: the set of all $U \in \mathrm{SU}(2)$ such that $\Re\alpha$ is some fixed value (hint: this was already done for $x = \pm 1$);
   (b) the longitudes: the set of $U \in \mathrm{SU}(2)$ cut out by any hyperplane ($\mathbb{R}^3$) in $\mathbb{C}^2$ which passes through $\pm I$.

3. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus.
   (a) Show that the linear automorphism of $\mathbb{R}^2$ represented by $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ descends to T. The resulting map on the torus is the **Arnold's cat map** α.
   (b) Draw the mapping torus of α, $M_\alpha \coloneqq (T \times [0,1])/((x,1) \sim (\alpha(x), 0))$. This manifold is a Sol-manifold.

4. [70, Exercise 4.7.1] If ϕ is an isometry of $\mathbb{S}^2$ then the mapping torus M_ϕ is an $(\mathbb{S}^2 \times \mathbb{E}^1)$-manifold. In fact it is the quotient of $\mathbb{S}^2 \times \mathbb{E}^1$ by the discrete group generated by the transformation $(v, t) \mapsto (\phi v, t+1)$ where $v \in \mathbb{S}^2$. The manifold is diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ when ϕ is orientation preserving and is non-orientable otherwise. What other manifolds admit $\mathbb{S}^2 \times \mathbb{E}^1$ structures?
   (a) Any discrete subgroup of isometries of $\mathbb{S}^2 \times \mathbb{E}^1$ acts discretely (but not necessarily freely or effectively) on $\mathbb{E}^1$.
   (b) An infinite discrete group of isometries of $\mathbb{E}^1$ is isomorphic to $\mathbb{Z}$ or $C_2 * C_2$.
   (c) There are only three closed 3-manifolds, up to diffeomorphism, that admit $(\mathbb{S}^2 \times \mathbb{E}^1)$-structures. Two are orientable and one is not.

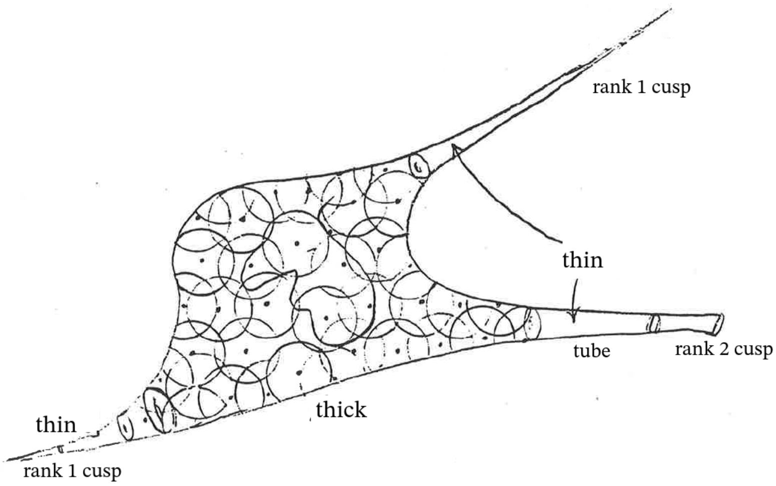


Figure 2.7: The thick-thin decomposition. Modified from from [68, §5.11].

2.2 Hyperbolic invariants and computation

We are now interested only in knots k whose complement is hyperbolic. Recall by this that we mean the following: there exists a Riemann metric on $\mathbb{S}^3 \setminus k$ which has constant sectional curvature -1. If this Riemann metric is complete, then we even have a faithful representation $\pi_1(k) \to \mathrm{PSL}(2, \mathbb{C})$ such that $\mathbb{S}^3 \setminus k = \mathbb{H}^3/\pi_1(k)$. We wish to study (i) invariants which we can define using hyperbolic geometry, and (ii) the space of incomplete structures on the knot complement.

Our main geometric invariant is volume, so we need to prove that knot complement manifolds (a) have finite volume, and (b) admit only one structure so that this volume is well-defined. We need to know some of the global geometry, which is controlled by the so-called **thick-thin decomposition**, a consequence of Margulis' lemma [8, Chapter D]. Suppose that M is a complete hyperbolic 3-manifold. For each $x \in M$ we define the **injectivity radius** of M at x, $\iota(x)$, to be the supremum of all $r \in \mathbb{R}$ such that the ball of radius r around x is isometric to the ball of radius r in $\mathbb{H}^3$. Let $\varepsilon > 0$. The **ε-thin part** of M is the set

$$M_\varepsilon = \{x \in M \,:\, \iota(x) < \varepsilon/2\}.$$

See Fig. 2.7 for a two-dimensional cartoon of the thick and thin parts of a hyperbolic 3-manifold.

2.20 Theorem (Structure of thin part). *There exists a universal (i.e. independent of M) constant $\varepsilon_3 > 0$ such that for $0 < \varepsilon \leq \varepsilon_3$ the ε-thin part of any complete hyperbolic 3-manifold M consists of tubes around short geodesics, rank 1 cusps, and rank 2 cusps.*

In the theorem, a **rank 1 cusp** is a piece of M which is isometric to the quotient of the horoball based at at $\xi \in \hat{\mathbb{C}}$ by a cyclic group of parabolic elements with fixed point at ξ, and a **rank 2 cusp** is isometric to the quotient of the horoball by a rank 2 Abelian group generated by two parabolics with fixed points at ξ. These three objects are shown in Thurston's cartoon Fig. 2.7: observe that a geodesic on the surface can be moved by free homotopy towards a rank 1 cusp, this is meant to depict how a geodesic can be shrunk in a 3-manifold around a rank 1 cusp. See also that a short geodesic on the tube part of the surface is surrounded by a cylinder on the surface to make up that piece of the thin part. It is hard to draw a cartoon of a rank 2 cusp, so it is represented by simply drawing a small torus: there are two independent directions of geodesics which can be shrunk to a point around this. Finally note that the different parts intersect along torus surfaces.

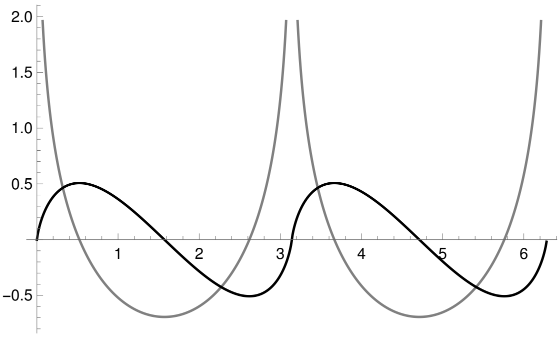


Figure 2.8: The Lobachevskii function Л and its derivative (in grey).

One can also ask for bounds on the radii of tubes around short geodesics [47, Theorem 3.3.4].

2.21 Theorem. *A complete hyperbolic 3-manifold M has finite volume iff either*

- *M is compact without boundary, or*
- *M is homeomorphic to the interior of a compact manifold $\overline{M}$ with torus boundary components, such that $\overline{M}$ is neither a solid torus or $\mathbb{T}^2 \times [0,1]$.*

In particular, knot complements which admit hyperbolic metrics have finite volume.

Proof. The proof goes via the thick-thin decomposition of 3-manifolds [8]. First if M is compact without volume then it is the image of a compact fundamental domain in $\mathbb{H}^3$ which is finite volume. If M is the interior of a manifold with only torus boundary components and is not elementary (the two excluded homeomorphism classes) then we can write it as the union of a compact piece (finite volume) and neighbourhoods of rank 2 cusps, and these neighbourhoods are finite volume. Conversely, if M has finite volume and is not compact without boundary then (i) the thick part of M must be compact (else it would be infinite volume), (ii) the number of components of M_ε is finite (since each of these is glued onto the boundary of the thick part, which is compact). Add the tubes to the thick part: this union continues to be compact. Attach a $\mathbb{T}^2 \times [0,1]$ to the boundary of each torus boundary component, and call the result N. Clearly $N = \operatorname{int} M$, and the boundary of N cannot be one of the two excluded cases since those manifolds are infinite volume.

For knot invariant construction the outlook is not very good unless the hyperbolic structure is unique (otherwise different structures on the same topological knot complement might give different volumes).

2.22 Theorem (Mostow-Prasad rigidity). *Let M be a hyperbolic 3-manifold. Then M admits at most one complete finite-volume hyperbolic structure.*

Here, **complete** means in the usual metric sense, and can also be detected locally in fundamental domains for the uniformising group. The conditions for a given structure to be complete are polynomial conditions, but are complicated.

2.23 Corollary. *The map* $\mathrm{Vol} : \mathsf{Knot} \to \hat{\mathbb{R}}_{>0}$ *which sends a hyperbolic knot to the hyperbolic volume of its complement and a torus or satellite knot to ∞ is a well-defined knot invariant.*

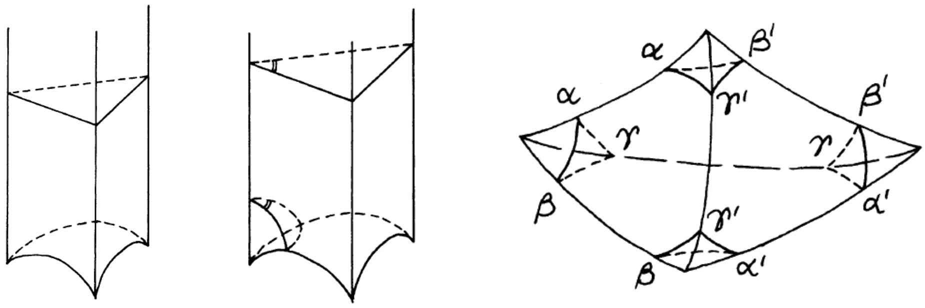


Figure 2.9: On the left, a hyperbolic tetrahedron with one vertex at ∞ and the corresponding level set. In the centre and on the right we see all four vertices adorned with these horocyclic triangles. Figure from [8, Figs. C.10–C.12].

2.24 Definition. The **Lobachevskii function** [44] $Л : [0, 2\pi] \to \mathbb{R}$ is defined by

$$Л(\theta) = -\int_0^\theta \log|2\sin u| du,$$

and is plotted in Fig. 2.8.

2.25 Proposition. *Let $\mathcal{T}$ be the set of $\mathbb{R}^2$-triangles up to similarity; that is, $\mathcal{T}$ is the set of unordered triples $\alpha, \beta, \gamma \in (0, \pi)$ such that $\alpha + \beta + \gamma = \pi$. Let $\mathcal{S}_3$ be the set of isometry classes of ideal simplices (tetrahedra) in $\mathbb{H}^3$. There exists a bijective map $T : \mathcal{S}_3 \to \mathcal{T}$ such that, if $\sigma \in \mathcal{S}_3$, then* $\mathrm{Vol}(\sigma) = Л(\alpha) + Л(\beta) + Л(\gamma)$ *where α, β, γ are the angles of $T(\sigma)$.*

We follow the proof given in [8, §C.2].

Proof. Let $\sigma \in \mathcal{S}_3$ be an ideal simplex with vertex set $\{p_0, p_1, p_2, p_3\}$. Define maps T_i for $0 \le i \le 3$ in the following way: send p_i to ∞ via an isometry; for large enough t, the set $\sigma \cap (\mathbb{R}^2 \times \{t\})$ is a Euclidean triangle (Fig. 2.9, left) which we take to be $T_i(\sigma)$. The similarity class of this triangle is independent of the choice of σ in $[\sigma]$: observe that isometries of $\mathbb{H}^3$ keeping ∞ fixed induce conformal maps $\mathbb{R}^2 \times \{t\} \to \mathbb{R}^2 \times \{\lambda t\}$ for some t (this is the essence of the definition of Poincaré extension actually). Hence the T_i are all well-defined.

We now claim that $T_i(\sigma)$ is independent of σ. To see this we draw all of the triangles $T_i(\sigma)$ at once. Move one of the vertices p_i is at ∞, and consider the horospherical triangle which T_j constructs near p_j. We see that the two angles indicated in the central image of Fig. 2.9 are equal; overall we only have the six distinct angles shown in the rightmost image of the figure. We obtain four equations in these six angles from the Euclidean angle sum formula: $\alpha+\beta+\gamma = \alpha+\beta'+\gamma' = \alpha'+\beta+\gamma' = \alpha'+\beta'+\gamma = \pi$. Reducing these we get $\alpha + \beta = \alpha' + \beta'$, $\alpha + \gamma = \alpha' + \gamma'$, and $\beta + \gamma = \beta' + \gamma'$, this is a system of three linear equations in six unknowns that has three-dimensional solution space $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$. This shows that all the images $T_i(\sigma)$ are similar Euclidean triangles and hence we can define $T(\sigma)$ to be 'the Euclidean triangle cut out by any sufficiently large horosphere around any vertex of σ'.

The equation on $Л$ is a technical exercise in hyperbolic trigonometry which we omit, see [8, Prop. C.2.8]. ⚞

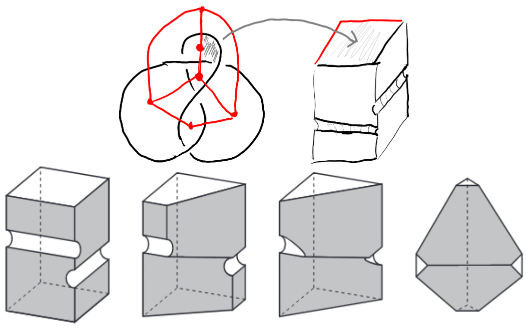


Figure 2.10: Weeks' algorithm for triangulating hyperbolic knot complements. Bottom figure from [72].

2.26 Example. By the proof of Theorem 2.5 the volume of the figure eight knot complement is twice the volume of the tetrahedron with all angles $\pi/3$, i.e. it is

$$6\,\text{Л}(\pi/3) = -6\int_0^{\pi/3} \log|2\sin u|du \approx 2.0299.$$

In fact the figure eight knot is the knot of minimal volume.

In order to compute the hyperbolic volume of a knot, then, it is enough to compute triangulations of the complement manifold.

2.27 Algorithm (SnapPea Algorithm (Jeff Weeks, c.1985))**.** Let k be a hyperbolic knot in $\mathbb{S}^3$. The algorithm computes a decomposition of $\mathbb{S}^3 \setminus k$ into hyperbolic ideal tetrahedra.

1. Embed the knot in $S^2 \times [-1, 1]$ 'flatly' around $S^2 \times \{0\}$.
2. Cut straight down along the dual graph & the knot graph (Fig. 2.10, top).
3. Collapse the quadrilateral slices to tetrahedra (Fig. 2.10, bottom).
4. Glue four cusps onto these vertices to get spherical tetrahedra.
5. Do a bit of fiddling to get the hyperbolic geometry back.

The details of this algorithm can be found in [72], and it is implemented in the `SnapPy` software [20]. We will give some detailed examples when we study two-bridge knots.

We would like to ask how 'good' this invariant actually is. The two main results in this area form the following theorem.

2.28 Theorem.
1. *Given some $v \in \mathbb{R}_{>0}$, the number of hyperbolic 3-manifolds with volume v is finite.*
2. *The set of all volumes $\mathcal{F}_3$ is a well-ordered non-discrete subset of $\mathbb{R}_{>0}$ (without the axiom of choice).*
3. *Given any $n \in \mathbb{N}$ there exists some volume $v \in \mathcal{F}_3$ such that $\left|\mathrm{Vol}^{-1}(v)\right| = n$. (Wielenberg, 1981)*

This theorem follows from Thurston's Dehn filling theorem (together with a lot of work) by taking sequences of Dehn surgeries of manifolds and looking at the convergence behaviour of their volumes. The motivation behind this is the classification of *incomplete* hyperbolic structures on hyperbolic manifolds, of which there are infinitely many.

2.29 Definition. Let M be a manifold with torus boundary component T, and let $\gamma_{p/q}$ be an isotopy class of simple closed curves on t. The manifold obtained by attaching a solid torus to T such that $\gamma_{p/q}$ bounds a disc is called the **Dehn filling** of M along $\gamma_{p/q}$.

2.30 Definition. Let M be a manifold, let k be a knot in M, and let $p/q \in \hat{\mathbb{Q}}$. The manifold M' obtained from M by drilling out a solid torus neighbourhood of k and performing a p/q Dehn filling along the result is called the result of **Dehn surgery** along k.

The result of Dehn surgery in a hyperbolic manifold is usually hyperbolic. This follows from the next theorem, which we first state in a rough sense: *Let M be a 3-manifold homeomorphic to the interior of a compact manifold with boundary a single torus $\mathbb{T}^2$ such that M admits a complete hyperbolic structure. Then the space of all Dehn surgeries on M contains an open neighbourhood of the complete structure.*

More precisely, let M be any 3-manifold with torus boundary C (C is called a **cusp torus**) and suppose that an incomplete hyperbolic structure is placed on M, so the holonomy group $\pi_1(C)$ is not generated by parabolic elements. Then there is a natural map $L : \pi_1(C) = H_1(C, \mathbb{Z}) \to \mathbb{C}$ given by the complex length function, and this admits a canonical extension $L : H_1(C, \mathbb{R}) \to \mathbb{C}$. (In other words we extend from simple closed curves to arbitrary laminations of one leaf.)

Suppose that the complex length around a simple closed curve γ is $2\pi i$. Then γ corresponds exactly to a rotation by 2π and in the completion of M the curve γ bounds a smooth hyperbolic disc. Hence the completion of M with this hyperbolic structure is a manifold homeomorphic to the Dehn filled manifold along γ and we therefore get a complete hyperbolic structure. On the other hand if the imaginary part is $\theta \neq 2\pi$, in the completion the curve γ will bound a hyperbolic cone of angle θ, the metric on the completion is not smooth, and so we don't get a structure. There is a unique $c \in H_1(C, \mathbb{R})$ with $L(c) = 2\pi i$, and this c is called the **Dehn surgery coefficient** of C. The subset of $H_1(C, \mathbb{R}) \simeq \mathbb{R}^2$ consisting of all Dehn filling coefficients for all possible hyperbolic structures (that is, $H_1(C, \mathbb{R})$ is a topological invariant so does not depend on the incomplete structure on M, but L does depend on this structure, so we get different coefficients for each structure that all lie in the same $\mathbb{R}^2$) is called the **hyperbolic Dehn filling space** for M, and by convention we let ∞ be the complete hyperbolic structure on M if it exists.

2.31 Theorem (Thurston's Dehn filling theorem)**.** *Let M be a 3-manifold homeomorphic to the interior of a compact manifold with boundary a single torus $\mathbb{T}^2$ such that M admits a complete hyperbolic structure. Then the Dehn filling space of M contains an open neighbourhood of ∞. More generally if M is the interior of a compact manifold with torus boundary components $T_1, \dots, T_n$ and if it admits a complete structure, then for each T_i the corresponding Dehn filling space contains an open neighbourhood of ∞.*

Sketch of proof. Suppose M admits a complete structure, and let $\rho : \pi_1(X) \to \mathrm{PSL}(2, \mathbb{C})$ be the representation verifying this. The fundamental group of $\mathbb{T}^2$ has image $\rho(\pi_1(\mathbb{T}^2))$ generated by two parabolics with the same fixed point. Now show that ρ admits a one-parameter family of deformations, and that each deformation sends these parabolics to a pair of loxodromics which share fixed points. Now this gives a distance measure which induces an incomplete complex structure that has Dehn filling coefficient varying continuously around ∞. In general, one can do high-dimensional deformations to get the result for n boundary components. ⊠

2.32 Corollary. *Let X be a complete hyperbolic manifold with n torus boundary components $T_1, \dots, T_n$. For each T_i, exclude finitely many Dehn fillings. The resulting Dehn fillings yield a manifold with a complete hyperbolic structure.*

Proof. For every i there are only finitely many elements of $H_1(C, \mathbb{Z})$ that lie outside the open neighbourhood of ∞ in $H_1(C, \mathbb{R})$ given by the theorem. ⊠

Conversely, all 3-manifolds arise by Dehn surgery:

2.33 Theorem (Lickorish/Wallace, 1960–1962)**.** *Let M be a closed orientable 3-manifold. Then M is the result of Dehn surgery along some link in $\mathbb{S}^3$.*

A slightly stronger version of this is:

2.34 Theorem (Jørgensen)**.** *Let $C > 0$. Among all hyperbolic 3-manifolds M with volume at most C, there are only finitely many homeomorphism types of M_ε. There is a universal link $L_C \subseteq \mathbb{S}^3$ such that every complete hyperbolic manifold with volume at most C is obtained by some Dehn surgery along L.*

The combination of Corollary 2.32 and Theorem 2.33 implies, roughly speaking, that most 3-manifolds are hyperbolic.

2.35 Exercises. 1. [47, Exercise 3-46] (Halpern's inequality) Suppose G is a torsion-free Fuchsian group (i.e. discrete subgroup of $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{Isom}^+(\mathbb{H}^2)$) acting on the upper half-plane $\mathbb{H}^2 = \{x+ti \in \mathbb{C} : t > 0\}$. Assume G has a parabolic fixed point at ∞ and that the parabolic subgroup is generated by $T : z \mapsto z+1$. Prove that for every $A \in G$ that does not fix ∞, $|c| \geq 2$ where c is the lower-left-hand entry of A. (Hint: compute $\operatorname{tr} TATA^{-1}$.)

2. [47, Exercise 3-5] A **Dirichlet region** for a Kleinian group G with centre $z \in \mathbb{H}^3$ is the closed convex hyperbolic polyhedron
$$\bigcap_{g\in G} H_g$$
where H_g is the relatively closed half-space which is bounded by the perpendicular bisector of $[z, gz]$ containing z. This is a fundamental polyhedron for G [48, §IV.G].

   Find a Dirichlet region for the rank two parabolic group generated by $z \mapsto z+1$ and $z \mapsto \tau$ for $\tau \in i\mathbb{R}_{>0}$. Show that generically it has six edges, but sometimes only four. Compute the hyperbolic volume of the part of the polyhedron lying above a general horosphere based at ∞. Show that the quotient $\mathbb{H}^3 \cup \mathbb{C}/G$ (this is OK since $\Omega(G) = \hat{C} \setminus \{\infty\}$!) is homeomorphic to $\{0 < |z| \leq 1 : z \in \mathbb{C}\} \times \mathbb{S}^1$, i.e. the complement of the core circle is the solid torus. This is the prototype of the local structure about a hyperbolic knot, the parabolic fixed point is 'stretched' onto the knot.

3. [47, Exercise 6-1] Let Γ be the group of isometries of $\mathbb{E}^3$ which is generated by $(x,y,t) \mapsto (x+1,y,t)$ and $(x,y,t) \mapsto (-x,y+1,-t)$. Let $\Gamma_0 = \langle (x,y,t) \mapsto (x+1,y,t), (x,y,t) \mapsto (x,y+1,t)\rangle$

   (a) The group Γ preserves $\mathbb{C}$ and $\mathbb{C}/\Gamma$ is the Klein bottle.

   (b) The interior of $\mathbb{T}^2 \times [0,1]$ obtained by thickening the torus $\mathbb{T}^2$ is almost hyperbolic except for the existence of hyperbolic essential cylinders with one boundary component on $\mathbb{T}^2 \times \{0\}$. The interior is $\mathbb{E}^3/\Gamma_0$. (This is the only manifold whose boundary components are tori whose interior does not have a complete finite-volume hyperbolic structure, by a theorem from the lecture.)

   (c) The torus $\mathbb{C}/\Gamma_0$ is the two-sheeted orientable cover of $\mathbb{C}/\Gamma$ and the cover transformation is 'flipping'. The corresponding 3-manifold $\mathbb{E}^3/\Gamma$ is called the **twisted *I*-bundle** over the Klein bottle and is the only homotopically atoroidal manifold whose interior does not have a hyperbolic structure.

Chapter 3

Braids

In this third we week we will study braids. These are important and central objects in geometric topology and their study arises naturally in knot theory and in the study of surface homeomorphisms. Representations of braid groups will be of particular interest to us, as this is the direction we need to head in order to introduce the Jones polynomials in the final week. As well as preparing for this we will classify all 2-bridge knots and explain their relationship to Lens spaces, and consider the link between knot theory and the theory of mapping classes. For braids in general, see the textbook of Kassel and Turaev [39] as well as the large monograph of Burde and Zieschang [15]. We will also use parts of Farb and Margalit's monograph on mapping classes [24] as an important reference.

3.1 4-plats and 2-bridge knots

In this section, we mainly follow the exposition of [15, Chapters 10–12] and Chapter 10 of [53] in classifying all 2-bridge links. We will also explain the Riley representations of the corresponding groups into $\mathbb{M}$.

Recall that a **2-bridge link** is a link $k \subseteq \mathbb{S}^3$ which can be arranged via isotopy in such a way that k intersects a fixed plane (taken to be $\mathbb{R}^2$) transversely in exactly four points such that the intersection of k with each half-space cut out by the plane (consisting of two space arcs) projects injectively to two disjoint arcs on the plane. The figure eight knot is an example, as seen in Fig. 3.1.

Without loss of generality (i.e. by applying an appropriate isotopy) we can assume that the image of the two arcs on one side of $\mathbb{R}^2$ is exactly the two invervals $I_1 = [0, 1]$ and $I_2 = [3, 2]$ (observe the orientation of I_2 is reversed), and the other two arcs projecting from the other side of $\mathbb{R}^2$ end up as two curves u, v winding in a spiral fashion like in Fig. 3.2; we always assume that such a **torsion diagram** is reduced. The number of double points is even since each of u and v intersects both I_1 and I_2 the same number of times: call this number of intersectons $\alpha - 1$. If α is odd then k is a knot and if α is even then it is a two-component link. Note that α does not determine the diagram uniquely, since the curve might wind around through the middle like in the diagram for the figure eight knot (Fig. 3.3); we will introduce a second invariant β via a topological argument and you are invited to supply a geometric interpretation in terms of the diagram as an exercise.

Now observe that there is a natural double cover $\mathbb{T}^2 \to \mathbb{S}^2$ given by the hyperelliptic involution $\tau \circlearrowleft \mathbb{T}^2$ (Fig. 3.4). Lifting u and v to $\hat{u}$ and $\hat{v}$ and I_1 and I_2 to $\hat{I}_1$ and $\hat{I}_2$, we see that $\hat{v} - \tau\hat{v}$ and $\hat{u} - \tau\hat{u}$ are isotopic homotopy chains (where the minus signs show only that the orientation needs to be reversed in order to obtain well-defined chains); they intersect alterately with the lifts of the two intervals, $(1-\tau)\hat{I}_1$ and $(1-\tau)\hat{I}_2$. These lifts are shown in Fig. 3.5. Choose a basis for $H_1(\mathbb{T}^2)$ consisting

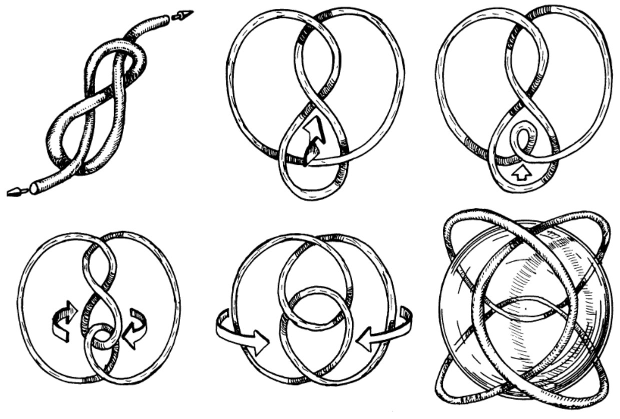


Figure 3.1: The figure eight knot is 2-bridge; a presentation is in the lower right corner. Figure from [27, p. 150].

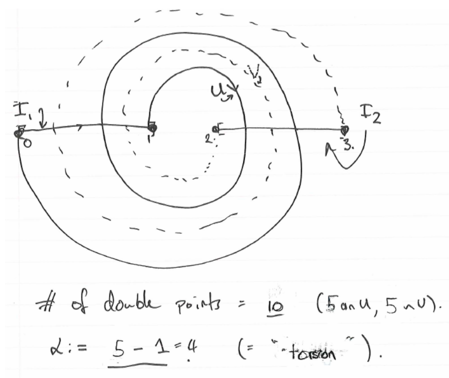


Figure 3.2: Torsion diagram for the (4, 1) 2-bridge link.

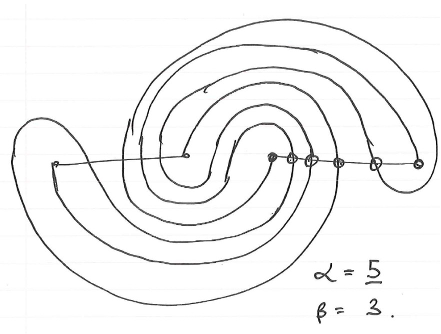


Figure 3.3: Torsion diagram for the figure eight knot, showing how impractical torsion diagrams are.

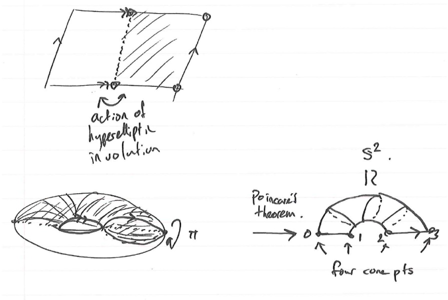


Figure 3.4: The hyperelliptic involution $\tau \circlearrowright \mathbb{T}^2$ induces an $\mathbb{S}^2$ as quotient.

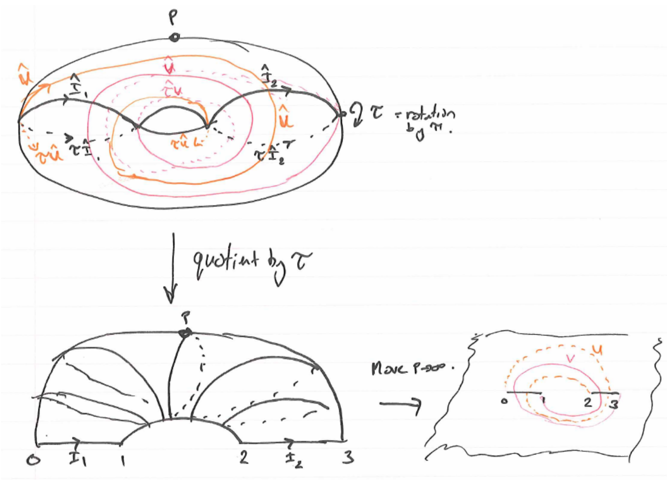


Figure 3.5: Computation of Heegard splitting invariants from a 2-bridge knot.

of a meridian M (isotopic to $(1-\tau)\hat{I}_1$) and a longitude L (isotopic to one of the lifts of a simple closed curve separating I_1 from I_2). Assume that $\alpha > 1$ (you are asked in the exercises for $\alpha \in \{0, 1\}$). Then $(1-\tau)\hat{u}$ (and $(1-\tau)\hat{v}$, being isotopic to it) is of $\mathbb{Z}$-homology type $\beta M + \alpha L$ where $|\beta| < \alpha$ and where β is positive or negative according to whether v crosses $[0, 1]$ in one direction or the other (in the sense of Fig. 1.4). We also see that $(\alpha, \beta) = 1$, as a consequence of Lemma 1.21.

Thus:-

3.1 Proposition. *For any 2-bridge link, there is a pair of integers* (α, β) *with*

(TBL) $\qquad \alpha > 0, \qquad |\beta| < \alpha, \qquad (\alpha, \beta) = 1, \qquad$ *and* β *odd.*

Further, the number of components of the link is $\mu \equiv \alpha \pmod 2$ *where* $1 \le \mu \le 2$.

The invariants are respectively called the **torsion** or **determinant** (α) and the **crossing number** (β).

There are two natural questions arising from Proposition 3.1.

1. Does the converse of Proposition 3.1, i.e. existence of a knot given a pair of integers, also hold?

2. Is the map from 2-bridge knots to pairs of integers a 1–1 correspondence?

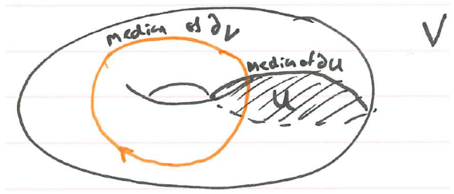


Figure 3.6: The 3-sphere admits a Heegard splitting.

The answer to (1) is *yes*, and in order to prove it (Corollary 3.4) we will consider 2-fold coverings of the 3-manifold $\mathbb{S}^3$ branched along the knot. The answer to (2) is *no*, and is the theorem of Schubert (Theorem 3.5).

3.2 Construction (Lens spaces and Heegard splittings). Identify $\mathbb{S}^{2n-1}$ with the set $\{z \in \mathbb{C}^n : \|z\| = 1\}$. Fix an integer p and set $\zeta = e^{2\pi i/p}$. Choose $q_1, \dots, q_n$ integers such that $(p, q_i) = 1$ for all i, and define an action of ζ on $\mathbb{S}^{2n-1}$ by the rule

$$\zeta.(z_1, \dots, z_n) := (\zeta^{q_1} z_1, \dots, \zeta^{q_n} z_n).$$

This action is isometric with respect to the angular metric on the sphere, and is properly discontinuous since $gx = x$ implies g is the identity. Hence the quotient $\mathbb{S}^{2n-1}/\langle\zeta\rangle$ is a spherical manifold, the **lens space** $L(p; q_1, \dots, q_n)$. In the special case $n = 2$ and $q_1 = 1$ we write $L(p, q) := L(p; 1, q)$, this is a smooth 3-manifold modelled on $\mathbb{S}^3$. In the sequel this will be the only class of lens spaces we want.

A (genus g) **Heegard splitting** of a compact oriented 3-manifold M is a decomposition $M \simeq_{\text{homeo.}} U \cup_f V$ where (i) both U and V are solid handlebodies of genus g, (ii) f is a orientation reversing homeomorphism $U \to V$ (the notation $\cup_f$ means 'take the disjoint union and quotient by the equivalence relation set up by f'). We will classify the 3-manifolds which admit a genus one splitting, for details see Hempel [32, pp. 20–23]. Before doing any work we immediately observe that $\mathbb{S}^3$ itself admits such a splitting, Fig. 3.6.

Suppose $M = U \cup_f V$ where U, V are solid genus 1 handlebodies. The homeomorphism $f : \partial U \to \partial V$ is isotopic to a map which glues a simple closed curve ω on ∂U to the curve $[\alpha]$ on ∂V, and different choices of ω (mod homotopy) give different homeomorphisms—this is just the classification of mapping classes on the torus. Hence by Lemma 1.21 the Heegard splittings are indexed by the pairs (p, q) of coprime integers.

We now claim that the manifold with Heegard splitting (p, q) is exactly the Lens space $L(p, q)$. To do this consider the **Clifford torus**

$$C = \frac{1}{\sqrt{2}}\mathbb{S}^1 \times \frac{1}{\sqrt{2}}\mathbb{S}^1 = \{\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\phi}) : 0 \le \theta, \phi < 2\pi\} \subseteq \mathbb{C}^2$$

which lies in $\mathbb{S}^3$. The action of ζ on C is

$$\zeta.\frac{1}{\sqrt{2}}(e^{i\theta}, e^{i\phi}) = \frac{1}{\sqrt{2}}(e^{i(\theta+2\pi/p)}, e^{i(\phi+2q\pi/p)})$$

so the quotient of C by $\langle\zeta\rangle$ sets up a p-fold cover of a torus T in $L(p, q)$ by C; the image of $[\alpha]$ on C is the meridian of T and the image of $[\beta]$ is a curve wrapping p times in the meridian direction and q times in the longitudinal direction (where 'meridian' and 'longitudinal' are with respect to looking at C from infinity). If one instead looks at the exterior of the Clifford torus then the (p, q) curve is the quotient of the meridian of this second solid torus. This completes the proof. (Draw some pictures, following [25, §4.3].)

Remark. By looking at the Klein bottle and not the torus, one sees that there is a unique non-orientable 3-manifold with genus one Heegard splitting, the non-orientable 2-sphere bundle over $\mathbb{S}^1$.

Remark. A lens space is exactly a Dehn surgery of $\mathbb{S}^3$ along the trivial knot.

3.3 Theorem. *If k is a 2-bridge knot with invariants (α, β) (in the sense of Proposition 3.1), then the two-fold covering of $\mathbb{S}^3$ branched along k is precisely $L(\alpha, \beta)$.*

Proof. Suppose we have such a 2-bridge knot embedded $\mathbb{S}^3$ such that it intersects some $\mathbb{S}^2$ exactly four times. By the construction given earlier we have a covering $p : \mathbb{T}^2 \to \mathbb{S}^2$. The idea is to extend this covering to a covering $\tilde{p} : M \to \mathbb{S}^3$ of order two branched along k. Denote by B_1 and B_2 the two 3-balls in $\mathbb{S}^3$ bounded by the $\mathbb{S}^2$, so $B_1 \cap k$ is the pair of intervals I_1 and I_2 pushed slightly into the interior and $B_2 \cap k = u \cup v$. For each i the 2-fold covering $\hat{B}_i$ of B_i branched along $B_i \cap k$ can be constructed by cutting B_i along two disjoint discs spanning the arcs and identifying two copies of the result in the obvious way. Each $\hat{B}_i$ is a solid torus and $(1-\tau)\hat{I}_1$ and $(1-\tau)\hat{u}$ represent the medians of $\hat{B}_1$ and $\hat{B}_2$ respectively. By construction we see that $B_1 \cup B_2$ is the Heegard splitting of $L(\alpha, \beta)$, since $(1-\tau)\hat{u} \simeq \beta M + \alpha L$.

As a corollary of the proof of Theorem 3.3 we get the following:

3.4 Corollary. *Given any pair (α, β) of integers satisfying the conditions (TBL) in Proposition 3.1, then there exists a 2-bridge link of μ components with the given invariants; we call it $\mathfrak{b}(\alpha, \beta)$.*

Proof. Consider the standard construction of $\mathbb{T}^2$ as $\mathbb{R}^2/\mathbb{Z}^2$ (c.f.). The curve $(1-\tau)\hat{u}$ produced from a 2-bridge knot with invariants (α, β) (if such a knot exists) is constructed by projecting the line through $(0, 0)$ and (β, α) to $\mathbb{T}^2$. Draw such a line in $\mathbb{R}^2$ and project it to the torus. Continue going backwards in the construction by quotienting by the hyperelliptic involution τ. This will give a spiral curve which loops the appropriate number of times from 0 to one of 1 or 3 depending on the value of α (it is covered twice, but we consider just the topological curve not its parameterisation). Take the projections of the lines through $(1, 0)$ and $(\beta + 1, \alpha)$ to get the other spiral curve. Now draw $[0, 1]$ and $[1, 1]$ in, and define the spiral curve to be an overcrossing at all the vertices of the resulting graph.

Remark. One can even visualise these branched coverings: see the Thurston lecture *Knots to Narnia* [67] and the software `Polycut` [11].

We will neglect the proof of the following for the sake of time; for references see [15, Theorem 12.6].

3.5 Theorem (Schubert, 1956). *Let β be odd, $(\alpha, \beta) = 1$, and let β^{-1} be the inverse of β mod 2α.*

1. *$\mathfrak{b}(\alpha, \beta)$ and $\mathfrak{b}(\alpha', \beta')$ are equivalent as oriented links iff $\alpha = \alpha'$ and $\beta^{\pm 1} = \beta' \pmod{2\alpha}$.*
2. *$\mathfrak{b}(\alpha, \beta)$ and $\mathfrak{b}(\alpha', \beta')$ are equivalent as unoriented links iff $\alpha = \alpha'$ and $\beta^{\pm 1} = \beta' \pmod{\alpha}$.*

Proof. Exercises!

There is an alternative construction of 2-bridge knots via braids. We will concern ourselves only with a geometric consideration of Artin's braid theory here; in the next lecture we will look at the algebra.

3.6 Definition. Fix n distinct points in the complex plane, and choose a permutation $\pi \in S_n$. A **braid on n strands** is an n-tuple $\mathfrak{F} = (f_1, \dots, f_n)$ of piecewise linear functions $I \to I \times \mathbb{C}$ such that (a) for each i, $f_i(0) = P_i$ and $f_i(1) = \pi P_i$; and (b) $f_i(t) = f_j(t)$ for some t if and only if $i = j$.

Figure 3.7: Closing a 4-braid to obtain a 4-plat [45, Fig. 1.8].

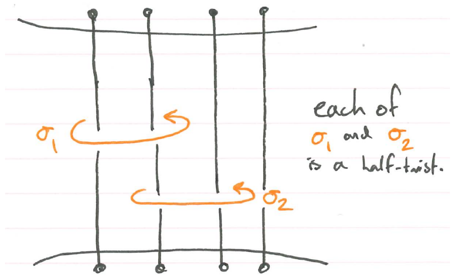


Figure 3.8: The two Artin generators σ_1 and σ_2 of the spherical braid group on four strands.

Braids are defined up to **level-preserving isotopy** (the reader should supply the obvious definition), and the set of n-braids admits (up to this isotopy) a natural group operation, namely the 'concatenation' operation familiar from homotopy theory (so $\mathfrak{G}\mathfrak{F}$ is the n-tuple of functions which consists of 'do f_i from $t = 0$ to $t = 1/2$ and then do $g_{\pi(i)}$ from $t = 1/2$ to $t = 1$). This is called the **braid group** B_n.

We now restrict ourselves to the case $n = 4$.

3.7 Definition. A **4-plat**,[1] or **Viergeflechte**, is obtained by taking a 4-braid and closing it by adding four arcs in the manner of Fig. 3.7.

Warning. The 'plat' manner of closing a braid (which makes sense for any braid on an even number of strands) should be contrasted with the **closure** of a braid (for which see the exercises and which makes sense for any braid at all).

Cut a 4-plat diagram in the orientation of Fig. 3.7 by a vertical line placed as far to the right as possible that cuts the diagram transversely in exactly four places. Lifting this back into 3-space, we have a representation of the 4-plat as the closure of a rational tangle:

3.8 Definition. A **tangle** in a 3-ball $\mathbb{B}^3 \subseteq \mathbb{S}^3$ in the sense of Conway is a collection of disjointly embedded (piecewise-linear) arcs in $\mathbb{B}^3$, with endpoints in $\partial\mathbb{B}^3$. The tangle is **rational** if it consists of exactly two arcs.

3.9 Proposition (Conway, 1970)**.** *There is a bijective correspondence between equivalence classes of rational tangles (i.e. up to isotopy with $\partial\mathbb{B}^3$ fixed) and the set $\hat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$.*

The proof of the proposition as stated will become clear as we continue our discussion; we will instead prove the analogous theorem for knots, Theorem 3.13 below.

Consider braids which are, instead of lying in $\mathbb{C} \times [0, 1]$, (as in Definition 3.6 above), lying in $\mathbb{S}^1 \times [0, 1]$. We will study this more carefully in the next section; all we need to know is that the braid group is generated by the two Artin generators shown in Fig. 3.8. Let $\mathfrak{b}(\alpha, \beta)$ be a 2-bridge knot, and view it as a 4-braid with four additional arcs; that is, we cut $\mathbb{S}^3$ into two 3-balls B_0 and B_1 and a

[1] Beware! the correct English is 'plait', yet the mathematical term for this general kind of object is $(2m$-)'plat'...

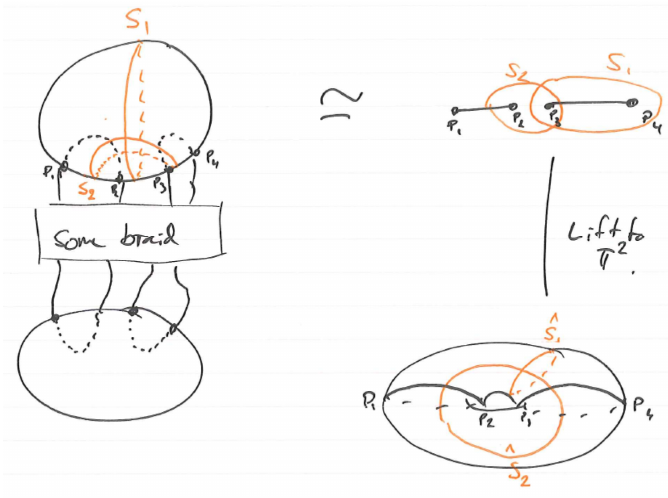


Figure 3.9: The two generators σ_1 and σ_2 of the spherical braid group on four strands induce homeomorphisms on the bridge plane B_0 (left), namely half-twists along the indicated curves. These lift to Dehn twists on the covering space $\mathbb{T}^2$ (right).

complement $[0,1] \times \mathbb{S}^2$ such that each B_0 and B_1 contains a pair of disjoint arcs and such that the braid is contained entirely in $[0,1] \times \mathbb{S}^2$. Consider the lens space $L(\alpha, \beta)$ which is the 2-fold cover of $\mathbb{S}^3$ branched along $\mathfrak{b}(\alpha, \beta)$. Let $\mathbb{T}^2$ be the torus of the Heegard splitting of this lens space which is the lift of the ball B_0.

3.10 Lemma. *With the notation as just described, the two homeomorphisms of B_0 induced by σ_1 and σ_2 respectively lift to Dehn twists about the curves $\hat{s}_1$ and $\hat{s}_2$ of Fig. 3.9.*

Hence, considering the action of the Dehn twists on the canonical basis of $H_1(\mathbb{T}^2)$ given by M and L (notation again as above), we have a natural representation of the braid group into $\mathrm{PSL}(2, \mathbb{C})$ given by

$$\sigma_1 \mapsto L = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \sigma_2 \mapsto R = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

In fact this is a faithful representation (it is exactly the orientation preserving part of the mapping class group of the four-punctured sphere).

The Heegard splitting which gives us $L(\alpha, \beta)$ is induced by some homeomorphism $h \circlearrowright \mathbb{T}^2$. With respect to some choice of bases for the homology groups of the tori, the induced map $h_* : H_1(T_1) \to H_1(T_2)$ (T_1 and T_2 the two tori which are glued to form the whole 3-manifold) is represented by some element of $\mathrm{SL}(2, \mathbb{Z})$,

$$A = \begin{bmatrix} \beta & \alpha' \\ \alpha & \beta' \end{bmatrix}.$$

The matrix entries are defined modulo multiplication on the right by powers of L, since these do not change the isotopy class of the knot. We can therefore replace A with a matrix that factors as a product of L's and R's ending on the right with a nonzero power of R:

$$A = R^{a_1} L^{-a_2} \cdots L^{-a_{m-1}} R^{a_m} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & a_2 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & a_{m-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_m & 1 \end{bmatrix}$$

where $a_m \neq 0$. Considering the entries of these matrices we obtain a Euclidean algorithm, in the sense that we obtain a sequence of equations

$$\begin{aligned} r_0 &= a_1 r_1 + r_2 \\ r_1 &= a_2 r_2 + r_3 \\ \vdots &= \vdots \\ r_{m-1} &= a_m r_m + 0, \qquad |r_m| = 1 \end{aligned} \tag{3.11}$$

where $r_0 = \alpha$ and $r_1 = \beta$ from the intermediate steps

$$R^{-a_i}\begin{bmatrix} r_i & * \\ r_{i-1} & * \end{bmatrix} = \begin{bmatrix} r_i & * \\ r_{i-1} - a_i r_i & * \end{bmatrix} =: \begin{bmatrix} r_i & * \\ r_{i+1} & * \end{bmatrix}$$
$$L^{a_{i+1}}\begin{bmatrix} r_i & * \\ r_{i+1} & * \end{bmatrix} = \begin{bmatrix} r_i - a_{i+1} r_{i+1} & * \\ r_{i+1} & * \end{bmatrix} =: \begin{bmatrix} r_{i+2} & * \\ r_{i+1} & * \end{bmatrix}.$$

i.e. we unwind the word A from left to right by multiplying by appropriate inverses.

Conversely, from any such Euclidean algorithm for β/α (i.e. any sequence of integers $a_1, \dots, a_m$ and such that there exist integers $r_0, \dots, r_m$ with $|r_m| = 1$ and $0 \leq r_i < r_{i-1}$ for all i such that $r_0 = \alpha$ and $r_1 = \beta$ satisfying Eq. (3.11)) we obtain a matrix factorisation

$$\begin{bmatrix} \beta & \alpha' \\ \alpha & \beta' \end{bmatrix} = \begin{cases} R^{a_1} L^{-a_2} \dots R^{a_m} \begin{bmatrix} \pm 1 & * \\ 0 & \pm 1 \end{bmatrix} & m \text{ odd} \\ R^{a_1} L^{-a_2} \dots L^{a_m} \begin{bmatrix} 0 & \pm 1 \\ \pm 1 & * \end{bmatrix} & m \text{ even.} \end{cases}$$

In the first case the induced Heegard splitting is the covering of the knot $\mathfrak{b}(\alpha, \beta)$ since the final matrix is the lift of a power of σ_1 and hence does not change the knot type. On the other hand when m is even we observe that the final factor can be 'fixed' by

$$\begin{bmatrix} 0 & -1 \\ 1 & b \end{bmatrix} = R^{-b}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{3.12}$$

and where the final factor corresponds to having to close the plat in a nontrivial way (exercises).

Observing that we have simply given the decomposition of β/α as a **continued fraction**,

$$\frac{\beta}{\alpha} = [a_1, \dots, a_m] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_m}}}}},$$

and that continued fraction decompositions of *odd* length always exist and are unique, we have the following result:

3.13 Theorem. *The knot $\mathfrak{b}(\alpha, \beta)$ with $0 < \beta < \alpha$ has a presentation as a 4-plat with a defining braid*

$$\sigma_2^{a_1} \sigma_1^{-a_2} \cdots \sigma_2^{a_m}$$

where each $a_i > 0$ and where m is odd, such that the a_i are the quotients of the continued fraction $[a_1, \dots, a_m] = \beta/\alpha$. Sequences $(a_1, \dots, a_m)$ and $(a'_1, \dots, a'_{m'})$ define the same knot iff $m = m'$ and $a_i = a'_i$ or $a_i = a'_{m-i}$ for $1 \leq i \leq m$.

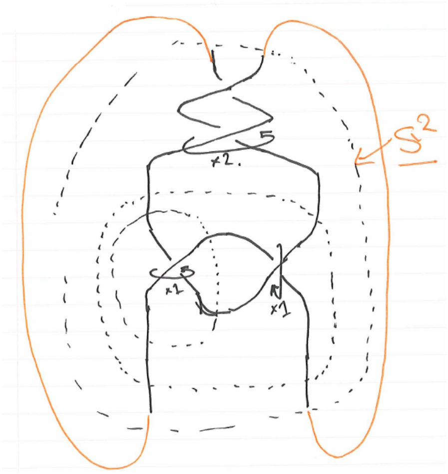


Figure 3.10: The 4-plat presentation of the figure eight knot.

(All that remains is to observe that the very final possibility—$a_i = a'_{m-i}$—comes from the fact that we assumed everything was defined with respect to B_0, while B_0 and B_1 are in fact symmetric.)

3.14 Example. We have seen (Fig. 3.3) that the figure eight knot is $\mathfrak{b}(5,3)$, but the method of torsion diagrams is hugely impractical. (Did anyone actually check that the cited figure is a diagram for the figure eight knot? The author certainly did not.) We can decompose 3/5 as the continued fraction

$$\frac{3}{5} = [1,1,2] = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2}}}.$$

We therefore have a 4-plat presentation as in Fig. 3.10. This is much easier!

We will now construct some representations $\pi_1(\mathfrak{b}(\alpha,\beta)) \to \mathrm{PSL}(2,\mathbb{C})$, following Riley [58]. From the torsion presentations Figs. 3.2 and 3.3 we can read off a presentation of the knot group in the following form:

3.15 Proposition ([58, Proposition 1]). *Fix a 2-bridge link $\mathfrak{b}(\alpha,\beta)$. For any $a \neq \alpha$ write $\bar{a}$ for the reprentative of a mod 2α in the interval $(-\alpha,\alpha)$. For each i set $\varepsilon_i = -\operatorname{sign}(\overline{i\beta})$. Define a word $W_{\beta/\alpha}$ in the symbols X and Y by*

$$\overline{W} = \overline{W}_{\beta/\alpha} = X^{\varepsilon_1} Y^{\varepsilon_2} \cdots (X \text{ or } Y \text{ depending on } \alpha)^{\varepsilon_{\alpha-1}},$$

so $\overline{W}$ is a word of length $\alpha - 1$. Then, if α is odd (so the link is a knot) we have

$$\pi_1(\mathfrak{b}(\alpha,\beta)) \simeq \langle X, Y \,:\, \overline{W}X = Y\overline{W}\rangle;$$

if α is even (so the link has two components) then

$$\pi_1(\mathfrak{b}(\alpha,\beta)) \simeq \langle X, Y \,:\, \overline{W}Y = Y\overline{W}\rangle.$$

Sketch of the proof. Let $k = \mathfrak{b}(\alpha,\beta)$ and choose a torsion diagram for it. Take the two intervals I_1 and I_2 to be always overcrossings, and let X and Y be the corresponding Wirtinger generators. Reduce the Wirtinger presentation to leave only these generators.

3.16 Definition. The relator of length 2α,

$$W_{\beta/\alpha} := \overline{W}_{\beta/\alpha} A \overline{W}_{\beta/\alpha}^{-1} Y^{-1},$$

where $A = X$ if α is odd and $A = Y$ if α is even is called the β/α-**Farey word** [22]. We also extend this definition to the case β is even (see the exercises) by taking the $(\alpha-\beta)/\alpha$ Riley word to be the same but with $y \leftrightarrow Y$.

3.17 Example. The 3/5 Riley word is $XY^{-1}X^{-1}Y$. The 3/5 Farey word is

$$XY^{-1}X^{-1}YXY^{-1}XYX^{-1}Y^{-1}.$$

3.18 Theorem ([58, Theorem 2])**.** *Let $k = \mathfrak{b}(\alpha,\beta)$ and consider the presentation for $\pi_1(k)$ of Proposition 3.15. Let us define a family of functions $\rho = \rho_\mu : \{X, Y, W\} \to \mathrm{PSL}(2,\mathbb{C})$ (indexed on $\mu \in \mathbb{C}$) by*

$$\rho(X) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \rho(Y) = \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix}.$$

and

$$\rho(W) = \rho(X)^{\varepsilon_1}\rho(Y)^{\varepsilon_2}\cdots\rho(X)^{\varepsilon_{\alpha-2}}\rho(Y)^{\varepsilon_{\alpha-1}}.$$

Define a polynomial

$$\Lambda_{\beta/\alpha}(\mu) := \rho(W)_{11}(\mu);$$

*this is the **Riley polynomial**. Then Λ is of the form*

$$\Lambda_{\beta/\alpha}(\mu) = 1 + c_1\mu + c_2\mu^2 + \cdots + c_{\lambda-1}\mu^{\lambda-1} + (-1)^\lambda\mu^\lambda$$

where $\lambda = (\alpha-1)/2$, and a necessary and sufficient condition on $\mu \in \mathbb{C}$ for ρ to extend to a homomorphism $\rho : \pi_1(k) \to \mathrm{PSL}(2,\mathbb{C})$ is that $\rho(W)_{11} = 0$.

We consider our favourite example.

3.19 Proposition (Riley, 1975 [56, 55, 13])**.** *The figure eight knot group admits a faithful representation $\langle X, Y : XyxYXyXYxY$ (where $x = X^{-1}$ and $y = Y^{-1}$) into $\mathrm{PSL}(2,\mathbb{C})$ given by*

$$X \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } Y \mapsto \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}.$$

where $\omega = \exp(2\pi i/3)$. In fact if $\Gamma_{-\omega}$ is the group generated by this pair of matrices, then the quotient $\mathbb{H}^3/\Gamma_{-\omega}$ has finite volume and hence by Mostow rigidity it is the figure eight complement. (Riley's cited paper gives an alternative proof that this is the figure eight knot complement.)

With some effort, Riley produced the diagram of Fig. 3.11, showing all μ such that the group

$$\Gamma_\mu := \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \mu & 1 \end{bmatrix} \right\rangle$$

is a two-bridge link. Also shown on this diagram are various related groups, called **Heckoid groups**, as well as the edge of the so-called **Riley slice** (the top-right region bounded by the diagonal fractal curve) of four-times punctured sphere groups. See [21] for a detailed account aimed at graduate students.

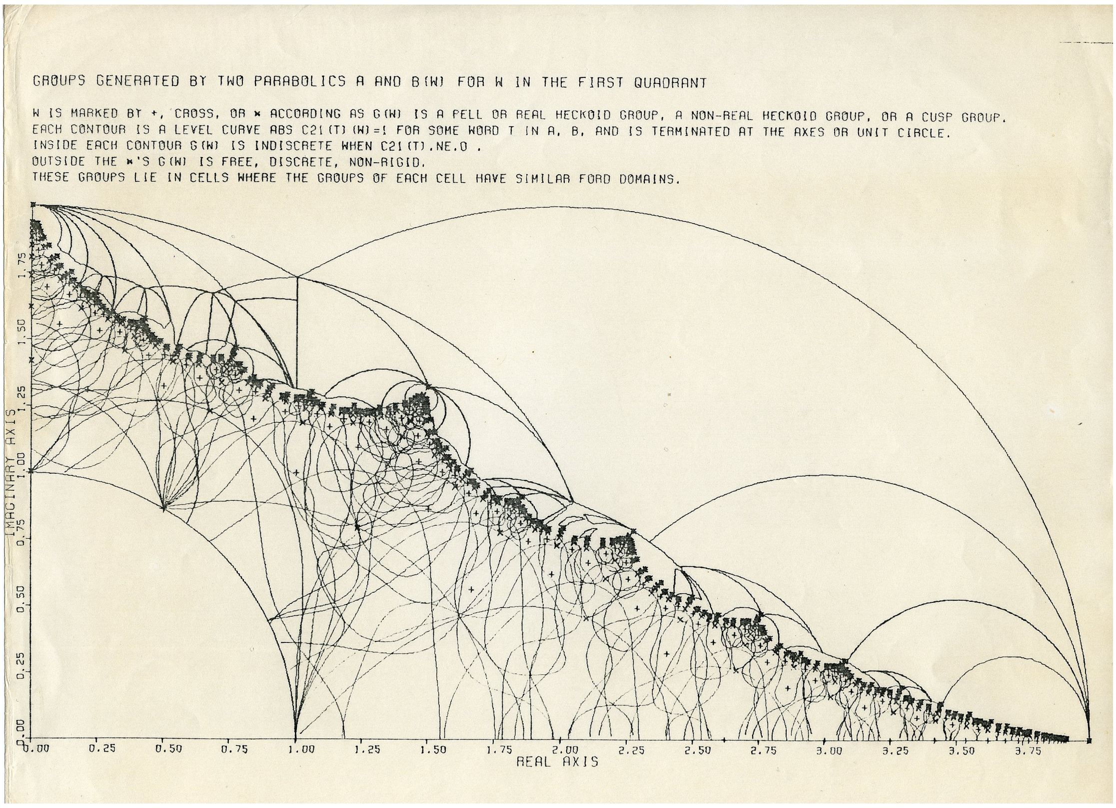


Figure 3.11: Riley's plot of two-bridge link groups in the $(+,+)$-quadrant of $\mathbb{C}$, reproduced from the monograph [4, p. VIII].

3.20 Exercises. 1. What does β determine in a torsion diagram? Hint: start at 0 and walk along the curve u according to its orientation. Where do you end up? Hence β is not the crossing number as in 'number of crossings', but in terms of 'number of *the* crossing'. The exercise is to check that this is actually what the homology number is measuring.

2. Show that the (α, β) torsion diagram with β *even* (so α is odd and the orientation of I_2 is reversed) is a diagram of the $(\alpha - \beta)/\alpha$ knot. Hence if the assumption 'β odd' is deleted then Theorem 3.5 should be modified to read .... and $\beta^{\pm 1} = \pm\beta'$... in both cases. Hint: rotate I_2 by an isotopy of $\mathbb{R}^2$. The corresponding presentation is now on X and y^{-1} so to obtain the $(\alpha-\beta)/\alpha$ Riley word from the β/α one swap Y and y; instead of $VX = YV$ we also have $VX = yV$ so the new Farey word is $VxvY$ up to inverses.

   (The point here is that the β/α and $(\alpha-\beta)/\alpha$ torsion diagrams correspond to different 2-bridge presentations of the same knot. The knot groups are the same, but you are picking a different presentation and a different unknotting tunnel to 'expand' to get a Riley group. There are six unknotting tunnels to a 2-bridge knot [43] and two unknotting tunnels to a hyperbolic 2-bridge link [1].)

3. Classify the 2-bridge links with $\alpha \in \{0, 1\}$. Draw the corresponding 4-plats.

4. Draw the 3/4 torsion diagram and write down the corresponding word.

5. Give the rational number corresponding to (a) the trefoil knot, (b) the knots of Fig. 3.7, (c) the stevedore's knot, Fig. 1.20. Compute their Farey words.

6. Verify that the 5/7 and the 3/7 knots are the same. What are the corresponding Riley words? Conjecture a rule relating the p/q Riley word with the p^{-1}/q Riley word (inverses taken mod $2q$).

7. Prove Proposition 3.15 formally. Hint: in the case of a 2-component link the Wirtinger presentation will give you two relations. These should correspond to the same relator, the Farey word.

8. On lens spaces, Construction 3.2.

   (a) $\pi_1(L(p, q)) = \mathbb{Z}/p\mathbb{Z}$.

   (b) A homeomorphism $h : \partial U \to \partial U$ extends to an autohomeomorphism of U iff $h_*(\beta) = [\beta]^{\pm 1}$. (Here β is one of the loops in the standard basis, same notation as above.)

   (c) $L(1, 0) = \mathbb{S}^3$ and $L(0, 1) = \mathbb{S}^2 \times \mathbb{S}^1$. In fact, $L(1, q) = \mathbb{S}^3$ for all q.

   (d) $L(p, q) = L(p, q')$ if and only if $q \equiv \pm q' \pmod{p}$ or $q \equiv \pm q'^{-1} \pmod{p}$. Hint:- under these conditions there is a homeomorphism $h : L_{p,q} \to L_{p,q'}$ which preserves the two handebodies in the first case and swaps them in the second case.

   (e) Computer project. Draw pictures of lens spaces [18].

9. Use (4) of the previous exercise to prove (2) of Theorem 3.5. Then prove (1) of Theorem 3.5.

10. If an n-braid is chosen with permutation π, as in the definition, then there exists a link with μ components obtained by identifying the P_i with $Q_{\pi(i)}$. Give a formal definition of this link (the **closure** of the braid). Prove (Alexander, 1928) that every link can be obtained as the closure of some braid [15, §2D].

11. Prove Proposition 3.9 from Theorem 3.13.

12. (Bankwitz-Schumann) All 2-bridge knots are alternating.
13. All 2-bridge knots are amphichiral.
14. Show that the group of Proposition 3.19 is isomorphic to Thurston's group from Eq. (2.6).
15. Computer project. Plot $\bigcup_{r\in\mathbb{Q}} \Lambda_r^{-1}(0)$.

3.2 Braids in general and mapping classes

Thanks to Josh Lehman for giving this guest lecture.

Last week we defined the braid group on n strands, B_n (Definition 3.6). We will study this group from the perspective of configuration spaces this week, following selected parts of Chapters 4 and 9 of [24]. Fix a surface S, and let $\mathrm{Conf}(S,n)$ be the **configuration space** of n distinct ordered points in S,

$$\mathrm{Conf}(S,n) \coloneqq S^{\times n} \setminus \Delta(S^{\times n})$$

where $\Delta(S^{\times n})$ is the **big diagonal**, the set of all $(x_1,\dots,x_n)$ such that for some $i \neq j$, $x_i = x_j$. The symmetric group S_n acts on $\mathrm{Conf}(S,n)$ by permuting the labels of the x_i: if $\pi \in S_n$ define $\pi \cdot (x_1,\dots,x_n) \coloneqq (x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$. This action is free, since

$$(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}) = (x_1,\dots,x_n) \iff x_{\pi^{-1}(i)} = x_i \text{ for all } i$$

and this is only the case if $\pi^{-1}(i) = i$ for all i since the big diagonal is gone. Thus the quotient $\mathrm{Conf}(S,n)/S_n$ is a manifold, called the **unordered configuration space** $\mathrm{UConf}(S,n)$.

Almost immediately, we see that $B_n = \pi_1(\mathrm{UConf}(\mathbb{C},n))$: a loop in $\mathrm{UConf}(\mathbb{C},n)$ is exactly a map $\gamma : [0,1] \to \mathrm{UConf}(\mathbb{C},n)$ such that $\gamma(0) = \gamma(1)$ and such that for each t we have n maps $\gamma_1,\dots,\gamma_n : [0,1] \to \mathbb{C}^n \setminus \Delta(\mathbb{C},n)$ where $\gamma_i(0) = x_i$ and $\gamma_i(1)$ is some x_j (the map $i \mapsto j$ defined in this way is the **permutation** of the braid.)

The **Artin generators** of B_n are the elements σ_i ($1 \leq i < n$) consisting of a crossing between the i and $(i+1)$th strands, with the $(i+1)$th strand passing in front. These correspond to the elements of $\pi_1(\mathrm{UConf}(\mathbb{C},n))$ which consist of the ith and $(i+1)$th elements swapping by moving in a clockwise way, while the other elements remain fixed. In fact, these generators fit into a finite presentation of B_n:

$$B_n = \langle \sigma_1, \dots \sigma_{n-1} : \forall_{|i-j|>1} \sigma_i\sigma_j = \sigma_j\sigma_i, \quad \forall_i \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}\rangle. \tag{3.21}$$

You are asked to prove this as an exercise.

Recall next that the **mapping class group** $\mathrm{Mod}(S)$ of a surface S with n marked points $x_1,\dots,x_n$ and b boundary components (deleted discs or punctures) $\beta_1,\dots,\beta_b$ is the group of orientation preserving homeomorphisms $S \to S$ which (i) pointwise fix each of the β_j and (ii) permute the x_i among themselves, modulo isotopies fixing all the x_i and all the β_j pointwise. More compactly, $\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(D_n,\partial D_n))$. We have already used implicitly that $\mathrm{Mod}(\mathbb{T}^2) = \mathrm{SL}(2,\mathbb{Z})$. Another important example is:

3.22 Lemma (Alexander lemma)**.** *The mapping class group of the disc* $\mathbb{B}^2 = \{x \in \mathbb{R}^2 : |x| \leq 1\}$, $\mathrm{Mod}(\mathbb{B}^2)$, *is trivial.*

Proof. Let $\phi : \mathbb{B}^2 \to \mathbb{B}^2$ be a homeomorphism which fixes $\partial\mathbb{B}^2$ pointwise. Define an isotopy from ϕ to the identity by

$$\Phi(x,t) = \begin{cases} (1-t)\phi(x/(1-t)) & 0 \le |x| < 1-t \\ x & 1-t \le |x| \le 1 \end{cases}$$

where $0 \le t < 1$ and set $F(x,1) = x$ for all x. Hence ϕ is trivial in $\mathrm{Mod}(\mathbb{B}^2)$.

The proof of this lemma is called the **Alexander trick**.

Let D_n be a closed 2-disc $\mathbb{B}^2$ with n marked points. We claim that

$$\mathrm{Mod}(D_n) \simeq B_n. \tag{3.23}$$

and we will spend a bit of time to get the machinery to prove this. We will deduce Eq. (3.23) from the generalised Birman exact sequence, Theorem 3.25, but in order to prove this theorem we will need the ordinary Birman exact sequence, Theorem 3.24. So we will do this first.

Let S be any surface, with no marked points. Denote by (S,x) the surface obtained by marking S with some point x in its interior. Then there is a natural map Forget : $\mathrm{Mod}(S,x) \to \mathrm{Mod}(S)$ given by forgetting the data of x, and this map is surjective since every homeomorphism of S may be modified by an isotopy so as to fix x. Alternatively one could look at Forget as being induced by the map $S \setminus \{x\} \to S$, since $\mathrm{Mod}(S,x)$ is naturally isomorphic to the subgroup of $\mathrm{Mod}(S \setminus \{x\})$ of mapping classes stabilising S, and there is a natural map $\mathrm{Mod}(S \setminus \{x\}) \to S$ given by filling in x.

We wish to find ker Forget. An element of the kernel (up to isotopy) is a nontrivial homeomorphism ϕ of S, fixing x, which is trivial up to isotopies that do not fix x: more precisely, there is an isotopy $\Phi : [0,1] \times S \to S$ such that $\Phi(0,s) = \phi(s)$ for all $s \in S$ and $\Phi(1,s) = s$ for all s. Such an isotopy must induce an isotopy of x along a loop α in S, defined by $\alpha(t) = \Phi(t,x)$: clearly $\alpha(0) = x$ and $\alpha(1) = x$, anyway. The idea now is that there is a bijection between elements of the kernel of Forget and such isotopies: i.e. every such isotopy extends to an element of ker Forget. In other words, we will describe a function Push that takes a loop α based at x and produces an element $\phi \in |\mathrm{Forget}\rangle$ that 'pushes' x along the loop. That such a map is well-defined is not very clear from this vague description, and it is the main content of the following theorem.

3.24 Theorem (Birman exact sequence)**.** *Let S be a hyperbolic surface (i.e. $\chi(S) < 0$), possibly with punctures and deleted discs. Let x be a point in the interior of S and denote by (S,x) the surface S with a marking at x. Then there exists a well-defined map* Push : $\pi_1(S,x) \to \mathrm{Mod}(S,x)$ *such that* $\mathrm{Push}(\alpha)$ *restricts to the final state of an isotopy of $x \in S$ around α and which makes the sequence*

$$1 \longrightarrow \pi_1(S,x) \xrightarrow{\mathrm{Push}} \mathrm{Mod}(S,x) \xrightarrow{\mathrm{Forget}} \mathrm{Mod}(S) \longrightarrow 1$$

exact.

Proof. Consider the system of maps

$$\begin{tikzcd} \mathrm{Homeo}^+(S,x) \arrow[r] & \mathrm{Homeo}^+(S) \arrow[d, "\mathcal{E}"] \\ & S \end{tikzcd}$$

where $\mathrm{Homeo}^+(S,x) \to \mathrm{Homeo}^+(S)$ is the natural inclusion and the map $\mathcal{E}$ is evaluation at x. We now show that this is a fibre bundle over S with fibres homeomorphic to $\mathrm{Homeo}^+(S,x)$: i.e. that $\mathrm{Homeo}^+(S)$ is locally homeomorphic to $\mathrm{Homeo}^+(S,x) \times U$ for some open $U \subseteq S$.

Let U be an open neighbourhood of x in S which is homeomorphic to a disc. For $u \in U$ we can choose $\phi_u \in \mathrm{Homeo}^+(U)$ such that $\phi_u(x) = u$ and such that the family $\{\phi_u\}_{u \in U}$ is continuous in u. Define a map $\mathrm{Homeo}^+(S, x) \times U \to \mathcal{E}^{-1}(U)$ by $(\psi, u) \mapsto \phi_u \circ \psi$: evaluation of $\phi_u \circ \psi$ at x gives $\phi_u(\psi(x)) = \phi_u(x) = u \in U$, so the image is as claimed. This map is continuous by construction and has continuous inverse $\mathcal{E}^{-1}(U) \to \mathrm{Homeo}^+(S, x) \times U$ given by $\psi \mapsto (\phi^{-1}_{\psi(x)} \circ \psi, \psi(x))$. This shows that small neighbourhoods $U \ni x$ are locally homeomorphic to $\mathrm{Homeo}^+(S, x) \times U$. Let y be an arbitrary point of S, and let $\Psi : S \to S$ be a homeomorphism with $\Psi(x) = y$. Then $\mathcal{E}^{-1}(U) \to \mathcal{E}^{-1}(\Psi(x))$ defined by $\psi \mapsto \Psi \circ \psi$ is a homeomorphism.

Given that $\mathcal{E}$ is a fibre bundle, we can write down the long exact sequence of homology [12, Theorem VII.6.7],

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_n(\mathrm{Homeo}^+(S)) \arrow[r] & \pi_n(S) \arrow[dll] \\
\pi_{n-1}(\mathrm{Homeo}^+(S,x)) \arrow[r] & \pi_{n-1}(\mathrm{Homeo}^+(S)) \arrow[r] & \pi_{n-1}(S) \arrow[dll, dotted] \\
\pi_1(\mathrm{Homeo}^+(S,x)) \arrow[r] & \pi_1(\mathrm{Homeo}^+(S)) \arrow[r] & \pi_1(S) \arrow[dll] \\
\pi_0(\mathrm{Homeo}^+(S,x)) \arrow[r] & \pi_0(\mathrm{Homeo}^+(S)) \arrow[r] & \pi_0(S) = 1.
\end{tikzcd}$$

By Hanstrom's theorem [24, Theorem 1.14], $\pi_1(\mathrm{Homeo}^+(S)) = 1$ (it is here that hyperbolicity of S is used). Thus the long exact sequence reduces to the short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_0(\mathrm{Homeo}^+(S, x)) \longrightarrow \pi_0(\mathrm{Homeo}^+(S)) \longrightarrow 1$$

and the maps in this short exact sequence are exactly Push and Forget.

One can now use the same argument to prove (exercise):

3.25 Theorem (Generalised Birman exact sequence)**.** *Let S be a surface with possible boundary components and no marked points, and let $\{x_1, \dots, x_n\}$ be n distinct points on the interior of S. Write* $\mathrm{Mod}(S, \{x_i\})$ *for the mapping class group of S equipped with the points x_i as marked points. Suppose also that $\chi(S) < 0$. Then the following sequence is exact:-*

$$1 \longrightarrow \pi_1(\mathrm{UConf}(S, n)) \xrightarrow{\text{Push}} \mathrm{Mod}(S, \{x_i\}) \xrightarrow{\text{Forget}} \mathrm{Mod}(S) \longrightarrow 1.$$

The proof of Eq. (3.23) is now straightforward:

3.26 Corollary. $\mathrm{Mod}(D_n) \simeq \pi_1(\mathrm{UConf}(\mathbb{C}, n)) \simeq \mathbb{B}_n$.

Proof. Take $S = \mathbb{B}^2$ in Theorem 3.25. We get an exact sequence

$$1 \longrightarrow \pi_1(\mathrm{UConf}(\mathbb{B}^2, n)) \xrightarrow{\text{Push}} \mathrm{Mod}(D_n) \xrightarrow{\text{Forget}} \mathrm{Mod}(\mathbb{B}^2) \longrightarrow 1.$$

We have $\mathrm{Mod}(\mathbb{B}^2) = 1$ by the Alexander trick, Lemma 3.22. Note also that $\pi_1(\mathrm{UConf}(\mathbb{B}^2, n)) \simeq \pi_1(\mathrm{UConf}(\mathbb{C}, n))$ is just B_n. Thus we have an exact sequence $1 \to \mathrm{Mod}(D_n) \to B_n \to 1$.

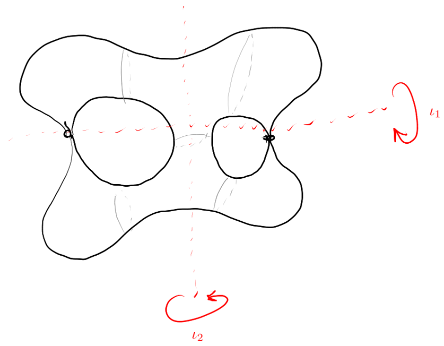


Figure 3.12: Two involutions of the four-times punctured sphere.

In the previous section, we used some properties of spherical braid groups: braids in $\mathbb{S}^2 \times [0,1]$ not $\mathbb{C} \times [0,1]$, or more formally $\pi_1(\mathrm{UConf}(\mathbb{S}^2, n))$. In this case, $\pi_1(\mathrm{Homeo}^+(S, \partial S))$ is non-trivial (the sphere is not hyperbolic!)—in fact, $\mathrm{Homeo}^+(S)$ has the same homotopy type as SO(3) and hence $\pi_1(\mathrm{Homeo}^+(S)) \simeq \pi_1(\mathrm{SO}(3)) \simeq \mathbb{Z}/2\mathbb{Z}$ (this was the belt trick, Trick 2.12). Substituting into the long exact sequence of Theorem 3.25 and using that $\mathrm{Mod}(\mathbb{S}^2) = 1$ gives us the short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathrm{UConf}(\mathbb{S}^2, n)) \to \mathrm{Mod}(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\}) \to 1 \tag{3.27}$$

where $x_1, \dots, x_n$ are distinct punctures on $\mathbb{S}^2$. The nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ is mapped to a 2π rotation of the points in the configuration space. Call this loop α; then the image of α in $\mathrm{Mod}(\mathbb{S}^2 \setminus \{x_1, \dots, x_n\})$ is trivial (it is a Dehn twist around the simple loop surrounding all n punctures). That $\alpha^2 = 1$ in the configuration space is exactly the belt trick.

3.28 Exercises. Many exercises courtesy of Josh Lehman.

1. Write down the relevant fibre bundle and prove Theorem 3.25 using the same argument as Theorem 3.24 [24, Theorem 9.1].
2. Supply a formal proof that $\pi_1(\mathrm{UConf}(\mathbb{B}^2, n)) \simeq \pi_1(\mathrm{UConf}(\mathbb{C}, n))$ (this was used in the proof of Corollary 3.26).
3. Show that the (p, q) torus knot is the closure of the braid $(\sigma_1, \dots, \sigma_{p-1})^q$ by embedding the latter braid on the torus.
4. (The four-times punctured sphere.) Let $S_{0,4}$ be the topological four-times punctured sphere.
   (a) Show that $\mathrm{Mod}(\mathbb{T}^2) = \mathrm{SL}(2, \mathbb{Z})$. [Hint: write $\mathbb{T}^2$ as the quotient of $\mathbb{C}$ by some lattice Λ, and show that $\mathrm{SL}(2, \mathbb{Z})$ is the maximal group which permutes all the different lattices that produce the same complex structure—see [33, §1.2] for details and pictures, and compare with Example 2.14.]
   (b) Observe that $\S_{0,4}$ and $\mathbb{T}^2$ are both produced by quotients of a quadrilateral in $\mathbb{C}$ and conclude that there is an induced surjective homomorphism $\mathrm{Mod}(S_{0,4}) \to \mathrm{SL}(2, \mathbb{Z})$ given by topological lifting; show that the kernel of this homomorphism is generated by two rotations by π of $S_{0,4}$ (Fig. 3.12).
   (c) Conclude that $\mathrm{Mod}(S_{0,4}) = \mathrm{SL}(2, \mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z})^2$.
   (d) Describe the maps in the spherical Birman exact sequence (Eq. (3.27)), $1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(\mathrm{UConf}(\mathbb{S}^2, 4)) \to \mathrm{Mod}(S_{0,4}) \to 1$.

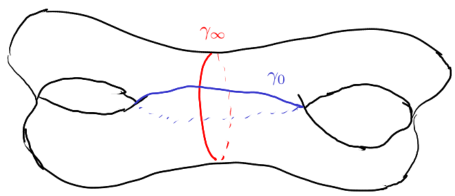


Figure 3.13: Two curves on the four-times punctured sphere.

(e) Recall that $\mathrm{SL}(2,\mathbb{Z})$ is generated by $R = (1,1 \mid 0,1)$ and $Q = (0,-1 \mid 1,0)$. Write $L = (1,0 \mid 1,0)$. Show that $\mathrm{SL}(2,\mathbb{Z}) = \langle R, L\rangle \rtimes \langle Q\rangle$.

(f) Let $\Gamma_1 = \langle L, R\rangle$. Describe the action of Γ_1 as a subset of the mapping class group on the curves γ_0 and γ_∞ of Fig. 3.13

(g) Compare with the discussion of the previous lecture on 2-bridge knots and links.

5. Prove that Eq. (3.21) is a presentation of the braid group.

6. Recall that S_1^1 denotes the torus with a single boundary component. Prove that $\mathrm{Mod}(S_1^1) \simeq B_3$. (Hint: Take the quotient of S_1^1 by the hyperelliptic involution).

7. (The Birman exact sequence, revisited) Here we outline an alternative proof of the Birman exact sequence, using only hyperbolic geometry and Alexander's method, avoiding appealing to the long exact sequence in homotopy and deep results about the contractibility of spaces of homeomorphisms of surfaces.

   Let S be a hyperbolic surface. Fix a point x in the interior of S.

   (a) Prove that $\pi_1(S,x)$ has trivial center (Hint: Use the representation $\pi_1(S,x) \to \mathrm{Isom}^+(\mathbb{H}^2)$ and the classification of isometries of $\mathbb{H}^2$).

   (b) Recall that if G is a group with $Z(G) = 1$, then, we have a short exact sequence,

$$1 \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1.$$

   (c) Show that the canonical homomorphism $\mathrm{Mod}(S,x) \to \mathrm{Aut}(\pi_1(S,x))$ is injective. (Hint: use Alexander's method [24, p. 59] applied to a suitable collection of curves based at the point x).

   (d) Show that there is a natural, well-defined injection $\mathrm{Mod}(S) \to \mathrm{Out}(\pi_1(S,x))$ (use the fact that S has contractible universal cover to see injectivity). What about surjectivity? See the Dehn–Nielsen–Baer Theorem [24, Chapter 8].

   (e) Consider the diagram

$$\begin{tikzcd}
 & \mathrm{Mod}(S,x) \arrow[r, "\text{Forget}"] \arrow[d] & \mathrm{Mod}(S) \arrow[d] & \\
1 \arrow[r] & \pi_1(S,x) \arrow[r] & \mathrm{Aut}(\pi_1(S,x)) \arrow[r] & \mathrm{Out}(\pi_1(S,x)) \arrow[r] & 1
\end{tikzcd}$$

   Show that the image of $\pi_1(S,x)$ is contained in the image of $\mathrm{Mod}(S,x)$ as follows: Let α be a simple loop in S, based at x. Push α to the left a bit, to get α^+, and to the right a bit to get α^-. Show that the composition of Dehn twists $T_{\alpha^+}T_{\alpha^-}^{-1}$ acts as conjugation by α on $\pi_1(S,x)$. Conclude that the image of $\pi_1(S,x)$ lies within the kernel of Forget.

(f) Verify that the map $\pi_1(S, x) \to \mathrm{Mod}(S, x)$ is actually the push map, and complete the statement.

8. This exercise comes from a paper of Farb [23]. It provides a geometric explanation for the existence of an exceptional surjection.

   (a) Let $n > m > 2$, and denote by Σ_n the symmetric group on n letters. Show that there exists an epimorphism $\Sigma_n \to \Sigma_m$ if and only if $(n, m) = (4, 3)$. (Hint: If $n \geq 5$, then A_n is simple).

   (b) Find a lift of the homomorphism $\Sigma_4 \to \Sigma_3$ obtained above to $B_4 \to B_3$, where B_n denotes the braid group on n strands.

   (c) Recall that $B_n = \pi_1(\mathrm{Poly}_n(\mathbb{C})) = \mathrm{Mod}(D_n)$, where $\mathrm{Poly}_n(\mathbb{C})$ denotes the space of monic, degree n, square free polynomials over $\mathbb{C}$. There is a map $\mathrm{Fer} : \mathrm{Poly}_4(\mathbb{C}) \to \mathrm{Poly}_3(\mathbb{C})$, called the resolving quartic map, induced via the following: send a configuration of 4 distinct points (q_1, q_2, q_3, q_4) to 3 *distinct* points (z_1, z_2, z_3) where,

$$z_1 = (q_1 - q_2 - q_3 + q_4)^2/4$$
$$z_2 = (q_1 - q_2 + q_3 - q_4)^2/4$$
$$z_3 = (q_1 + q_2 - q_3 - q_4)^2/4$$

   Investigate the induced map $\mathrm{Fer}_* : B_4 \to B_3$.

9. (Capping and realizing B_3 as homeomorphisms) This exercise involves the braid group on 3 strands, and in particular, some consequences of viewing it as the mapping class group $\mathrm{Mod}(D_3)$. Is there a section to the projection $\mathrm{Homeo}^+(D_3, \partial D) \to B_3$? That is, can you realise the braid group on 3 strands as a group of homeomorphisms? The answer is *yes*, proven by Thurston (`https://mathoverflow.net/q/55555`).

   The following discussion might be helpful regarding the above. We can cap the boundary of D_3 with a disk. If this disk is marked, then one has the following **capping exact sequence** [24, p. 82]:

$$1 \to \mathbb{Z} \to B_3 \to \mathrm{Mod}(S_{0,4}) \to 1$$

   The homomorphism $\mathrm{Mod}(D_3) \to \mathrm{Mod}(S_{0,4})$ is simply given by extending as the identity, and the kernel is a Dehn twist about the boundary of D_3 (which, as you should check, generates the center of B_3). We show in another exercise that $\mathrm{Mod}(S_{0,4}) \simeq \mathrm{PSL}(2, \mathbb{Z}) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

   Switching gears a bit, suppose we cap the boundary component of D_3 with just a disk (no marked point). It can be shown (see for example [24, p. 104]) that one has an exact sequence,

$$\cdots \to \pi_1(\mathrm{Diff}^+(S^2)) \to \pi_1(UTS^2) \to B_3 \to \mathrm{Mod}(S_{0,3}) \to 1$$

   where UTS^2 is the unit tangent bundle of S^2. Recall also that $\mathrm{Diff}^+(S^2)$ has the homotopy type of SO(3). Now, the unit tangent bundle of the 2-sphere UTS^2 can be identified with $\mathbb{RP}^3$, so, in particular, $\pi_1(UTS^2) = \mathbb{Z}/2\mathbb{Z}$. One can try and place this exact sequence into the context of the Birman exact sequence for the 3-stranded spherical braid group and obtain a similar picture to the four-punctured sphere case.

Chapter 4

Knot polynomials

In the final week, we will study polynomial invariants. These arise from representation theory: in the case of the classical invariant, the Alexander polynomial, the representations to consider are those of the knot group onto the cyclic group, and the geometric interpretation is in terms of homology groups of infinite cyclic covers of the knot (compare with the work of Riley which we studied earlier in Section 1.2). In the case of the quantum invariants that arose following the work of V.F.R. Jones in the 1980s, it is the restriction of a representation from a tensor category to the category of vector spaces, restricted to the automorphism groups of the unit object.

4.1 The Alexander–Conway polynomial

Thanks to Lavender Marshall for giving this guest lecture.

We have all seen physicists get very excited about minimal surfaces (Fig. 4.1). A minimal surface spanning a knot is called a Seifert surface. More precisely:-

4.1 Definition. A **Seifert surface** for a link $L \subseteq \mathbb{S}^3$ is an embedded orientable surface S in $\mathbb{S}^3$ such that $\partial S = L$. The **genus** of a link is the minimal genus of a Seifert surface for the link.

4.2 Example. See three views of a Seifert surface for the figure eight knot drawn by `Polycut` in Fig. 4.2.

Remark. One cannot compute the genus easily. The algorithm below does not usually give a surface of minimal genus. The knot genus is additive with respect to the operation $\oplus$ (Construction 1.10) and is NP-complete [2]. One can also try to visualise the genus; see the interesting discussion in [73].

4.3 Algorithm. Let K be an oriented knot and let δ be an oriented diagram of K. (It will be clear that one can work with each component of a link 'separately'.) Then the following algorithm produces a Seifert surface for K [40, Proposition 5.8].

1. For every vertex of δ, cut and deform the two intersecting arcs into two disjoint arcs while respecting orientation. The result will be a collection of disjoint topological circles in the plane of δ called the **Seifert circles**.

2. Colour the resulting partition of the plane with two colours such that the two sides of each circle are different colours, and declare one of the colours to mean 'there is part of a surface

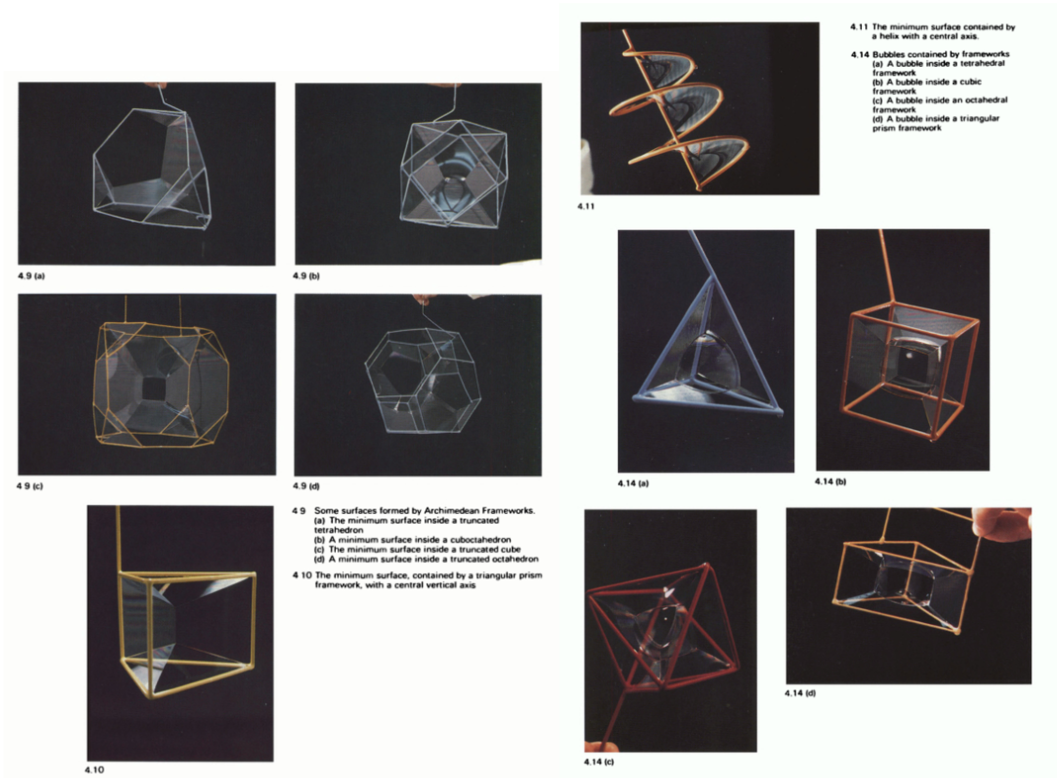


Figure 4.1: Minimal surfaces spanned by soap films [34].

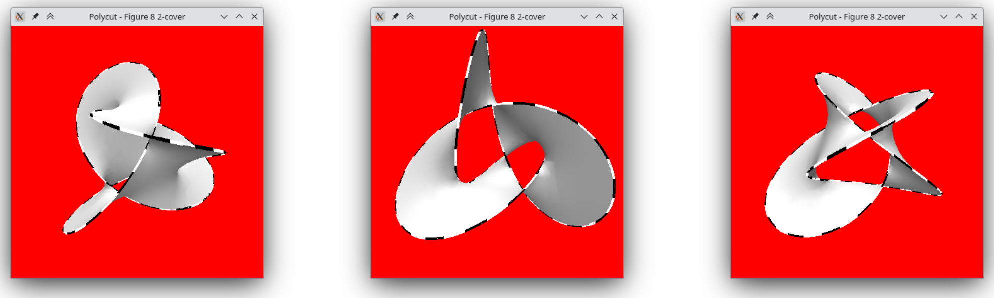


Figure 4.2: Three views of a Seifert surface for the figure eight knot, using the 'soapfilm' feature of `Polycut` [11].

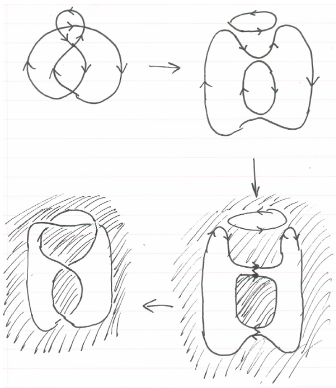


Figure 4.3: A Seifert surface for the figure eight knot following Seifert's algorithm.

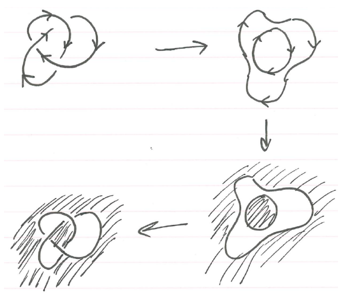


Figure 4.4: A Seifert surface for the trefoil knot following Seifert's algorithm.

here'.[1]

3. At each vertex of the diagram glue in a twisted band whose edges agree with the crossing of the knot at that vertex.

Historical remark. Proof of existence of Seifert surfaces was given originally by Frankl and Pontryagin in 1930 [28] and the above algorithm was given by Seifert in 1934 [62].

4.4 Example. See the figure eight knot (Fig. 4.3) and the trefoil knot (Fig. 4.4).

We begin by following the discussion of Chapter 6 of [45], but an alternative (slower) presentation is given in Chapter VII of [19].

Let M be a module over a (commutative with unity) ring R. An R-module is **free** if there exists a subset B called a **basis** such that every element of the module admits a unique expression as an R-linear combination of elements of B. A **finite presentation** for an R-module M is an exact sequence $F \to E \to M \to 0$ where F and E are free R-modules with finite bases Suppose that the bases for F and E are respectively $(f_1, \dots, f_m)$ and $(e_1, \dots, e_n)$, and let A be the matrix with respect to these bases for the map $F \to E$; we say that A is a **presentation matrix** for M. The images of (e_i) in M generate M, and the images of (f_i) in E give linear relations between these generators; since these relations are encoded in A we may simply speak of the presentation matrix in leiu of carrying F around.

[1] Kauffman simply says 'glue in discs', but it is allowed for the circles to be nested as in Fig. 4.3 below, in which case the things being glued are not actually discs.

Let M be a finitely presented R-module with $m \times n$ presentation matrix A. The rth **elementary ideal** of M, $\mathcal{E}_r$, is the ideal of R generated by all the $(m-r+1) \times (m-r+1)$ ideals of A. One can show that $\mathcal{E}_r$ is independent of the choice of presentation matrix. Since a finite abelian group G is a $\mathbb{Z}$-module, and if it is finitely generated then it has a square presentation matrix, we have $\mathcal{E}_1$ in this case being the determinant of the presentation matrix which is exactly the order of the group (proof: 320).

The special case of the latter which we are interested in is the integer homology group of an oriented compact connected surface S of genus g with n boundary components. Algebraic topology tells us that this homology is

$$H_1(S, \mathbb{Z}) = \bigoplus_{2g+n-1} \mathbb{Z}$$

where the summed cyclic groups are generated by the $[\alpha_i]$ depicted in the figure.

4.5 Proposition. *Suppose that $S \subseteq \mathbb{S}^3$ is a piecewise linear connected, compact, orientable surface with non-empty boundary. Then the homology groups $H_1(\mathbb{S}^3 \setminus S, \mathbb{Z})$ and $H_1(S, \mathbb{Z})$ are isomorphic and there is a unique nonsingular bilinear form*

$$\beta : H_1(\mathbb{S}^3 \setminus S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

such that $\beta([c], [d]) = \mathrm{lk}(c, d)$ for any oriented simple closed curves c and d in $\mathbb{S}^3 \setminus S$ and S respectively.

Restrict now to the case that S is Seifert surface for an oriented link L. Delete a collar neighbourhood of $L = \partial S$ from S—i.e. let X be $\mathbb{S}^3 \setminus N$ for N a regular neighbourhood of L and take $S \cap X$. This new surface (which we will also call S) admits a regular neighbourhood $S \times [-1, 1]$, where the orientation is chosen so that medians to L enter the neighbourhood across $S \times -1$ and leave across $S \times 1$. Let $i^{\pm} : S \to \mathbb{S}^3 \setminus S$ denote the two embeddings defined by $x \mapsto x \times \pm 1$ and if c is an oriented simple closed curve in S wrote $c^{\pm}$ for $i^{\pm}c$ respectively. This identification of curves on S with nearby curves in $\mathbb{S}^3 \setminus S$ induces a bilinear form:

4.6 Definition. Let S be the Seifert surface of an oriented link L; the **Seifert form** of L is the bilinear form

$$\alpha : H_1(S, \mathbb{Z}) \times H_1(S, \mathbb{Z}) \to \mathbb{Z}$$

defined by $\beta(x, y) = \alpha((i^-)_* x, y)$.

Let us now perform some magic. Let Y be obtained by taking X and cutting out $S \times (-1, 1)$. Then Y can be turned back into X by gluing $S \times -1$ to $S \times 1$, but instead we will take infinitely many copies of Y, $(Y_i : i \in \mathbb{Z})$, and form a space X_∞ by identifying $S_i^+ \subset Y_i$ with $S_i^- \subset Y_{i+1}$ (Fig. 4.5).

Remark. The construction of X_∞ is intended to be reminiscent of the construction of the developing map of a manifold. It is known as the **cyclic covering** of the knot or link complement by Rolfsen [59, §5C].

On X_∞ there is a natural automorphism t given by a one-unit shift sending each $Y_i \mapsto Y_{i+1}$. We therefore have an action of $\langle t \rangle$ on $H_1(X_\infty, \mathbb{Z})$ and hence an action of the group algebra $\mathbb{Z}[\langle t \rangle]$ on $H_1(X_\infty, \mathbb{Z})$ given by

$$\left(\sum_{n \in \mathbb{Z}} \lambda_n t^n \right) x = \sum_{n \in \mathbb{Z}} \lambda_n (t^n x)$$

where the outer summation and multiplication by integers λ_n are the group addition and integer multiplication in the abelian group $H_1(X_\infty, \mathbb{Z})$. Also recall that the group R-algebra of an infinite cyclic group is just the R-algebra of Laurent polynomials $R[t, t^{-1}]$ and hence we have constructed an action of the ring of integer Laurent polynomials $\mathbb{Z}[t, t^{-1}]$ on $H_1(X_\infty, \mathbb{Z})$.

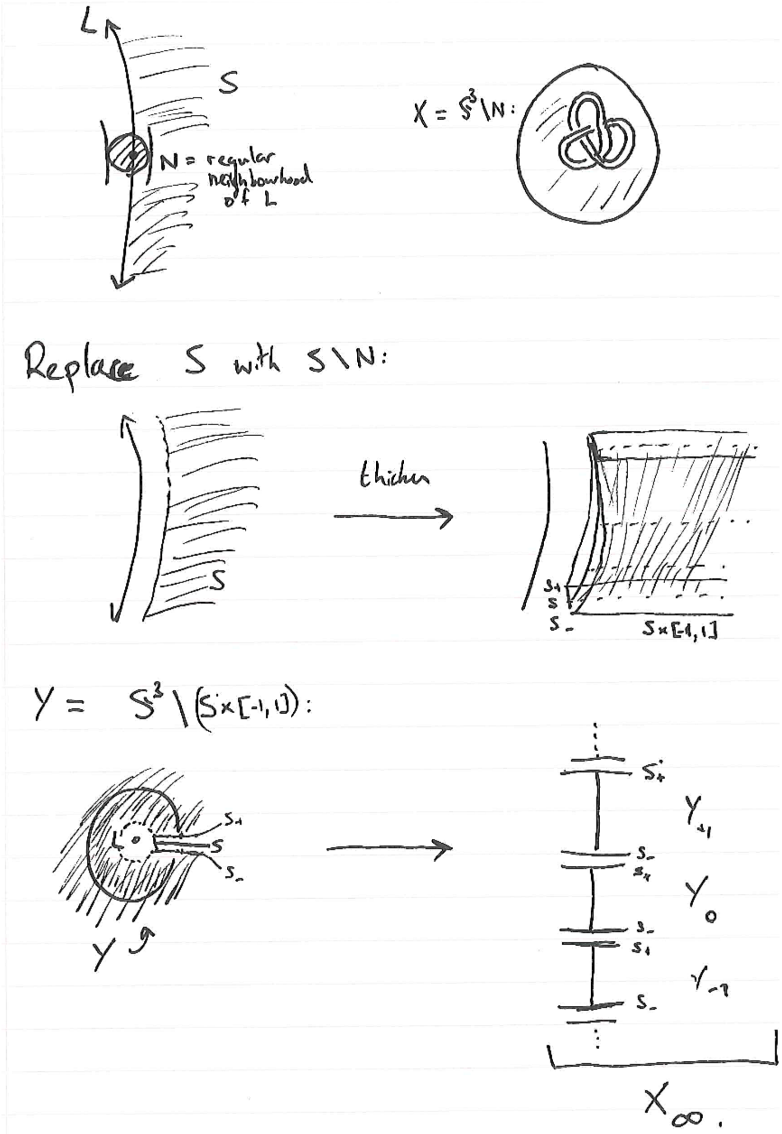


Figure 4.5: The construction of the cyclic covering of the complement of L via the collared Seifert surface S.

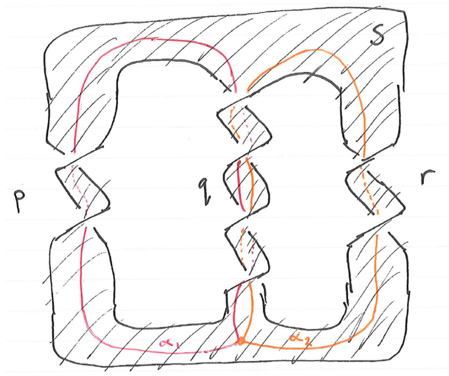


Figure 4.6: The (p, q, r) pretzel knot and a Seifert surface spanning it. If any of the parameters is negative, the twisting direction for the corresponding 2-braid is reversed.

The covering space X_∞ and the action on it by $\langle t \rangle$ are determined up to orientation preserving homeomorphism entirely by the link L and so the $\mathbb{Z}[t, t^{-1}]$-module $H_1(X_\infty, \mathbb{Z})$ is an invariant of L called the **Alexander module**. The rth elementary ideal of the Alexander module of a link L is called the rth **Alexander ideal** of L. Every Alexander ideal is contained in a minimal principal ideal (generated by the gcd of all elements in the ideal), and the generator of this ideal is the rth **Alexander polynomial**. The first Alexander polynomial is called *the* Alexander polynomial $\Delta_L(t)$.

4.7 Lemma. *Let A be a matrix for the Seifert form of L with respect to any basis of $H_1(S, \mathbb{Z})$ (S any Seifert surface). Then $tA - A^\top$ is a presentation matrix for the Alexander module of L.*

By the lemma, we see that that $\mathcal{E}_1$ itself is principal:- the Alexander module has a square presentation matrix, $tA - A^\top$, hence a unique minor of maximal rank and so $\Delta_L(t) = \det(tA - A^\top)$ (up to multiplication by a unit, i.e. a power of $\pm t$, so we normalise such that no power of t divides Δ_L).

4.8 Example. $\Delta_{\text{unknot}}(t) = 1$.

4.9 Example. Let $P(p, q, r)$ ($p, q, r \in \mathbb{Z}$ odd) be the (p, q, r) **pretzel knot** shown in Fig. 4.6 and choose the basis (α_1, α_2) for $H_1(S, \mathbb{Z})$ depicted. Then, since the Seifert prodict $\alpha([c], [d])$ is defined by taking the linking number of c and d with d shifted slightly off S in a consistent way, we can see by inspection that

$$\alpha([\alpha_1], [\alpha_1]) = \frac{1}{2}(p + q)\alpha([\alpha_1], [\alpha_2]) = \frac{1}{2}(q + 1)$$
$$\alpha([\alpha_2], [\alpha_1]) = \frac{1}{2}(q - 1)\alpha([\alpha_2], [\alpha_2]) = \frac{1}{2}(q + r)$$

(where the 1/2 factors come from the definition of lk, Lemma 1.8). If A is the corresponding matrix we have

$$\Delta_{P(p,q,r)}(t) = \det\big(tA - A^\top\big) = \frac{1}{4}\left((pq + qr + rp)\big(t^2 - 2t + 1\big) + t^2 + 2t + 1\right).$$

We see that for $(p, q, r) = (-3, 5, 7)$ then the corresponding knot has polynomial $\Delta(t) = t$, equal up to units to that of the unknot. This pretzel knot is called **Seifert's knot**, and to see that it is nontrivial we can use the Jones polynomial (next lecture). Anyway, the Alexander polynomial is still a fairly good invariant:- it completely classifies all knots with at most eight crossings (see the table on p. 59 of [45]).

Figure 4.7: A Conway triple: three links (L_+, L_-, L_0) which differ only in the three small balls shown. Figure from [45, Fig. 8.1].

Remark. An alternative characterisation of the Alexander polynomial: it is the characteristic polynomial of the linear map $t_* : H_1(X_\infty, \mathbb{Q}) \to H_1(X_\infty, \mathbb{Q})$ where t is the translation map of the cyclic cover.

One can compute the Alexander polynomial inductively using the **skein relations** discovered by Conway [17]—actually, the relations were given by Alexander [5] but Conway was the first (according to Birman [9, §2]) to observe that they allow the reconstruction of the Alexander polynomial without the ambiguity of divisibility by units in $\mathbb{Z}[t^{\pm 1}]$. We say that a triple (L_+, L_-, L_0) of oriented links is a **Conway triple** if they are the same except in the neighbourhood of one point where they differ as shown in Fig. 4.7.

4.10 Theorem. *For oriented links L, there is a polynomial $f_L \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$ characterised by*

(a) (inductive base) $f_{\text{unknot}}(t) = 1$, and

(b) (skein relation) whenever three orientable links L_+, L_-, and L_0 are the same except in , then

$$f_{L_+} - f_{L_-} = (t^{-1/2} - t^{1/2}) f_{L_0}.$$

Further, the polynomial f_L agrees with the Alexander polynomial Δ_L up to units in $\mathbb{Z}[t^{\pm 1/2}]$.

The polynomial f_L (which we will from now on denote Δ_L) is called the **Alexander–Conway polynomial**. A triple which are equal apart from the nei

Proof. Construct a Seifert surface F_0 for L_0 which locally looks like the one shown in Fig. 4.7, let A_0 be a Seifert matrix for F_0, and define $f_{L_0} = \det\big(t^{1/2} A_0 - t^{-1/2} A_0^\top\big)$. This surface extends to Seifert surfaces $F_\pm$ for $L_\pm$ by adding twisted strips as shown. Choose a generating set $\{f_2, \ldots, f_n\}$ for $H_1(F_0, \mathbb{Z})$ and extend it to a generating set for $H_1(F_\pm, \mathbb{Z})$ by adding the additional curves f_1 shown. Let A_0 be the Seifert form matrix for F_0. Then the Seifert matrices for $F_\pm$ are respectively

$$A_+ = \begin{bmatrix} n-1 & * \\ * & A_0 \end{bmatrix} \text{ and } A_- = \begin{bmatrix} n & * \\ * & A_0 \end{bmatrix}$$

for some integer n. It is easily checked that the corresponding determinants f_{L_+} and f_{L_-} satisfy the skein relation.

4.11 Exercises. 1. Draw a Seifert surface for the $(3, 4, 3)$ pretzel knot.

2. Show that the skein relations are invariant under Reidemeister moves.

3. Give the genera for the Seifert surfaces of Figs. 4.3 and 4.4.

4. Let L be the link consisting of two parallel trefoil knot complements. Construct a surface spanned by L by taking a narrow rectangular strip of paper and tying it up in a trefoil knot with the two short ends suitably identified. Show that this surface is non-orientable. Draw a Seifert surface for L.

5. Show that for any oriented link L, $\Delta_L(t) = \Delta_L(t^{-1})$; and for any oriented knot k, $\Delta_k(1) = \pm 1$.

6. In this long exercise, we will compute the Alexander–Conway polynomials (or 'A–C polynomials' for short) for all 2-bridge knots, following [15, §12C].

   Define the **Fibonacci polynomials** $\mathsf{fib}_n(z)$ by

$$\mathsf{fib}_0(z) = 0, \qquad \mathsf{fib}_1(z) = 1,$$
$$\mathsf{fib}_{n+1}(z) = z\,\mathsf{fib}_n(z) + \mathsf{fib}_{n-1}(z),$$
$$\mathsf{fib}_{-n}(z) = (-1)^{n+1}\,\mathsf{fib}_n(z) \text{ for } n \geq 0.$$

   (a) Show that the Fibonacci polynomials for $n \geq 0$ are of the form

$$f_{2n-1} = 1 + a_1 z^2 + a_2 z^4 + \cdots + a_{n-1} z^{2n-2}$$
$$f_{2n} = z(b_0 + b_1 z^2 + b_2 z^4 + \cdots + b_{n-1} z^{2n-2})$$

   for some $a_i, b_i \in \mathbb{Z}$, with $a_{n-1} = b_{n-1} = 1$.

   Let $k = \mathfrak{b}(\alpha, \beta)$ be a two-bridge *knot*—so $\alpha \equiv \beta \equiv 1 \pmod 2$. Represent this knot by the braid

$$\sigma_2^{a_1}\sigma_1^{-2b_1} \cdots \sigma_2^{a_k}$$

   where $k = (m+1)/2$. (There is always a unique generalised Euclidean algorithm of this form, [15, Proposition 12.7].)

   (b) Using the skein relations, show that the A–C polynomial of the 4-plat defined by σ_2^a for $a > 0$ is $\Delta_a(z) = (-1)^{a+1}\,\mathsf{fib}_a(z)$.

   (c) Show that $\Delta_{-a} = (-1)^{a+1}\Delta_a$.

   (d) Assume that $a > 0$, $b > 0$, and $c \neq -1$. Show that the A–C polynomial of the 4-plat defined by $\sigma_2^a\sigma_1^{-2b}\sigma_2^c$ is

$$\Delta_{abc}(z) = \Delta_{a-1}(z)\Delta_c(z) + \Delta_a(z)\Delta_{c+1}(z) - bz\Delta_a(z)\Delta_c(z).$$

   Hint: use the skein relations on the double points of σ_2^a from top to bottom.

   (e) Use (1) to show that if $c > 0$,

$$\deg \Delta_{abc} = a + c - 1 \text{ and } |\mathrm{LC}(\Delta_{abc})| = |b+1|$$

   and if $c < 0$ then

$$\deg \Delta_{abc} = a - c - 1 \text{ and } |\mathrm{LC}(\Delta_{abc})| = |b|.$$

   One can also show that if $a < 0$, $b < 0$, and $c \neq 1$ then

$$\Delta_{abc} = \Delta_{a+1}\Delta_c + \Delta_a\Delta_{c-1} - bc\Delta_a\Delta_c$$

   and so $\deg \Delta_{abc} = |a| + |c| - 1$, $C(\Delta_{abc}) = |\beta| + 1 - \eta$ where $\eta = 1$ or 0 according to whether $c > 0$ or $c < 0$. But the proof is boring so just take it for granted. We continue.

(f) Suppose that $\beta = \sigma_2^{a_1}\sigma_1^{-2b_1}\beta'$ and $\beta' = \sigma_2^{a_2}\sigma_3^{-2b_2}\cdots$ where $a_1 > 0$ and $a_2 > 0$. Show that the A–C polynomial of β is

$$\Delta_\beta = \Delta_{a_1}\Delta_{\sigma_2\beta'} + \Delta_{a_1-1}\Delta_{\beta'} - b_1 z\Delta_{a_1}\Delta_{\beta'}$$

and $\deg\Delta_\beta = \deg\Delta_{a_1}\Delta_{\sigma_1\beta'}$.

(g) Conclude by induction that

$$\deg\Delta_k = |a_1| - 1 + \sum_{j>1}|a_j| \text{ and } |\mathrm{LC}(\Delta_k)| = \prod_{j=1}^{k-1}(|b_j| + 1 - \eta_j).$$

(h) It is a classical theorem that the genus of an alternating knot is $(d+1)/2$ where d is the degree of its A–C polynomial. Compute the genus of every 2-bridge knot. Conclude also that there are infinitely many knots of positive genus.

4.2 Quantum invariants

We will give a very fast introduction to the theory of quantum knot invariants, following various sections of [38]. The philosophy is that these invariants arise from the representation theory of quantum groups, and so we will first introduce these objects. It is remarkable that, despite including the Alexander polynomials as a special case, entirely new techniques (beyond those of classical topology or Thurston's geometric ideas) are required to understand them.

4.12 Definition. Let V be a vector space over a field k. A linear automorphism $c \in \mathrm{Aut}(V \otimes V)$ is called an R-**matrix** if it is a solution of the **Yang-Baxter equation**,

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c).$$

Remark. The Yang-Baxter equation arises naturally when studying factorisations of S-matrices; these are matrices which represent quantum scattering operators.

Computing solutions to the Yang-Baxter equation is incredibly hard.

4.13 Example. If $\tau \in \mathrm{Aut}(V \otimes V)$ is the map $\tau(u \otimes v) := v \otimes u$, then τ is an R-matrix since $(1,2)(2,3)(1,2) = (2,3)(1,2)(2,3)$ in S_3.

4.14 Example. Let V be a finite dimensional vector space over k with basis $\{e_1, \dots, e_N\}$. For $p, q \in k^*$ and $\{r_{i,j}\}_{1\le i,j\le N}$ any $N \times N$ matrix of scalars with $r_{ii} = q$ for all i and $r_{ij}r_{ji} = q$ when $i \neq j$, define an automorphism $c \in \mathrm{Aut}(V \otimes V)$ by

$$c(e_i \otimes e_i) := qe_i \otimes e_i$$
$$c(e_i \otimes e_j) := \begin{cases} r_{ji}e_j \otimes e_i & \text{if } i < j; \\ r_{ji}e_j \otimes e_i + (q - pq^{-1})e_i \otimes e_j & \text{if } i > j. \end{cases}$$

This can be shown by direct computation(!) to be an R-matrix. Further, it satisfies a quadratic minimal polynomial

$$c^2 - (q - pq^{-1})c - p\,\mathrm{id}_{V\otimes V} = 0.$$

As a special case, take $p = q^2$ and $r_{ij} = q$ for all i, j. Then $c(T) = qT$ for all $T \in V \otimes V$.

It will turn out that solutions to the Yang-Baxter equation may be manufactured from objects called 'braided bialgebras'—the first hint to this is that the Yang-Baxter equation 'looks' like the Artin braid relations, Eq. (3.21). We therefore take some time to define bialgebras, and what it means for them to be braided. The most useful example to keep in mind is the classical duality from algebraic geometry:

4.15 Example. Throughout much of the following, keep in mind that we can view affine n-space $\mathbb{A}^n_k$ as being the space of algebraic maps $k[x_1, \dots, x_n] \to k$; sticking to the affine line $n = 1$, we can define maps $\Delta : k[x] \to k[x, y] = k[x] \otimes k[y]$ and $\varepsilon : k[x] \to k$ by

$$\Delta(x) = x + y \text{ and } \varepsilon(x) = 0;$$

identifying $k[x, y]$ with $\mathbb{A}^2_k$, Δ is exactly encoding addition and ε encodes existence of additive identity in $\mathbb{A}^1$:- given $P : k[x] \to k$ and $Q : k[x] \to k$, the addition $P + Q : k[x] \to k$ is the map defined by the diagram

$$\begin{tikzcd}
k[x] \arrow[r, "\Delta"] \arrow[d, "P"] & k[x] \otimes k[y] \arrow[d, "P"] & k[y] \arrow[l, "\Delta"'] \arrow[d, "Q"] \\
k & k \otimes k[y] \arrow[d, "Q"] & k \\
 & k \otimes k = k. &
\end{tikzcd}$$

Thus $k[x]$ has two algebraic structures:- the usual algebra structure with the usual multiplication $k[x] \times k[x] \to k[x]$, and some kind of comultiplication $k[x] \to k[x] \otimes k[x]$ which is encoding the abelian group structure on $\mathbb{A}^2$.

4.16 Definition. An **algebra** is a triple (A, μ, η) where A is a vector space and $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps which make the following diagrams commute:

(Ass.)
$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

(Un.)
$$\begin{tikzcd}
k \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\simeq"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\simeq"] \\
 & A &
\end{tikzcd}$$

4.17 Definition. A **coalgebra** is a triple (C, Δ, ε) where C is a vector space and $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$ are linear maps making the following diagrams commute:

(Coass.)
$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id} \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes \mathrm{id}"] & C
\end{tikzcd}$$

(Coun.)
$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes \mathrm{id}"'] \arrow[r, "\varepsilon \otimes \mathrm{id}"] & C \otimes k \\
 & C. \arrow[ul, "\simeq"] \arrow[u, "\Delta"] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

4.18 Example. Let X be a set, let $C = k\{X\} = \oplus_{x \in X} kx$ be the vector space with basis X. There is a coalgebra structure on $k\{X\}$ given by $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$. There is also a natural algebra structure on $C^\vee$: it is the algebra of functions $X \to k$, with unit the constant function $\varepsilon : X \to k$.

4.19 Example. Let $A = M_n(k)$ be the algebra of $n \times n$ matrices over k. Let $\{E_{ij}\}$ be the usual basis, and let $\{x_{ij}\}$ be the dual basis. Then we have a coalgebra structure on $A^\vee$ given by

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj} \text{ and } \varepsilon(x_{ij}) = \delta_{ij}.$$

The tensor product of two coalgebras (C, Δ, ε) and $(C', \Delta', \varepsilon')$ has a natural coalgebra structure: the comultiplication is $C \otimes C' \to C \otimes C' \otimes C \otimes C$ given by $(\mathrm{id} \otimes \tau_{C,C'} \otimes \mathrm{id}) \circ (\Delta \otimes \Delta')$ and the counit is $\varepsilon \otimes \varepsilon'$.

4.20 Example. $k\{X\} \otimes k\{Y\} \simeq k\{X \times Y\}$ as a coalgebra: the isomorphism is $(u \otimes v) \mapsto (u, v)$.

4.21 Lemma. *Let H be a vector space equipped simultaneously with an algebra structure (H, μ, η) and a coalgebra structure (H, Δ, ε). Give $H \otimes H$ the induced structures of a tensor product of algebras and of coalgebras.*

The following two statements are equivalent:

1. *The maps μ and η are morphisms of coalgebras.*

2. *The maps Δ and ε are morphisms of algebras.*

Proof. Diagram pushing.

Such a vector space satisfying the conditions of Lemma 4.21 is a **bialgebra**.

4.22 Example. Le X be a set with a unital monoid structure, i.e. there is an associative map $\mu : X \times X \to X$ with left and right unit e. Then μ induces an algebra structure on $k\{X\}$ with unit e. Now $\Delta(xy) = xy \otimes xy = (x \otimes x)(y \otimes y) = \Delta(x)\Delta(y)$ so Δ is a morphism of the algebra structure. Similarly $\varepsilon(xy) = 1 = \varepsilon(x)\varepsilon(y)$ so ε is a morphism. Thus $k\{X\}$ is a bialgebra.

4.23 Definition. Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We say that it is **quasi-cocommutative** if there exists an invertible element R of the algebra $H \otimes H$ such that for all $x \in H$, $\Delta^{\mathrm{op}}(x) = R\Delta(x)R^{-1}$ where $\Delta^{\mathrm{op}} = \tau_{H,H} \circ \Delta$ is the opposite coproduct. An element R verifying this condition is called a **universal R-matrix**.

The quasi-commutative bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$ is called **braided** if the universal R-matrix $R = \sum_i s_i \otimes t_i$ satisfies the two additional conditions

$$(\Delta \otimes \mathrm{id})(R) = \left(\sum_i s_i \otimes 1 \otimes t_i\right)\left(\sum_i 1 \otimes s_i \otimes t_i\right) = \sum_{i,j} s_i \otimes s_j \otimes t_i t_j$$

$$(\mathrm{id} \otimes \Delta)(R) = \left(\sum_i s_i \otimes 1 \otimes t_i\right)\left(\sum_i s_i \otimes t_i \otimes 1\right) = \sum_{i,j} s_i s_j \otimes t_i \otimes t_j.$$

To make this somewhat less opaque, let us introduce some notation.

Notation. Let H be an algebra, and let $R = \sum_i s_i \otimes t_i \in H \otimes H$. For any pair (a, b) of elements of $\{1, 2, 3\}$ write R_{ab} for the element of $H \otimes H \otimes H$ which is of the form $\sum_i u_{1,i} \otimes u_{2,i} \otimes u_{3,i}$ where $u_{a,i} = s_i$, $u_{b,i} = t_i$, and $u_{c,i} = 1$ (c being the index which is neither a nor b).

In this notation, the braiding conditions become

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$$
$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}.$$

4.24 Example (Sweedler's four-dimensional bialgebra)**.** Let H be the algebra generated by x and y with relations $x^2 = 1$, $y^2 = 0$, $xy = -yx$. The underlying vector space has basis $\{1, x, y, xy\}$. Define a coalgebra structuure on H by

$$\Delta(x) = x \otimes x, \;\; \varepsilon(x) = 1;$$
$$\Delta(1) = 1 \otimes y + y \otimes x, \;\; \varepsilon(y) = 0.$$

For any scalar λ, set

$$R_\lambda = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \frac{\lambda}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y).$$

Then any R_λ is a universal R-matrix making H into a braided bialgebra.

As promised, these braided bialgebras are machines that can manufacture many more solutions to the Yang-Baxter equation:

4.25 Theorem. *Fix a braided bialgebra $(H, \mu, \eta, \Delta, \varepsilon, R)$. Let V and W be two H-modules. The universal R-matrix produces a natural map $V \otimes W \to W \otimes V$ defined by*

$$c^R_{V,W}(v \otimes w) := \tau_{V,W}\left(R(v \otimes w)\right)$$

with inverse

$$(c^R_{V,W})^{-1}(w \otimes v) := R^{-1}(v \otimes w).$$

1. *The map $c^R_{V,W}$ is an isomorphism of H-modules.*

2. *For any triple (U, V, W) of H-modules:*

$$(c^R_{V,W} \otimes \mathrm{id}_U)(\mathrm{id}_V \otimes c^R_{U,W})(c^R_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_W \otimes c^R_{U,V})(c^R_{U,W} \otimes \mathrm{id}_V)(\mathrm{id}_U \otimes c^R_{V,W})$$

Hence if $U = V = W$ we conclude that $c^R_{V,V}$ satisfies the Yang-Baxter equation for any H-module V.

For a proof, see [38, §VIII.3]. This shows that the representation theory of these braided bialgebras is something one might want to study in its own right, and this is how Vaughan Jones came upon his polynomial (actually he came via von Neumannn algebras, which are related quantum objects: braided categories give representations of von Neumann algebras).

That aside, our main theorem is the following.

4.26 Theorem. *Let* Link *be the set of all oriented links in $\mathbb{S}^3$. There exists a unique map* Link $\to$ *$\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ denoted by $L \mapsto P_L$, defined up to ambient isotopy, such that*

1. *The value of P on the unknot is* 1*, and*

2. *Whenever (L_+, L_-, L_0) is a Conway triple, we have $xP_{L_+} - x^{-1}P_{L_-} = yP_{L_0}$.*

The polynomial P_L is called (variously) the **two-variable Jones polynomial** and the **HOMFLY polynomial** of L; it was introduced following V.F.R. Jones' discovery of a new one-variable knot invariant [35, 36] by various authors simultaneously [29]. The Alexander polynomial is the special case $\Delta_L(t) = P_L(1, t)$ and the classical Jones polynomial is $V_L(t) = P_L(t^{-1}, t^{1/2} - t^{-1/2})$.

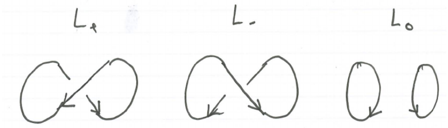


Figure 4.8: The unknot fits into a Conway triple.

4.27 Definition. Let $\Lambda = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ be the Laurent polynomial ring, let $\Lambda[\mathsf{Link}]$ be the free Λ-module generated by the isotopy classes of oriented links, and let Υ be the quotient of $\Lambda[\mathsf{Link}]$ by the Λ-submodule generated by the relators

$$xL_+ - x^{-1}L_- - yL_0 \tag{4.28}$$

where (L_+, L_-, L_0) runs over all Conway triples. This module is the **skein module** (of $\mathbb{S}^3$).

Theorem 4.26 can be deduced from the following proposition:

4.29 Proposition. *Let $Q : \Lambda \to \Upsilon$ be the Λ-linear map sending $1 \mapsto [O]$, where O denotes the unknot and where $[\cdot]$ denotes taking the class of a knot in the quotient module Υ. The map Q is an isomorphism of Λ-modules.*

In other words, the Λ-module generated by links modulo skein conditions is cyclic.

4.30 Corollary. *The map $L \mapsto P_L := Q^{-1}(L)$ clearly satisfies the relations of Theorem 4.26 and is unique by injectivity of Q.*

Thus it 'suffices' to prove Proposition 4.29.

Proof of surjectivity in Proposition 4.29. Step I: We show that Υ is generated by the isotopy classes $[O^{\otimes n}]$ for $n > 0$, where $\otimes$ is used to denote isolated disjoint union. This is done by induction on crossing number: set Υ_m is the submodule of Υ generated by links with crossing number $\leq m$ and observe that Υ is the direct limit of $\cdots \to \Upsilon_m \to \Upsilon_{m+1} \to \cdots$ (all maps inclusions). Clearly Υ_0 is generated by the unknot. For $[L] \in \Upsilon_m \setminus \Upsilon_{m-1}$, draw a diagram for L with m crossings. There is a Conway triple (L_+, L_-, L_0) with $L = L_+$ or $L = L_-$ and such that L_0 has less than m crossings. Using the skein module relation Eq. (4.28) we see $[L_+] = x^{-2}[L_-] \pmod{\Upsilon_{m-1}}$. Thus changing one of the crossings in L changes its class modulo Υ_{m-1} by a unit. But every link is obtained by taking a union of disjointly embedded unknots and swapping crossings; hence L is generated over Λ by classes of the form $[O^{\otimes n}]$ and classes in Υ_{m-1}.

Step II: Now it remains to see that every $[O^{\otimes n}]$ is generated in Υ by the class of the unknot. To see this observe that $([O^{\otimes n}], [O^{\otimes n}], [O^{\otimes (n+1)}])$ is a Conway triple (Fig. 4.8) so

$$[O^{\otimes (n+1)}] = \frac{x - x^{-1}}{y}[O^{\otimes n}]$$

and the result follows by induction.

Remark. In the surjectivity proof, we did not really use any properties of Λ, just topological facts and the skein relations.

For the injectivity part of Proposition 4.29 we will really need some machinery. It will follow from the following proposition (which is somehow defining the correct inverse of Q on a basis for $\Lambda[\mathsf{Link}]$):

4.31 Proposition. *Let $q \in \mathbb{C}^*$ not be a root of unity and let $m > 1$ be an integer. There exists a unique map $\Phi_{m,q} : \mathsf{Link} \to \mathbb{C}$ which is well-defined up to isotopy, such that*

1. *The value of $\Phi_{m,q}$ on the unknot is*
$$\Phi_{m,q}(O) = \frac{q^m - q^{-m}}{q - q^{-1}}$$
(and this is non-zero by the assumption on q), and

2. *whenever (L_+, L_-, L_0) is a Conway triple, we have*
$$q^m \Phi_{m,q}(L_+) - q^{-m}\Phi_{m,q}(L_-) = (q - q^{-1})\Phi_{m,q}(L_0).$$

Proof of injectivity of Proposition 4.29 from Proposition 4.31. Define a ring map $\zeta_{q,m} : \Lambda \to \mathbb{C}$ by $\zeta_{q,m}(x) = q^m$ and $\zeta_{q,m}(y) = q - q^{-1}$; this gives a Λ-module structure on $\mathbb{C}$ and so we extend $\Phi_{m,q}$ to a Λ-linear map $\Phi'_{m,p} : \Lambda[\mathsf{Link}] \to \mathbb{C}$. By the skein-like relation defining $\Phi_{m,p}$ we have for every Conway triple (L_+, L_-, L_0) that

$$\begin{aligned}
\Phi'_{m,p}\left(xL_+ - x^{-1}L_- - yL_0\right) &= \zeta_{m,p}(x)\Phi_{m,p}(L_+) - \zeta_{m,p}(x^{-1})\Phi_{m,p}(L_-) - \zeta_{m,p}\Phi_{m,p}(L_0) \\
&= q^m\Phi_{m,q}(L_+) - q^{-m}\Phi_{m,q}(L_-) - (q - q^{-1})\Phi_{m,q}(L_0) \\
&= 0.
\end{aligned}$$

Thus $\Phi'_{m,p}$ factors through the projection $\Lambda[\mathsf{Link}] \to \Upsilon$, call the factor map $\Phi''_{m,p} : \Upsilon \to \mathbb{C}$.

Using this, we will prove that $Q : \Lambda \to \Upsilon$ is injective. Suppose $f \in \ker Q$, i.e. $f \in \Lambda$ is a Laurent polynomial such that $Q(f) = f[O]$ is zero in Υ. Map this into $\mathbb{C}$ with $\Phi''_{m,p}$: $0 = \Phi''_{m,p}(0) = \Phi''_{m,p}(f(x,y)[O]) = f(q^m, q^1 - q^{-1})\Phi_{m,p}(O)$. Since $\Phi_{m,p}(O)$ is non-zero, $f(q^m, q^1 - q^{-1}) = 0$. Since this is true for infinitely many m (so infinitely many values in the first parameter), f must have a factor $x - (q^1 - q^{-1})$. But this must hold for all q, hence $f = 0$.

The injectivity of Q is thus reduced to the study of the (purported) maps $\Phi_{m,p}$ of Proposition 4.31. This is what we will concern ourselves with for the remainder, and it is here that we get to a connection with quantum groups. The path is roughly the following:

(I) Form a category of tangles (which contains in particular all links, as automorphisms).

(II) Show that a certain class of R-matrices induces representations from this category to the category of vector spaces.

(III) Exhibit a particular R-matrix which induces a representation, sending automorphisms of the tangle category (representing links) to automorphisms which are dilations and hence can be represented by scalars.

Things should be clearer as we go. First we define tangles, which generalise braids (and which are not to be confused with the rational tangles we studied earlier).

Let k, l be nonnegative integers. A **tangle** L of type (k, l) is the (isotopy class of a) union of a finite number of pairwise disjoint simple oriented piecewise linear arcs in $\mathbb{C} \times [0, 1]$ such that the boundary ∂L is a subset of $\mathbb{C} \times \{0, 1\}$ and in fact $L \cap \mathbb{C} \times \{0\} = \{1, 2, \ldots, k\}$ and $L \cap \mathbb{C} \times \{1\} = \{1, 2, \ldots, l\}$; the example in Fig. 4.9 is of type $(3, 5)$. We allow k or l to be zero, in which case there are no boundary points of L on the corresponding copy of $\mathbb{C}$. We also allow polygonal loops which do not intersect $\mathbb{C} \times \{0, 1\}$ at all. *The isotopy classes of tangles without boundary are exactly the isotopy classes of links.*

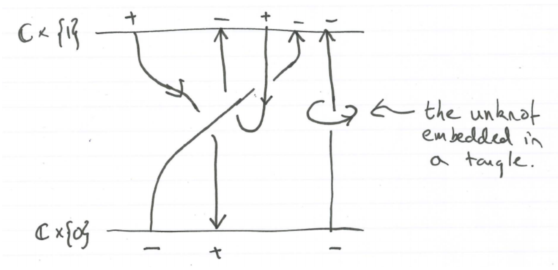


Figure 4.9: A tangle of type $(3, 5)$.

Given a tangle L of type (k, l) we define two finite sequences $s(L)$ and $b(L)$ of symbols from the alphabet $\{+, -\}$. If $k = 0$ (resp. $l = 0$) then set $s(L) = ()$ (resp. $b(L) = ()$). Otherwise we define $s(L) = (\varepsilon_1, \dots, \varepsilon_k)$ and $b(L) = (\eta_1, \dots, \eta_l)$ where $\varepsilon_i = +$ (resp. $\eta_j = -$) if $(i, 0)$ is an endpoint (resp. $(j, 1)$ is a starting point) of L, otherwise $\varepsilon_i = -$ (resp. $\eta_i = -$): the tangle of Fig. 4.9 has $b(L) = (+, -, +, -, -)$ and $s(L) = (-, +, -)$.

Define a category Tan, the **tangle category**, to have object set consisting of finite (possibly empty) sequences of symbols from the alphabet $\{+, -\}$ and morphism set consisting of all isotopy classes of tangles, such that the origin of a morphism L is $s(L)$ and the target of L is $b(L)$.

We also equip Tan with a tensor product structure; if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_k)$ and $\eta = (\eta_1, \dots, \eta_l)$ are two objects then we set $\varepsilon \otimes \eta := (\varepsilon_1, \dots, \varepsilon_k, \eta_1, \dots, \eta_l)$ and if L and M are two tangles we set $L \otimes M$ to be the tangle consisting of 'L and M placed side-by side'. This tensor product has a unit, namely $()$, and *endomorphisms of this unit are exactly the tangles without boundary*.

We wish to study representations (i.e. functors) from Tan to the category $\mathsf{Vec}(k)$ of finite dimensional vector spaces over k.[2] More precisely, a **representation** $F : \mathsf{Tan} \to \mathsf{Vec}(k)$ is a functor which commutes with the tensor products *up to equality, not just isomorphism* and which preserves the tensor unit (sends $() \mapsto k$). Observe that if L is a link, so an automorphism of $()$, then $F(L)$ is an automorphism of k: it is therefore multiplication by a scalar, and so F induces a map $\mathsf{Link} \to k$. This concludes (I).

Given a vector spaces U, V, we define:

- the **evaluation map**, $\mathrm{ev}_V : V^\vee \otimes V \to k$ by $\mathrm{ev}_V(v^i \otimes v_j) = \langle v^i, v_j \rangle = \delta_{i,j}$;
- the **coevaluation map**, $\delta_V : k \to V \otimes V^\vee$ by $\delta_V = \sum_i v_i \otimes v^i$;
- the **partial tanspose** $f^+ : U^\vee \otimes V \to U^\vee \otimes V$ of $f : V \otimes U \to V \otimes U$ is defined on block matrices by:

$$\text{if } [f] = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$
$$\text{then } [f^+] = \begin{bmatrix} a_{11}B & \cdots & a_{n1}B \\ \vdots & \ddots & \vdots \\ a_{1m}B & \cdots & a_{nm}B \end{bmatrix}.$$

- the **partial trace** $\mathrm{tr}_2 : \mathrm{End}(U \otimes V) \to \mathrm{End}(U)$ to be the unique linear map such that for all $\phi \in \mathrm{End}(U)$ and $\psi \in \mathrm{End}(V)$, $\mathrm{tr}_2(\phi \otimes \psi) = \mathrm{tr}(\phi)\psi$.

[2]Actually, we take $\mathsf{Vec}(k)$ to be an equivalent category such that tensor associativity $U \otimes (V \otimes W) \simeq (U \otimes V) \otimes W$ is actually an equality not just isomorphism, but this is a technical point [38, Proposition XI.5.1].

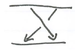
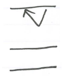
Tan	$\xrightarrow{F}$	$\mathsf{Vec}(k)$
$X_+ =$ 	$\mapsto$	c
$X_- =$ 	$\mapsto$	c^{-1}
$\cup =$ 	$\mapsto$	δ_V
$\cap =$ 	$\mapsto$	ev_V
$\overleftarrow{\cup} =$ 	$\mapsto$	$(\mathrm{id}_{V^\vee} \otimes \mu^{-1})\delta_{V^\vee}$
$\overleftarrow{\cap} =$ 	$\mapsto$	$\mathrm{ev}_{V^\vee}(\mu \otimes \mathrm{id}_{V^\vee})$.

Table 4.1: The six elementary tangles and images under the tensor representation F. This labelling is consistent with Fig. 4.7.

4.32 Definition. Let V be a finite dimensional vector space. An **enhanced R-matrix** on V is a pair (c, μ) where $c \in \mathrm{Aut}(V \otimes V)$ is an R-matrix and $\mu \in \mathrm{Aut}(V)$ satisfies

$$c(\mu \otimes \mu) = (\mu \otimes \mu)c$$
$$\mathrm{tr}_2(c^{\pm 1}(\mathrm{id}_V \otimes \mu)) = \mathrm{id}_V$$
$$(\tau c^{\mp 1})^+(\mathrm{id}_{V^\vee} \otimes \mu)(c^{\pm 1}\tau)^+(\mathrm{id}_{V^\vee} \otimes \mu^{-1}) = \mathrm{id}_{V^\vee \otimes V}\,.$$

The point is that these conditions on an R-matrix allow us to manufature representations of Tan.

4.33 Theorem. *Given an enhanced R-matrix (c, μ) on a finite dimensional vector space V there exists a unique representation $\mathsf{Tan} \to \mathsf{Vec}(k)$ such that on objects, $F((+)) = V$ and $F((-)) = V^\vee$; and on morphisms the six relations of Table 4.1 (depending on the data of the R-matrix) hold.*

Proof. We can actually write down generators and relations for the morphisms in the tangle category [38, Theorem XII.2.2] and the generators are exactly the six elementary tangles of Table 4.1, so uniqueness holds so long as the map respects the relations. It is a lengthy computation to see that this respectfulness condition is encoded by the definition of an enhanced R-matrix [38, Theorem XII.4.2].

This completes part (II) of our plan.

We will now prove Proposition 4.31 and thus injectivity of Q and thus well-definedness of the HOMFLY polynomial.

Proof of Proposition 4.31. Let k be a field, $q \in k^*$, and $m > 1$ an integer. Let $V_m = k\{v_1, \dots, v_m\}$. Define a linear map $c_m \in \mathrm{Aut}(V_m \otimes V_m)$ by

$$c_m(v_i \otimes v_i) := q^{-m} q e_i \otimes e_i$$
$$c_m(v_i \otimes v_j) := q^{-m} \begin{cases} e_j \otimes e_i & \text{if } i < j; \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{if } i > j. \end{cases}$$

This is a (scalar multiple of a) special case of Example 4.14 where all the $r_{ij} = 1$ and $p = 1$. Hence c_m is an R-matrix that additionally satisfies the relation

$$q^m c_m - q^{-m} c_m^{-1} = (q - q^{-1})\, \mathrm{id}_{V_m \otimes V_m}\,. \tag{4.34}$$

Define an automorphism $\mu_m \in V_m$ by $\mu_m(v_i) = q^{m-2i+1}v_i$. This automorphism has trace

$$\operatorname{tr}\mu_m = \frac{q^m - q^{-m}}{q - q^{-1}}$$

(it is easy to calculate this since μ_m is diagonal).

One can now show that the pair (c_m, μ_m) is an enhanced R-matrix for V_m. In particular, by Theorem 4.33 we have a unique representation $F_{m,q} : \mathsf{Tan} \to \mathsf{Vec}(k)$; and one can also show by directly applying $F_{m,q}$ to the quadratic Eq. (4.34) that

$$(*) \qquad q^m F_{m,q}(X_+) - q^{-m}F_{m,q}(X_-) = (q - q^{-1})(F_{m,q})(X_0)$$

where (X_+, X_-, X_0) are the three tangles of Fig. 4.7, and

$$F_{m,q}(O) = \operatorname{tr}(\mu_m) = \frac{q^m - q^{-m}}{q - q^{-1}}$$

(this is a computation using the relations of Eq. (4.28)).

Now, take $k = \mathbb{C}$ and $q \in \mathbb{C}^*$ a non-root-of-unity. Restrict $F_{m,q}$ to oriented links in $\mathbb{C} \times (0, 1)$ (i.e. tangles of type $(0, 0)$); call this map F. Since these links are endomorphisms of () in Tan, $F(L)$ for a tangle L is a linear $\mathbb{C}$-endomorphism of $\mathbb{C}$ and hence is a complex number. We have just seen that $F(O) = \frac{q^m-q^{-m}}{q-q^{-1}}$, and so we only need to prove that F satisfies condition (2) of Proposition 4.31: that if (L_+, L_-, L_0) is a Conway triple, then

$$(\dagger) \qquad q^m F(L_+) - q^{-m}F(L_-) = (q - q^{-1})F(L_0).$$

As an exercise, you can show that there exist tangles L_1, L_2, L_3, L_4 such that

$$\begin{aligned} L_+ &= L_1 \circ (L_2 \otimes X_+ \otimes L_3) \circ L_4 \\ L_- &= L_1 \circ (L_2 \otimes X_- \otimes L_3) \circ L_4 \\ L_0 &= L_1 \circ (L_2 \otimes X_0 \otimes L_3) \circ L_4 \end{aligned}$$

Substitute these into (†), and use the fact that F commutes with $\otimes$, to find that

$$(\dagger) = F(L_1)\,(F(L_2) \otimes S \otimes F(L_3))\,F(L_4)$$

where S is exactly the relation of (*), so is killed, and hence (†) vanishes.

4.35 Exercises. 1. Show that if $c \in \operatorname{Aut}(V \otimes V)$ is an R-matrix then so are λc, c^{-1}, and $\tau \circ c \circ \tau$ where τ is the map of Example 4.13 and λ is a scalar.

2. Show that the map c of Example 4.14 is an R-matrix and verify that it satisfies the given polynomial.
3. (a) Show that the dual vector space of a coalgebra C is an algebra: consider the map $\overline{\lambda} : C^\vee \otimes C^\vee \to (C \otimes C)^\vee$ defined by $(f \otimes g)(u \otimes v) := g(v) \otimes f(u)$ and define $A = C^\vee$, $\mu = \Delta^\vee \circ \overline{\lambda}$ and $\eta = \varepsilon^\vee$.
   (b) Show that in Example 4.18, the algebra structure defined on $k\{X\}^\vee$ is indeed the natural one we gave in (a) above.
   (c) Show that the dual vector space of a finite dimensional algebra is a coalgebra. Hint: in the finite dimensional setting, $\overline{\lambda}$ is an isomorphism. Compare Example 4.19.

4. Define **cocommutativity** of a bialgebra A. Show that the flip $\tau_{V,W} : V \otimes W \to W \otimes V$ is an isomorphism of A-modules when A is cocommutative. Show that addition in $\mathbb{A}^1$ is cocommutative in $k[x]$.

5. (Fun for 334 students.) Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. Define a **convolution** operation on $\mathrm{Hom}(H, H)$ by the composition

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

An endomorphism $S \in \mathrm{Hom}(H, H)$ is a **antipode** for the bialgebra H if $S * \mathrm{id}_H = \mathrm{id}_H * S = \eta \circ \varepsilon$. A **Hopf algebra** is a bialgebra with an antipode. Define the commutative algebras

$$M(2) = k[a, b, c, d]$$
$$GL(2) = M(2)[t]/((ad - bc)t - 1)$$
$$SL(2) = GL(2)/(t - 1) = M(2)/(ad - bc - 1).$$

(a) Show that for any commutative algebra A there are bijections $\mathrm{Hom}(GL(2), A) \simeq \mathrm{GL}_2(A)$ and $\mathrm{Hom}(SL(2), A) \simeq \mathrm{SL}_2(A)$, where GL_2 and SL_2 are the classical matrix algebras over A.

(b) Define $\Delta : M(2) \to M(2) \otimes M(2) \simeq k[a', a'', b', b'', c', c'', d', d'']$ by

$$\Delta(a) = a'a'' + b'c'', \qquad \Delta(b) = a'b'' + b'd''$$
$$\Delta(c) = c'a'' + d'c'', \qquad \Delta(d) = c'b'' + d'd''.$$

Show that for any commutative algebra A, Δ corresponds to usual matrix multiplication in $M_2(A)$.

(c) Show that $\Delta(ad - bc) = (a''d'' - b''c'')(a''d'' - b''c'')$.

(d) Observe that Δ induces maps $GL(2) \to GL(2) \otimes GL(2)$ and $SL(2) \to SL(2) \otimes SL(2)$.

(e) Define suitable morphisms $GL(2) \to k$ and $SL(2) \to k$ corresponding to units, and suitable automorphisms of $GL(2)$ and $SL(2)$ corresponding to inversions. Check that you now have a Hopf algebra structure on $GL(2)$ and $SL(2)$.

6. (Fun for 334 students who also like quantum field theory.) The affine plane is the algebra generated freely by x and y modulo the relation $yx = xy$. The **quantum commutation relation** is the relation $yx = qxy$, where $q \in k^*$. Let I_q be the two-sided ideal of the free algebra $k\langle x, y\rangle$ generated by $yx - qxy$, and let the **quantum plane** be the quotient $k_q[x, y] := k\langle x, y\rangle / I_q$.

(a) Let R be an algebra without zero divisors. If α is an algebra endomorphism of R, then an α-**derivation** of R is a linear map $\delta : R \to R$ such that for all $a, b \in R$, $\delta(ab) = \alpha(a)\delta(b) + \delta(a)\alpha(b)$. Given an injective algebra endomorphism $\alpha : R \to R$ and an α-derivation δ of R there exists a unique algebra structure on the free module of polynomials $R[t]$ such that the natural inclusion $R \to R[t]$ is an algebra morphism and $ta = \alpha(a)t + \delta(a)$. This algebra structure is called the **Ore extension** $R[t, \alpha, \delta]$. (A proof of existence and uniqueness is [38, Theorem I.7.1].) If R is (left) Noetherian, then so is the Ore extension [38, Theorem I.8.3].

Show that if α is the automorphism of $k[x]$ determined by $\alpha(x) = qx$, then $k_q[x, y]$ is isomorphic to the Ore extension $k[x][y, \alpha, 0]$. Conclude that $k_q[x, y]$ is Noetherian with no zero divisors and has basis $\{x^i y^j\}_{i,j \geq 0}$.

(b) Show also that for any pair (i, j) of nonnegative integers, $y^i x^j = q^{ij} x^j y^i$ and for any k-algebra R there is a natural bijection between $\mathrm{Hom}(k_q[x, y], R)$ and $\{(X, Y) \in R \times R : YX = qXY\}$. These pairs are R-**points of the quantum plane**.

(c) Let A be the algebra of smooth complex functions on $\mathbb{C} \setminus \{0\}$. Let $q \in \mathbb{C} \setminus \{0, 1\}$. Consider the elemenets of $R = \mathrm{End}_{\mathrm{lin.}}(A)$ given by

$$\tau_q(f)(x) = f(qx) \text{ and } \delta_q(f)(x) = \frac{f(qx) - f(x)}{qx - x}.$$

Show that (τ_q, δ_q) is an R-point of $k_q(x, y)$ and justify the equation $\lim_{q\to 1} \delta_q = d/dx$.

7. (a) Compute the HOMFLY polynomial of the trefoil knot and the Hopf link.

   (b) Show that if L is a link and L' is its mirror image then $P_{L'}(x, y) = P_L(x^{-1}, y^{-1})$. Conclude that the trefoil knot is not amphichiral.

8. Show that the HOMFLY polynomial is invariant under mutation, hence does not distinguish between the Kinoshita–Terasaka and Conway knots (Construction 1.15).

9. Let (L_+, L_-, L_0) be a Conway triple. Show that there exist tangles L_1, L_2, L_3, L_4 such that

$$\begin{aligned} L_+ &= L_1 \circ (L_2 \otimes X_+ \otimes L_3) \circ L_4 \\ L_- &= L_1 \circ (L_2 \otimes X_- \otimes L_3) \circ L_4 \\ L_0 &= L_1 \circ (L_2 \otimes X_0 \otimes L_3) \circ L_4. \end{aligned}$$

10. On representations of B_n, [38, §X.6.2]. Let V be a vector space, $c \in \mathrm{Aut}(V \otimes V)$, and $n > 1$ an integer. For $1 \le i \le n-1$ define $c_i \in \mathrm{Aut}(V^{\otimes n})$ by

$$c_i = \begin{cases} c \otimes \mathrm{id}_{V^{\otimes n-2}} & \text{if } i = 1 \\ \mathrm{id}_{V^{\otimes n-1}} \otimes c \otimes \mathrm{id}_{V^{\otimes n-i-1}} & \text{if } 1 < i < n-1 \\ \mathrm{id}_{V^{\otimes n-2}} & \text{if } i = n-1. \end{cases}$$

   (a) Show that if $|i - j| > 1$ then $c_i c_j = c_j c_i$.

   (b) Show that $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$ for all i if and only if c is an R-matrix (i.e. a solution of the Yang-Baxter equation).

   (c) Let $c \in \mathrm{Aut}(V \otimes V)$ be an R-matrix. Show that for any $n > 0$ there exists a unique group morphism $\rho_n^c : B_n \to \mathrm{Aut}(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $1 \le i \le n-1$. (In other words, R-matrices manufacture representations from B_n onto $V^{\otimes n}$ for all $n \ge 2$.)

∫∫∫∫

Since the university has only seen fit to provide us with two-dimensional blackboards, this is the best I can do.

Bibliography

[1] C.C. Adams and A.W. Reid. “Unknotting tunnels in two-bridge knot and link complements”. In: *Commentarii Mathematici Helvetici* 71 (1996), pp. 617–627. DOI: 10.1007/BF02566439 (cit. on p. 56).

[2] Ian Agol, Joel Hass, and William Thurston. “3-manifold knot genus is NP-complete”. In: *STOC ’02: Proceedings of the thiry-fourth annual ACM symposium on Theory of computing*. Ed. by John Reif. 2002, pp. 761–766. ISBN: 978-1-58113-495-7. DOI: 10.1145/509907.510016 (cit. on p. 63).

[3] Lars Ahlfors. *Complex analysis*. 2nd ed. Mc-Graw Hill, Inc., 1966 (cit. on p. 35).

[4] Hirotaka Akiyoshi, Makoto Sakuma, Masaaki Wada, and Yasushi Yamashita. *Punctured torus groups and 2-bridge knot groups I*. Lecture Notes in Mathematics 1909. Springer, 2007. ISBN: 978-3-540-71807-9. DOI: 10.1007/978-3-540-71807-9 (cit. on p. 55).

[5] J.W. Alexander. “Topological invariants of knots and links”. In: *Transactions of the American Mathematical Society* 30.2 (1928), pp. 275–306. DOI: 10.1090/s0002-9947-1928-1501429-1 (cit. on p. 69).

[6] M.A. Armstrong. *Basic topology*. Undergraduate Texts in Mathematics. Springer, 1983. ISBN: 978-0-387-90839-7 (cit. on pp. 19–21).

[7] Alan F. Beardon. *The geometry of discrete groups*. Graduate Texts in Mathematics 91. Springer-Verlag, 1983. ISBN: 0-387-90788-2. DOI: 10.1007/978-1-4612-1146-4 (cit. on pp. 27, 33).

[8] Riccardo Benedetti and Carlo Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, 1992. ISBN: 978-0-387-55534-8. DOI: 10.1007/978-3-642-58158-8 (cit. on pp. 27, 38–40).

[9] Joan Birman. “New points of view in knot theory”. In: *Bulletin of the American Mathematical Society (N.S.)* 28.2 (1993), pp. 253–287. DOI: 10.1090/S0273-0979-1993-00389-6 (cit. on p. 69).

[10] Francis Bonahon. “Geometric structures on 3-manifolds”. In: *Handbook of geometric topology*. Ed. by R.J. Daverman and R.B. Sher. North-Holland, 2002, pp. 93–164. ISBN: 0-444-82432-4 (cit. on pp. 27, 36).

[11] Ken Brakke. *Polycut: Connecting Multiple Universes*. Susquehanna University. 1997. URL: https://facstaff.susqu.edu/brakke/polycut/polycut.htm (cit. on pp. 49, 64).

[12] Glen E. Bredon. *Geometry and topology*. Graduate Texts in Mathematics 139. Springer, 1993. ISBN: 3-540-97926-3 (cit. on pp. 16, 33, 59).

[13] Matthew G. Brin, Gareth A. Jones, and David Singerman. “Commentary on Robert Riley’s article “A personal account of the discovery of hyperbolic structures on some knot complements””. In: *Expositiones Mathematicae* 31.2 (2013), pp. 99–103. DOI: 10.1016/j.exmath.2013.01.002 (cit. on pp. 30, 54).

[14] Gerhard Burde and Kunio Murasugi. "Links and Seifert fiber spaces". In: *Duke Mathematical Journal* 37.1 (1970), pp. 89–93. DOI: 10.1215/S0012-7094-70-03713-0 (cit. on p. 36).

[15] Gerhard Burde and Heiner Zieschang. *Knots*. 2nd ed. Studies in Mathematics 5. de Gruyer, 2003 (cit. on pp. 17, 45, 49, 56, 70).

[16] William Burnside. *Theory of groups of finite order*. Project Gutenberg, 2012. URL: https://www.gutenberg.org/ebooks/40395 (cit. on p. 22).

[17] J.H. Conway. "An enumeration of knots and links, and some of their algebraic properties". In: *Computational problems in abstract algebra*. Ed. by John Leech. Pergamon Press, Oxford, 1970, pp. 329–358. ISBN: 978-0-08-012975-4. DOI: 10.1016/B978-0-08-012975-4.50034-5 (cit. on pp. 14, 69).

[18] Rémi Coulon, Elisabetta A. Matsumoto, Henry Segerman, and Steve J. Trettel. "Ray-marching Thurston geometries". In: *Experimental Mathematics* 31.4 (2022), pp. 1197–1277. DOI: 10.1080/10586458.2022.2030262. arXiv: 2010.15801 [math.GT] (cit. on pp. 34, 56).

[19] Richard H. Crowell and Ralph H. Fox. *Introduction to knot theory*. Dover Publications, 2008. ISBN: 978-0-486-46894-5 (cit. on pp. 9, 10, 65).

[20] Marc Culler, Nathan M. Dunfield, Matthias Goerner, and Jeffrey R. Weeks. *SnapPy, a computer program for studying the geometry and topology of* 3*-manifolds*. Available at http://snappy.computop.org (12/07/2023) (cit. on p. 41).

[21] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals*. Springer, 2023. arXiv: 2204.11422 [math.GT]. To appear (cit. on p. 54).

[22] Alex Elzenaar, Gaven J. Martin, and Jeroen Schillewaert. *The combinatorics of Farey words and their traces*. 2022. arXiv: 2204.08076 [math.GT] (cit. on p. 54).

[23] Benson Farb. *Rigidity of moduli spaces and algebro-geometric constructions*. 2022. URL: https://www.math.uchicago.edu/~farb/papers/miracles.pdf (cit. on p. 62).

[24] Benson Farb and Dan Margalit. *A primer on mapping class groups*. Princeton Mathematical Series. Princeton University Press, 2012. ISBN: 978-0-691-14794-9 (cit. on pp. 45, 57, 59–62).

[25] A.T. Fomenko and S.V. Matveev. *Algorithmic and computer methods for three-manifolds*. Mathematics and Its Applications 425. Springer Dordrecht, 1997. ISBN: 978-0-7923-4770-5 (cit. on p. 48).

[26] Ralph H. Fox. "A remarkable simple closed curve". In: *Annals of Mathematics* 50.2 (1949), pp. 264–265. DOI: 10.2307/1969450 (cit. on pp. 10, 24).

[27] George K. Francis. *A topological picturebook*. Springer, 1987. ISBN: 0-387-34542-6 (cit. on pp. 30, 31, 36, 46).

[28] F. Frankl and L. Pontrjagin. "Ein Knotensatz mit Anwendung auf die Dimensionstheorie". In: *Mathematische Annalen* 102 (1930), pp. 785–789. DOI: 10.1007/BF01782377 (cit. on p. 65).

[29] P. Freyd et al. "A new polynomial invariant of knots and links". In: *Bulletin of the American Mathematical Society (N.S.)* 12.2 (1985), pp. 239–246. DOI: 10.1090/S0273-0979-1985-15361-3 (cit. on p. 74).

[30] William Fulton and Joe Harris. *Representation theory. A first course*. Graduate Texts in Mathematics 129. Springer, 1991. ISBN: 978-0-387-97495-8 (cit. on p. 32).

[31] C. McA. Gordon and J. Luecke. "Knots are determined by their complements". In: *Journal of the American Mathematical Society* 2.2 (1989), pp. 371–414. DOI: 10.2307/1990979 (cit. on p. 15).

[32] John Hempel. *3-Manifolds*. Annals of Mathematics Studies 86. Princeton University Press, 1976. ISBN: 0-691-08178-6 (cit. on p. 48).

[33] Yoichi Imayoshi and Masahiko Taniguchi. *An introduction to Teichmüller spaces*. Springer, 1987. ISBN: 978-4-431-68174-8. DOI: 10.1007/978-4-431-68174-8 (cit. on p. 60).

[34] Cyril Isenberg. *The science of soap films and soap bubbles*. Advanced Educational Toys Ltd, 1978. ISBN: 0-905028-02-3 (cit. on p. 64).

[35] V.F.R. Jones. "A polynomial invariant for knots via von Neumann algebras". In: *Bulletin of the American Mathematical Society (N.S.)* 12.1 (1985), pp. 103–111. DOI: 10.1090/S0273-0979-1985-15304-2 (cit. on p. 74).

[36] V.F.R. Jones. "Hecke algebra representations of braid groups and link polynomials". In: *Annals of Mathematics* 126.2 (1987), pp. 335–388. DOI: 10.2307/1971403 (cit. on p. 74).

[37] Michael Kapovich. *Hyperbolic manifolds and discrete groups*. Progress in Mathematics 183. Birkhaüser, 2001. ISBN: 978-0-8176-4913-5. DOI: 10.1007/978-0-8176-4913-5 (cit. on p. 30).

[38] Christian Kassel. *Quantum groups*. Graduate Texts in Mathematics 155. Springer, 1995. ISBN: 978-0-387-94370-1 (cit. on pp. 71, 74, 77, 78, 80, 81).

[39] Christian Kassel and Vladimir Turaev. *Braid groups*. Graduate Texts in Mathematics 247. Springer, 2008. ISBN: 978-0-387-33841-5 (cit. on p. 45).

[40] Louis H. Kauffman. *On knots*. Annals of Mathematics Series 115. Princeton University Press, 1987. ISBN: 0-691-08434-3 (cit. on pp. 9, 10, 14, 63).

[41] Jr. Kenneth A. Perko. "On the classification of knots". In: *Proceedings of the American Mathematical Society* 45.2 (1974), pp. 262–266. DOI: 10.2307/2040074 (cit. on p. 13).

[42] Shin'ichi Kinoshita and Hidetaka Terasaka. "On unions of knots". In: *Osaka Mathematical Journal* 9 (1957), pp. 131–153 (cit. on p. 14).

[43] Tsuyoshi Kobayashi. "Classification of unknotting tunnels for two bridge knots". In: *Proceedings of the Kirbyfest*. Ed. by Joel Hass and Martin Scharlemann. Geometry and Topology Mongraphs 2. Mathematical Sciences Publishers, 1999, pp. 259–290. DOI: 10.2140/gtm.1999.2 (cit. on p. 56).

[44] Tom Lehrer. *Lobachevsky*. Nov. 24, 2008. URL: https://youtu.be/UQHaGhC7C2E (visited on 05/08/2023) (cit. on p. 40).

[45] W.B. Raymond Lickorish. *An introduction to knot theory*. Graduate Texts in Mathematics 175. Springer, 1997. ISBN: 978-1-4612-0691-0 (cit. on pp. 9, 15, 50, 65, 68, 69).

[46] Wilhelm Magnus and Ada Peluso. "On knot groups". In: *Communications on Pure and Applied Mathematics* 20 (1967), pp. 749–770. DOI: 10.1002/cpa.3160200407 (cit. on p. 24).

[47] Albert Marden. *Hyperbolic manifolds. An introduction in 2 and 3 dimensions*. 2nd ed. First edition was published under the title "Outer circles". Cambridge University Press, 2016. ISBN: 978-1-107-11674-0 (cit. on pp. 39, 43).

[48] Bernard Maskit. *Kleinian groups*. Grundlehren der mathematischen Wissenshaften 287. Springer-Verlag, 1987. ISBN: 978-3-642-61590-0. DOI: 10.1007/978-3-642-61590-0 (cit. on pp. 37, 43).

[49] John Milnor. *Introduction to algebraic K-theory*. Annals of Mathematics Studies 72. Princeton University Press, 1971. ISBN: 0-691-08101-8 (cit. on p. 35).

[50] John Milnor. *Singular points of complex hypersurfaces*. Annals of Mathematics Studies 61. Princeton University Press, 1969. ISBN: 0-691-08065-8 (cit. on pp. 15, 25).

[51] Tiago Novello, Vinícius da Silva, and Luiz Velho. “Visualization of Nil, Sol, and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ geometries”. In: *Computers & Graphics* 91 (2020). See also https://www.visgraf.impa.br/ray-vr/?page_id=252, pp. 219–231. DOI: 10.1016/j.cag.2020.07.016 (cit. on p. 34).

[52] Ian R. Porteous. *Topological geometry*. 2nd ed. Cambridge University Press, 1981. ISBN: 978-0-521-29839-1 (cit. on p. 33).

[53] Jessica S. Purcell. *Hyperbolic knot theory*. Graduate Studies in Mathematics 209. American Mathematical Society, 2020. ISBN: 978-1-4704-5499-9 (cit. on pp. 27, 30, 45).

[54] John G. Ratcliffe. *Foundations of hyperbolic manifolds*. Graduate Texts in Mathematics 149. Springer, 1994. ISBN: 978-3-030-31597-9. DOI: 10.1007/978-3-030-31597-9 (cit. on p. 30).

[55] Robert Riley. “A personal account of the discovery of hyperbolic structures on some knot complements”. In: *Expositiones Mathematicae* 31.2 (2013), pp. 104–115. DOI: 10.1016/j.exmath.2013.01.003. arXiv: 1301.4601 (cit. on pp. 30, 54).

[56] Robert Riley. “A quadratic parabolic group”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 77 (1975), pp. 281–288. DOI: 10.1017/s0305004100051094 (cit. on pp. 30, 54).

[57] Robert Riley. “Homeomorphisms of knot groups on finite groups”. In: *Mathematics of Computation* 25.115 (1971), pp. 603–619. DOI: 10.2307/2005224 (cit. on pp. 22–24).

[58] Robert Riley. “Parabolic representations of knot groups, I”. In: *Proceedings of the London Mathematics Society*. 3rd ser. 24 (1972), pp. 217–242. DOI: 10.1112/plms/s3-24.2.217 (cit. on pp. 30, 53, 54).

[59] Dale Rolfsen. *Knots and links*. Corrected edition. AMS Chelsea, 2003. ISBN: 0-8218-3436-3 (cit. on pp. 13, 16, 32, 35, 66).

[60] Joseph J. Rotman. *An introduction to the theory of groups*. 4th ed. Graduate Texts in Mathematics 148. Springer, 1995. ISBN: 978-0-387-94285-8 (cit. on p. 24).

[61] Peter Scott. “The geometries of 3-manifolds”. In: *Bulletin of the London Mathematical Society* 15.5 (1983), pp. 401–487. DOI: 10.1112/blms/15.5.401 (cit. on pp. 27, 37).

[62] H. Seifert. “Über das Geschlecht von Knoten”. In: *Mathematische Annalen* 110 (1935), pp. 571–592. DOI: 10.1007/BF01448044 (cit. on p. 65).

[63] Jean-Paul Serre. *Trees*. Trans. by John Stillwell. Springer, 2002. ISBN: 978-3-540-44237-0 (cit. on pp. 25, 35).

[64] John Stillwell. *Classical topology and combinatorial group theory*. 2nd ed. Graduate Texts in Mathematics 72. Springer-Verlag, 1993. ISBN: 3-540-97970-0 (cit. on p. 9).

[65] Marta Sved. *Journey into geometries*. Spectrum Series. Mathematical Association of America, 1991. ISBN: 0-88385-500-3 (cit. on pp. 7, 27).

[66] William P. Thurston. “Hyperbolic structures on 3-manifolds I: Deformation of acylindrical manifolds”. In: *Annals of Mathematics* 124 (1986), pp. 203–246. DOI: 10.2307/1971277 (cit. on p. 30).

[67] William P. Thurston. *Knots to Narnia*. Recorded lecture at UC Berkeley in March 1992, published to YouTube by Anthony Phillips. 1992. URL: https://www.youtube.com/watch?v=IKSrBt2kFD4 (cit. on p. 49).

[68] William P. Thurston. *The geometry and topology of three-manifolds*. Unpublished notes. 1979. URL: http://library.msri.org/books/gt3m/ (cit. on pp. 27, 30, 37, 38).

[69] William P. Thurston. "Three dimensional manifolds, Kleinian groups, and hyperbolic geometry". In: *Bulletin (new series) of the American Mathematical Society* 6.3 (1982), pp. 357–381. DOI: 10.1090/s0273-0979-1982-15003-0 (cit. on pp. 27, 30, 32).

[70] William P. Thurston. *Three-dimensional geometry and topology*. Ed. by Silvio Levy. Vol. 1. Princeton University Press, 1997. ISBN: 0-691-08304-5 (cit. on pp. 27, 28, 34, 37).

[71] J.C. Turner and P. van de Griend. *History and science of knots*. Knots and Everything 11. World Scientific, 1996. ISBN: 981-02-2469-9 (cit. on p. 9).

[72] Jeff Weeks. "Computation of hyperbolic structures in knot theory". In: *Handbook of knot theory*. Ed. by William Menasco and Morwen Thistlethwaite. Elsevier, 2005, pp. 461–480. ISBN: 978-0-444-51452-3. DOI: 10.1016/B978-044451452-3/50011-3 (cit. on p. 41).

[73] J.J. van Wijk and A.M. Cohen. "Visualization of the genus of knots". In: *VIS 05. IEEE Visualization, 2005*. 2005, pp. 761–766. ISBN: 0-7803-9462-3. DOI: 10.1109/VISUAL.2005.1532843 (cit. on p. 63).

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