Moduli spaces of Kleinian groups II

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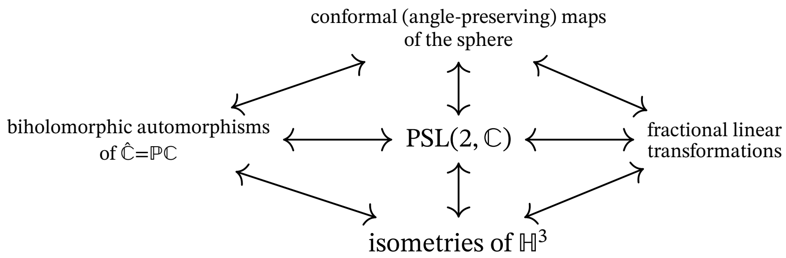
§0. Introduction

In these notes, we continue the study of the deformation spaces of Kleinian groups.

Recall, a **Kleinian group** is a discrete subgroup $\Gamma \le \mathrm{PSL}(2,\mathbb{C})$; we may identify $\mathrm{PSL}(2,\mathbb{C})$ with various groups of geometric interest, as in the following figure.

(0.1)



For the basic theory of these groups, one may look at the books by Beardon [3] and Maskit [16]. The hyperbolic geometry is discussed further in Kapovich [9], Matsuzaki–Taniguchi [19], Ratcliffe [24], and Thurston [26]; the dynamical systems viewpoint is discussed in Beardon [2] and McMullen [21, 22]; and the arithmetic viewpoint is discussed in Maclauchlan–Reid [15].

We shall use $\mathbb{H}^3$ to denote the halfspace model of hyperbolic 3-space, and $\mathbb{B}^3$ to denote the ball model. The sphere at infinity of $\mathbb{H}^3$ is the Riemann sphere $\hat{\mathbb{C}}$; the sphere at infinity of $\mathbb{B}^3$ is the usual sphere $\mathbb{S}^2$. We write $\overline{\mathbb{H}^3} := \mathbb{H}^3 \cup \hat{\mathbb{C}}$ and $\overline{\mathbb{B}^3} := \mathbb{B}^3 \cup \mathbb{S}^2$.

Let Γ be a Kleinian group, and let $\gamma \in \Gamma$ be nontrivial. We say variously that:-

- γ is **parabolic** if any of the following equivalent conditions hold:
  - $\mathrm{tr}^2\,\gamma = 4$;
  - γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to the translation $z \longmapsto z+1$;
  - γ has a unique fixed point on $\hat{\mathbb{C}}$.
- γ is **elliptic** if any of the following equivalent conditions hold:
  - $\mathrm{tr}^2\,\gamma \in [0,4)$;
  - γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a rotation $z \longmapsto \lambda^2 z$ for $|\lambda| = 1$ and $\lambda \ne \pm 1$;
  - γ has a fixed point in $\mathbb{H}^3$;
  - γ has two fixed points in $\hat{\mathbb{C}}$, and if z_0 is such a fixed point then $d(z, z_0) = d(\gamma z, z_0)$ for all $z \in \hat{\mathbb{C}}$;
  - $\mathrm{Fix}_{\overline{\mathbb{H}^3}}\,\gamma$ is the closure of a hyperbolic line.
- γ is **loxodromic** if any of the following equivalent conditions hold:
  - $\mathrm{tr}^2\,\gamma \notin [0,4]$;
  - γ is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a map $z \longmapsto \lambda^2 z$ for $|\lambda| \ne 1$;
  - γ has exactly two fixed points in $\overline{\mathbb{H}^3}$.

If $\gamma \in \Gamma$ is loxodromic with real trace, it is called **hyperbolic**; otherwise it is **strictly loxodromic**. Given an element which is either elliptic or loxodromic, the hyperbolic line joining its two fixed points is called its **axis**.

The **limit set** of a Kleinian group Γ, denoted $\Lambda(\Gamma)$, is the set of accumulation points of its orbits. If $|\Lambda\Gamma| < \infty$, then Γ is called **elementary** (equivalent conditions include $|\Lambda(\Gamma)| \le 2$, or that Γ has a global fixed point in $\overline{\mathbb{H}^3}$). The complement of the limit set, $\Omega(\Gamma)$, is the **ordinary set**.

0.2 Proposition. *Suppose* Γ *is a non-elementary Kleinian group. Then* $\Lambda(\Gamma)$ *is the minimal closed* Γ*-invariant subset of* $\hat{\mathbb{C}}$*; and if* $x \in \Lambda(\Gamma)$*, then* Γx *is dense in* $\Lambda(\Gamma)$. ∎

The main result is the following:

0.3 Theorem. *If* Γ *is a Kleinian group, then* $\Omega(\Gamma)/\Gamma$ *is a marked Riemann surface, with cone points corresponding to elliptic fixed points and punctures corresponding to parabolic fixed points. Further,* $\mathbb{H}^3/\Gamma$ *is a hyperbolic 3-manifold, and the boundary is naturally identified with* $\Omega(\Gamma)/\Gamma$. ∎

In order to compute the fundamental group of these manifolds, we recall the following two facts from algebraic topology; for proofs, see Bredon [6] (corollary III.6.9 and proposition III.7.2 resp.), or Lee [14] (corollary 12.8 and theorem 12.14 resp.):

0.4 Proposition. *If* $p : X \to Y$ *is a regular covering map, with* $x_0 \in X$ *and* $y_0 = p(x_0)$*, then* $\mathrm{Aut}(p) \simeq \pi_1(Y, y_0)/p_*(X, x_0)$. ∎

0.5 Proposition. *If* G *acts freely discontinuously on a path connected and locally path connected Hausdorff space* X*, then* $p : X \to X/G$ *is a regular covering map such that* $\mathrm{Aut}(p) = G$. ∎

As an easy corollary, we see that if X is simply connected, then $\pi_1(X/G) \simeq G$.

§1. Some groups of interest

In this first lecture, we shall discuss a number of examples of groups of interest.

§1.1. A digression: the Poincaré polyhedron theorem

We follow the presentation of Ratcliffe [24, section 13.5] for the statement of the Poincaré polyhedron theorem; alternative presentations are Beardon [3, section 9.8] (for the Fuchsian case) or Maskit [16, section IV.H].

Let $\mathcal{P}$ be a (closed convex) polyhedron in $\mathbb{H}^3$, and suppose that to each facet S of $\mathcal{P}$ we assign an isometry f_S such that $f_S(S) = S'$ for some other facet S'; further, suppose that these assignments are compatible, in the sense that if $f_S(S) = S'$ then $f_{S'} = f_S^{-1}$. Let Φ be the set of these **facet-pairing transformations**. We shall continue to notate the (unique) facet paired with some facet S by S'.

We say that two points $x, y \in \mathcal{P}$ are **paired** by Φ if there exists a facet S of $\mathcal{P}$ such that $x \in S$, $y \in S'$, and $y = f_S(x)$; in this case, we write $x \simeq y$. Observe that $\simeq$ is a symmetric relation. We extend it to an equivalence relation in the following way: if $x, y \in \mathcal{P}$, we say that x and y are **related** by Φ and write $x \sim y$ if either $x = y$ or there is a finite sequence $x_1, ..., x_m$ of points of $\mathcal{P}$ such that

$$x = x_1 \simeq \cdots \simeq x_m = y.$$

An equivalence class of related points is called a **cycle** of Φ; the cycle containing $x \in \mathcal{P}$ is denoted $[x]$. If $x \in \operatorname{int} \mathcal{P}$, then $[x] = \{x\}$. If $x \in \operatorname{relint} S$ for some facet S of $\mathcal{P}$, then $[x] = \{x, f_S(x)\}$. If x lies on an edge or vertex of $\mathcal{P}$, then the cycles become less trivial.

Suppose $x \in \operatorname{relint} e$ for some edge e of $\mathcal{P}$; then every point of $[x]$ lies in the relative interior of some edge of $\mathcal{P}$, and we call $[x]$ an **edge cycle**. Let $[x] = \{x_1, ..., x_m\}$ be a finite edge cycle of Φ. For each i, the element x_i is paired to at most two other elements of $[x]$ by Φ (since each edge bounds exactly two facets of $\mathcal{P}$) and so we can reindex $[x]$ such that $x_1 \simeq x_2 \simeq \cdots \simeq x_m$. Such a cycle is said to be **dihedral** if there is a facet S of $\mathcal{P}$ containing x_1 such that $S = S'$ and $f_S(x_1) = x_1$. An edge cycle which is not dihedral is called **cyclic** (!). In either case, we may define the **dihedral angle sum** of $[x]$ to be

$$\theta[x] \coloneqq \theta(x_1) + \cdots + \theta(x_m)$$

where $\theta(x_i)$ is the dihedral angle of $\mathcal{P}$ along the edge x_i for each i.

In order for Φ to induce a tessellation of $\mathcal{P}$, the dihedral angles must have particularly nice forms:

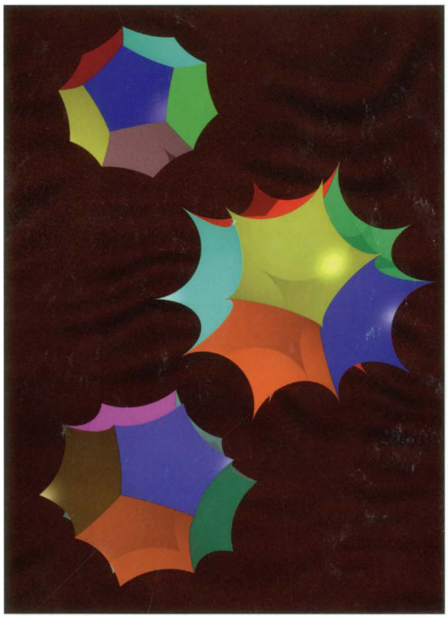


Figure 1: Three hyperbolic dodecahedra centered at $0 \in \mathbb{B}^3$ with respective dihedral angles 60°, 72°, and 90°, as drawn by Matthias Weber [28].

1.1 Definition. A side-pairing transformation Φ for a polyhedron $\mathcal{P}$ is said to be **subproper** if

- each cycle of Φ is finite,
- each dihedral edge cycle of Φ has dihedral angle sum a submultiple of π, and
- each cyclic edge cycle of Φ has dihedral angle sum a submultiple of 2π.

For a proof of the following theorem, see Ratcliffe [24, theorem 13.4.2].

1.2 Theorem. *Let G be a group of similarities of $\mathbb{H}^3$, and let M be the space obtained by gluing a hyperbolic polyhedron $\mathcal{P}$ according to a subproper facet pairing Φ (more precisely, let M be the space $\mathcal{P}/\sim$ of cycles endowed with the quotient topology). Then M is an $(\mathbb{H}^3, G)$-orbifold such that the natural injection* $\operatorname{int}\mathcal{P} \hookrightarrow M$ *is an $(\mathbb{H}^3, G)$-map.* ∎

1.3 Example. Let $\mathcal{P}$ be a regular hyperbolic dodecahedron in $\mathbb{H}^3$ with dihedral angles all $\pi/2$ (see Fig. 1). We define facet-pairing transformations as follows. Pick a set E of six edges of $\mathcal{P}$ with the property that, given any choice of two distinct facets F, G of $\mathcal{P}$, there is a unique edge $e \in E$ such that $F \sim e \sim G$. (That is, E is a perfect matching on the adjacency graph of the facets of $\mathcal{P}$; e.g. the orange edges of Fig. 2.) For each facet F of $\mathcal{P}$, if e is the edge in E incident with F and G is the other facet adjacent to e, let f_F be the rotation by $\pi/2$ around the axis e that rotates F onto G; let $\Phi = \{f_F : F \in \mathcal{P}(2)\}$.

If $x \in \operatorname{relint} e$ for $e \in E$, observe that $\theta[x] = \pi/2$ (blue points of Fig. 2); if $x \in \operatorname{relint} e$ for $e \in \mathcal{P}(1) \setminus E$, observe that $\theta[x] = 4(\pi/2) = 2\pi$ (green points of Fig. 2).

In particular, the pairing structure Φ is subproper; thus we can obtain a quotient orbifold M by gluing along Φ. By gluing one pair at a time, we see that M is topologically a 3-sphere; and that the singular points of M are the images in M of the perfect matching E, and after gluing these edges are the Borromean rings.

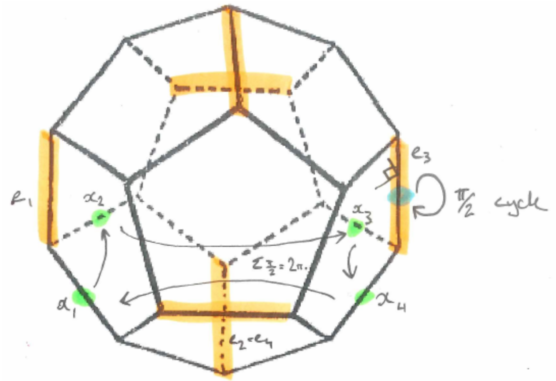


Figure 2: The cycles of an edge pairing on the dodecahedron.

The content of the Poincaré polyhedron theorem is that, if one is given a polyhedron $\mathcal{P}$ together with a facet pairing structure Φ which has sufficient regularity properties, then the group $\Gamma = \langle\Phi\rangle$ has a presentation determined exactly by the combinatorial properties of Φ and $\mathcal{P}$ is a fundamental polyhedron for Γ.

1.4 Theorem (Poincaré (1883))**.** *Let Φ be a subproper facet pairing for a polyhedron $\mathcal{P}$ in $\mathbb{H}^3$, such that the glued orbifold M is complete. Then:*

1. *$\Gamma := \langle\Phi\rangle$ is a Kleinian group with $M = \mathbb{H}^3/\Gamma$;*
2. *$\mathcal{P}$ is a (convex) fundamental polyhedron for Γ;*
3. *$\mathcal{P}$ is exact, that is for each facet $S \in \mathcal{P}(2)$ there exists some $\gamma \in \Gamma$ such that $S = \mathcal{P} \cap \gamma\mathcal{P}$;*
4. *If R is the set of words in the symbols $\mathcal{P}(2)$ corresponding to all of the side-pairing and cycle relations of Φ, then $\langle \mathcal{P}(2) : R\rangle$ is a presentation for Γ under the isomorphism $\mathcal{P}(2) \ni S \longmapsto f_S \in \Phi$.* ∎

In order to apply this theorem, we need a criterion for completeness of hyperbolic manifolds. Let $\mathcal{P} \subseteq \mathbb{B}^3$ be a convex polyhedron; a **cusp point** of $\mathcal{P}$ is a point $c \in \overline{\mathcal{P}} \cap \S^2$ such that there exists a neighbourhood U of c in $\mathbb{R}^3$ such that the intersection of the closures in $\overline{B^3}$ of all the facets of $\mathcal{P}$ which meet U is $\{c\}$. (Compare the proof of proposition VI.A.10 in Maskit [16].)

Suppose c is such a cusp point, and let $b \in [c]$. The **link** of b is the (Euclidean) convex polygon $L(b)$ obtained by intersecting $\mathcal{P}$ with a horosphere Σ_b based at b which meets only the sides of $\mathcal{P}$ incident with b. It is easy to see that we may choose the horospheres Σ_b to be sufficiently small that the $L(b)$ are mutually disjoint (suppose not; then there must be a sequence (b_n) of points of $[c]$ such that $b_n \to c$; in particular, some subsequence of the b_n must lie on an edge incident with c; and the two facets of $\mathcal{P}$ incident with that edge intersect at infinitely many points in any neighbourhood in the sense above of c). We now show that if Φ is a facet-pairing for $\mathcal{P}$, then Φ induces a set Ψ of Euclidean similarities which acts as a side-pairing for the disjoint union of the set of polygons $\{L(b) : b \in [c]\}$ after they have been embedded into $\mathbb{R}^2$. Suppose e is an edge of $L(b)$; we define the side-pairing transformation g_e. The edge e lies in some facet S of $\mathcal{P}$; now take $f_S(e)$, this lies on some facet $S' = f_S(S)$ incident with $b' = f_S(b) \in [c]$; and take g_e to be the Euclidean similarity in $\mathbb{R}^2$ which sends e to the edge corresponding to the radial projection of $f_S(e)$ onto the horosphere $\Sigma_{b'}$. Define $L[c]$ to be the space obtained by taking the quotient of $\{L(b) : b \in [c]\}$ according to Ψ; this

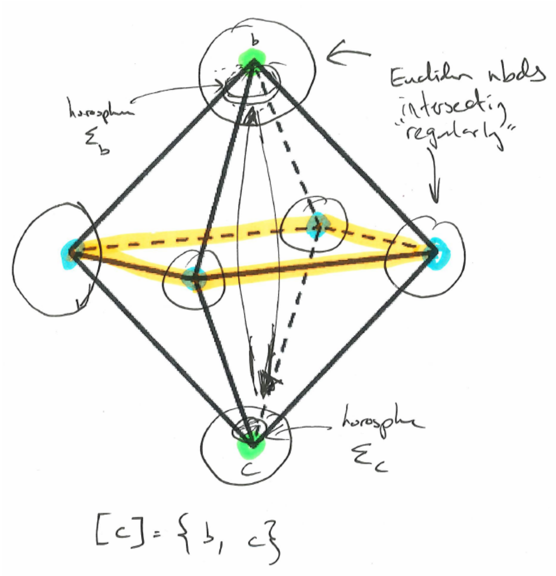


Figure 3: The cycles of an edge pairing on the octahedron.

space is called the **link of the cusp point** $[c]$. By the more general form of Theorem 1.2 for gluing in arbitrary geometric manifolds, the link $L[c]$ is a connected $(\mathbb{R}^2, S(\mathbb{R}^2))$-orbifold.

The following theorem is proved as theorem 13.4.7 of Ratcliffe [24].

1.5 Theorem. *With the above notation, the link $L[c]$ for a cusp point $[c]$ of $\mathcal{P}$ is complete iff each $L(b)$ for $b \in [c]$ can be chosen such that Φ restricts to a side-pairing for $\{L(b) : b \in [c]\}$ (i.e. if the radial projections in the definition are trivial). The manifold M obtained by gluing $\mathcal{P}$ is complete iff $L[c]$ is complete for each cusp point $[c]$ of $\mathcal{P}$.* ∎

1.6 Example. The manifold of Example 1.3 is trivially complete, as the dodecahedron has no vertices at infinity and hence no cusp points.

1.7 Example. Let $\mathcal{O}$ be the regular hyperbolic octahedron h.conv$\{\pm e_1, \pm e_2, \pm e_3\} \subseteq \mathbb{B}^3$ (the e_i being the standard basis vectors of $\mathbb{R}^3$). The dihedral angles of $\mathcal{O}$ are $\pi/2$. The horizontal edges of $\mathcal{O}$ in $\mathbb{R}^2$ induce a perfect matching on the facets of $\mathcal{O}$; define a facet-pairing Φ by sending a facet to the matched facet by a $\pi/2$ rotation with axis the horizontal edge incident to both. The relative interior points of horizontal edges form edge cycles with dihedral angle sum $\pi/2$, and the relative interior points of the other edges form edge cycles with dihedral angle sum π; thus Φ is subproper, and we may define the gluing manifold M of $\mathcal{O}$ with respect to Φ (see Fig. 3). Observe that the polyhedron has five cusps: four corresponding to the (singleton) cycles of the vertices in the horizontal plane, and one corresponding to the 2-cycle of the upper and lower vertices (c and d in the figure). Choose links for these cusps which are equidistant from the origin: then Φ restricts to a side-pairing on the set of these cusps, and M is complete.

§1.2. Schottky groups

A **(classical) Schottky group** on two generators is defined in the following way: let S, S', R, R' be disjoint circles in $\hat{\mathbb{C}}$, and let $f, g \in \mathrm{PSL}(2, \mathbb{C})$ be transformations such that f (resp. g) maps the exterior of S (resp. R) onto the interior of S' (resp. R'). Since everything is disjoint, the side-pairing trans-

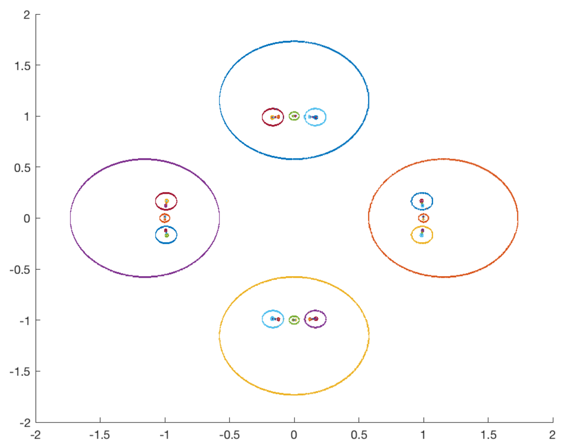


Figure 4: The $\pi/6$-Schottky group (Example 1.9).

formation of the polyhedron $\mathcal{P}$ with sides the extensions of S, S', R, R' into $\mathbb{H}^3$ generates a Kleinian group Γ with fundamental polyhedron $\mathcal{P}$.

1.8 Lemma. *A Schottky group Γ on two generators is free of rank 2 and purely loxodromic, and $\Omega(\Gamma)$ is dense in $\hat{\mathbb{C}}$; the common exterior of S, S', R, R' is a fundamental domain for Γ.* ■

For a proof in the general case of n-generator Schottky groups, see sections VIII.A and X.H of Maskit [16]. For the simple case of two generated classical Schottky groups, see section 5.3 of Beardon [3].

1.9 Example. The **θ-Schottky group** is the group generated by

$$\frac{1}{\sin\theta}\begin{bmatrix} 1 & i\cos\theta \\ -i\cos\theta & 1 \end{bmatrix} \text{ and } \frac{1}{\sin\theta}\begin{bmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{bmatrix};$$

in Fig. 4, we see the paired circles of the $\pi/6$-Schottky group along with their images (we observe that, for instance, the limit set of the group is contained within the circles, as guaranteed by the Poincaré polyhedron theorem). For more detail, see "project 4.2" of Mumford-Series-Wright [23].

It is easy to see from the fundamental polyhedron that the quotient surface $\Omega(\Gamma)/\Gamma$ is a 2-torus T_2; since the fundamental polyhedron is simply connected, we have that $\pi_1(M_\Gamma) = \Gamma$, with the two generating loops (one around each 'torus hole') lifting naturally to the axes of f and g. For the surface, the group computation is more complicated since the fundamental domain $W := \mathrm{int}(\partial\mathcal{P} \cap \hat{\mathbb{C}})$ is not simply connected. In Fig. 5 we observe that $\pi_1(T_2) = \langle a, b, c\rangle$ (for a, b, c the three indicated loops), by using the Seifert-Van Kampen theorem. In Fig. 6, we compute the image $p_*\pi_1(W)$, and find that it is the normal subgroup $\langle\langle b\rangle\rangle$. Hence,

$$F(2) \simeq \frac{\langle a, b, c\rangle}{\langle\langle b\rangle\rangle} = \frac{\pi_1(T_2)}{p_*\pi_1(W)} \simeq \Gamma$$

(as expected, of course).

Suppose we deform this quotient surface such that these geodesics are pinched to zero and the surface becomes a 4-punctured sphere. This is equivalent to deforming f and g to parabolics (that is,

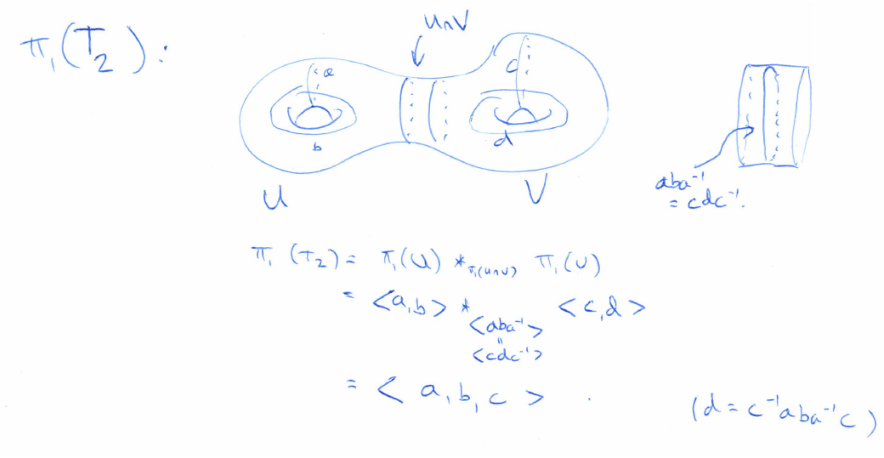


Figure 5: The fundamental group of the 2-torus T_2.

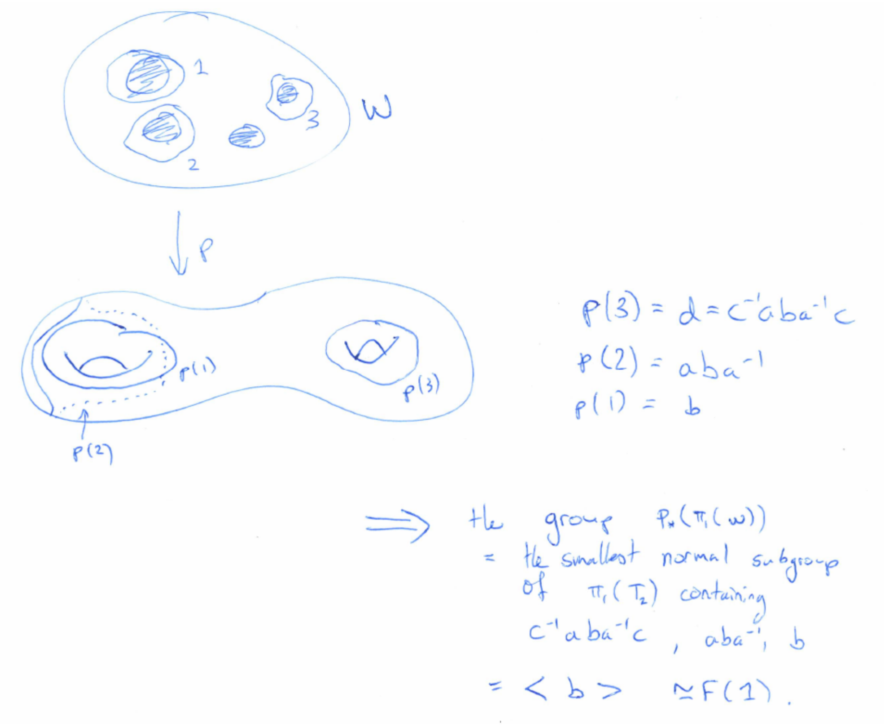


Figure 6: The fundamental group of T_2 in relation to the cover $p : W \to T_2$.

moving the fixed points of each generator together); or to moving the paired disjoint circles S, S' and R, R' to become tangent circles. The space of these groups lies on the boundary of the deformation space of classical Schottky groups on two generators (see [8]), and is called the **Riley slice**.

We may also 'deform' the generators f and g to elliptics (by sending the translation length to zero but not modifying the rotational component — this will move the paired circles to intersect transversely in their pairs); now the generators correspond to cone points.

§1.3. Groups generated by two parabolic elements

In this section we begin to study the Riley slice; more precisely, we will introduce the coordinate system which we shall use initially. Let $\rho \in \mathbb{C} \setminus \{0\}$, and let Γ_ρ be the group $\langle f, g\rangle$ where $f(z) \coloneqq z+1$ and $g(z) \coloneqq \frac{z}{\rho+1}$ for all $z \in \hat{\mathbb{C}}$. It is clear that every group generated freely by two parabolic elements may be normalised into this form.

The group Γ_ρ is non-elementary: clearly $0, \infty \in \Lambda(\Gamma_\rho)$; in addition, the element fgf^{-1} is parabolic, with fixed points distinct from 0 and ∞; so $|\Lambda(\Gamma_\rho)| \geq 3$. We may therefore compute approximations to the limit set by looking at the translates of 0 under the elements of the group. Some examples may be found in Fig. 7.

The isometric circles of g and g^{-1} are $S(|\rho|^{-1}, -\rho^{-1})$ and $S(|\rho|^{-1}, \rho^{-1})$; let S_1 and S_2 be the two hyperbolic planes in $\mathbb{H}^3$ which meet $\hat{\mathbb{C}}$ at these circles, and let P_1 and P_2 be the two hyperplanes which meet $\hat{\mathbb{C}}$ at the lines $-1/2+\mathbb{R}i$ and $1/2+\mathbb{R}i$. Let $\mathcal{P}$ be the convex polyhedron with facets P_1, P_2, S_1, S_2; then f is a facet-pairing transformation sending $P_1 \to P_2$, and g is a facet-pairing transformation sending $S_1 \to S_2$. For convenience, we consider the case that $P_1 \cup P_2$ and $S_1 \cup S_2$ are disjoint. This occurs if (and only if) the following equations hold:

$$\begin{aligned} \operatorname{Re}(-\rho^{-1} - |\rho|^{-1}) > 1/2 &\iff 2(-\cos(-\theta) - 1) > r \\ \operatorname{Re}(-\rho^{-1} + |\rho|^{-1}) < 1/2 &\iff 2(-\cos(-\theta) + 1) < r \\ \operatorname{Re}(\rho^{-1} - |\rho|^{-1}) > 1/2 &\iff 2(\cos(-\theta) - 1) > r \\ \operatorname{Re}(\rho^{-1} + |\rho|^{-1}) < 1/2 &\iff 2(\cos(-\theta) + 1) < r. \end{aligned}$$

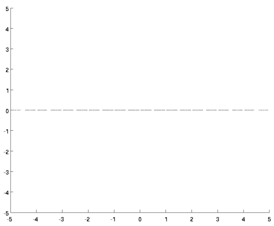
One can plot the region in which all these equations hold: it is the common exterior of the cardioids of Fig. 8. Let $\mathcal{R}^*$ be this common exterior.

1.10 Theorem. *When ρ lies in $\mathcal{R}^*$, the quotient surface $\Omega(\Gamma_\rho)/\Gamma_\rho$ is a four-times punctured sphere and the quotient manifold $\mathbb{H}^3/\Gamma_\rho$ is a hyperbolic 3-ball with four cusps and deleted arcs joining the arcs in pairs.*

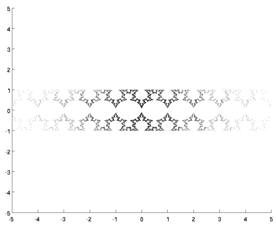
The gluing procedure for the surface is depicted in Fig. 9, and the 3-manifold is depicted in Fig. 10. It is clear that the fundamental group of the 3-manifold is a free group on two generators — one loop about each of the two arcs — and this is as expected, since $\Gamma \simeq \mathbb{H}^3/\Gamma_\rho$ by the algebraic topology (as the fundamental polyhedron $\mathcal{P}$ is simply connected). We now compute the relationship between Γ and $\pi_1(\Omega(\Gamma_\rho)/\Gamma_\rho)$. Consider the fundamental domain $W = \operatorname{int}(\hat{\mathbb{C}} \cap \mathcal{P})$; it is topologically an annulus, so $\pi_1(W) = \langle a\rangle$. In Fig. 11, we compute that $\pi_1(W/\Gamma_\rho) = \langle \alpha, \beta, \gamma\rangle$ for three specific loops α, β, γ, and that with this choice of generators $p_*\pi_1(W) = \langle\!\langle \gamma \rangle\!\rangle$. In particular,

$$\Gamma_\rho \simeq \frac{\langle \alpha, \beta, \gamma\rangle}{\langle\!\langle \gamma \rangle\!\rangle} = \langle \alpha, \beta\rangle.$$

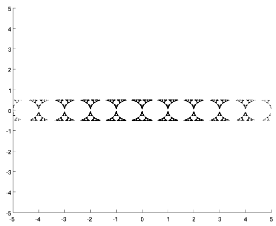
Using the figure, we observe also that the curves α and β are obtained as the (homotopy classes of the) images under p of the invariant circles of f and g; and the nontrivial loops about the arcs in the 3-manifold come from the invariant horocircles in $\mathbb{H}^3$ based at the parabolic fixed points.



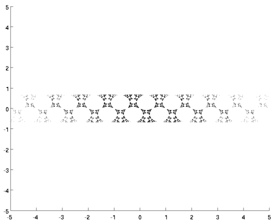
(a) $\rho = 4$



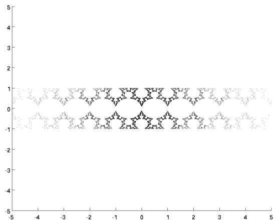
(b) $\rho = i$



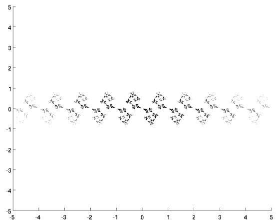
(c) $\rho = 2i$



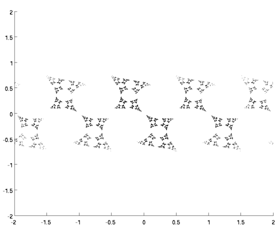
(d) $\rho = 1.5 + 1.3i$



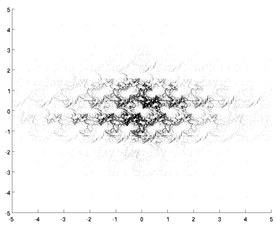
(e) $\rho = i$



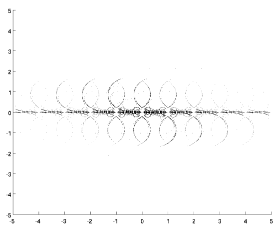
(f) $\rho = 1 + 2i$



(g) $\rho = \sqrt{2}(1 + i)$



(h) $\rho = -\exp(2\pi i/3)$ (dense in $\hat{\mathbb{C}}$)



(i) $\rho = 3 + 0.1i$ (dense in $\hat{\mathbb{C}}$?)

Figure 7: Some limit sets of 2-parabolic groups.

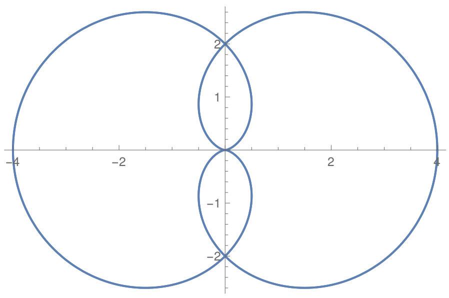


Figure 8: Cardioids bounding the region of $\mathbb{C}$ in which the Klein combination theorem works for us.

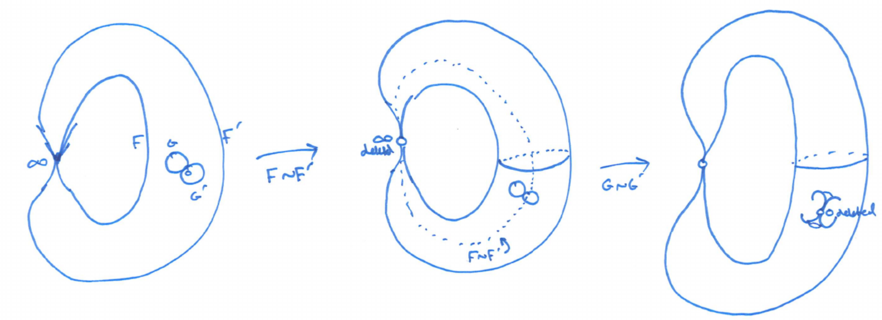


Figure 9: The 4-punctured sphere obtained from the 2-parabolic groups.

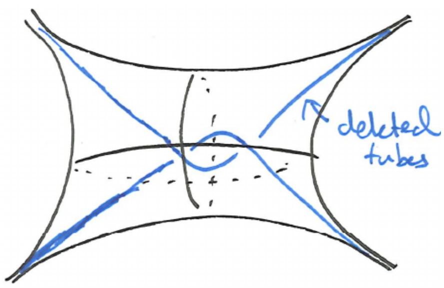


Figure 10: The 3-manifold corresponding to the 2-parabolic groups.

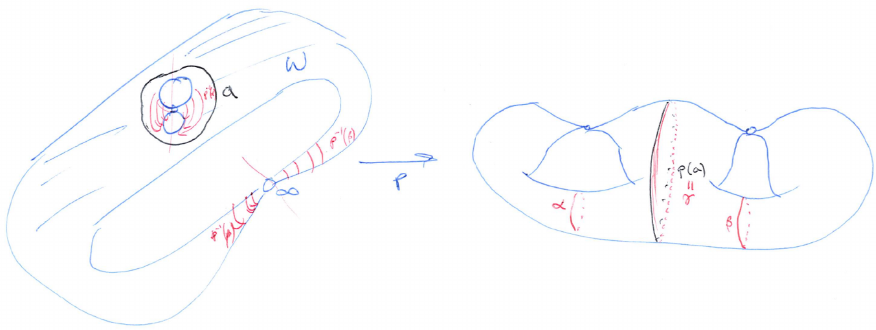


Figure 11: The fundamental group of the 4-punctured sphere in relation to the cover W.

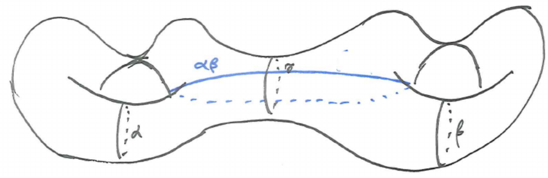


Figure 12: A second view of the 4-punctured sphere obtained from the 2-parabolic groups, showing the geodesic $\alpha\beta$ which is pinched as we move to the boundary.

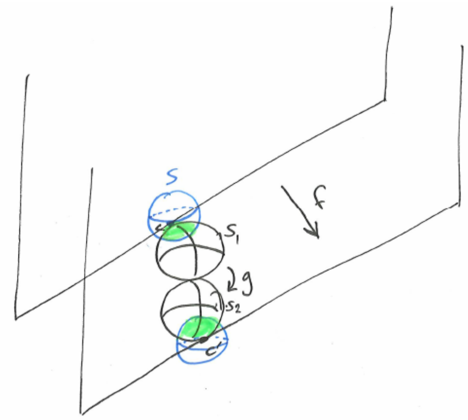


Figure 13: For arbitrary horospheres S at the cusps, the links (boundaries of the green horocircles) can never be paired by f and g due to the lack of symmetry: more precisely, the green circles are not even mapped onto each other, so their edges are not paired.

The curve γ essentially determines the geometry of the four-times punctured sphere. By this, we mean that in order to determine the complex geometry of the sphere it suffices to give the data of the length of γ, and the angle made by the geodesics joining the pairs of punctures in each hemisphere determined by γ. The space of the four-times punctured spheres of interest is 2-dimensional over $\mathbb{R}$ (namely, the real and complex parts of ρ) and so we have found all of the dimensions. (The distances between the punctures can be varied biholomorphically, of course.)

If we pinch off the geodesic $\alpha\beta$ (see Fig. 12) by sending the distance to zero then we split the surface into a pair of 3-punctured spheres. One might naïvely try to do this by expanding the isometric circles of g so that they become tangent to the lines paired by f; for instance, this occurs when $\rho = 4$. It is easy to show by the reasoning above that the circles are tangent in this way precisely when ρ lies on the shared boundary of the cardioids of Fig. 8; e.g. if $\rho = r\exp(\theta i)$ (the centre of the isometric circle) satisfies $2(\cos\theta + 1) = r$. The problem is, of course, that we may not apply the Poincaré polyhedron theorem to this situation: consider the points c, c' of tangency between the isometric circles of g and the lines paired by f. By inspection, $[c] = \{c, c'\} = [c']$. If we take any small horosphere S based at c, then the obstruction is illustrated by Fig. 13. This obstruction occurs for all possible orientations of the circles for ρ on the boundaries of the region bounded by Fig. 8, except for the extremal values $\rho = \pm 4$ or $\rho = \pm 2i$ (i.e. the only values of ρ for which the paired horocircles are symmetric with respect to f and g). This shows us that the region of four-punctured sphere groups is *not* just the exterior of the cardioids of Fig. 8 — we are simply drawing the 'wrong'

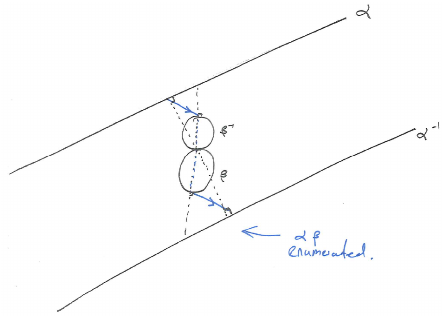


Figure 14: A lift of the geodesic $\alpha\beta$ from the four-punctured sphere to the fundamental domain.

fundamental domain in the cases within the cardioids, except at the four extremal points (the only points where the polyhedron theorem continues to hold).

Another way of seeing this is to consider the 'lift' of $\alpha\beta$ to the fundamental domain, as in Fig. 14; it is clear from the picture that if the circles expand to meet the boundary then the lift is still not deformed to zero length, as the tangency does not occur at the points where the curve meets the isometric circles.

1.11 Exercise. Draw the lifts of $\alpha\beta$ for $\rho = \pm 2i$ and $\rho = \pm 4$. Reconcile this with the above discussion.

One can deduce better bounds by considering 'slanted' fundamental domains instead of choosing vertical lines. That is, the quotient is a four-times punctured sphere if the pairs of isometric circles of g and their f-translates are disjoint. (This is inspired by the fundamental domains drawn by Jørgensen [7].) One shows that the pair of isometric circles tangent at 0 and the translated pair centred at -1 become tangent when ρ satisfies one of the following:

$$\begin{aligned} \left|\rho^{-1} - (\rho^{-1} - 1)\right| &= \left|\rho^{-1}\right| \\ \left|\rho^{-1} - (-\rho^{-1} - 1)\right| &= \left|\rho^{-1}\right| \\ \left|-\rho^{-1} - (\rho^{-1} - 1)\right| &= \left|\rho^{-1}\right| \\ \left|-\rho^{-1} - (-\rho^{-1} - 1)\right| &= \left|\rho^{-1}\right| \end{aligned}$$

Plotting this bound, we obtain the diagram included as Fig. 15.

The Riley slice boundary was first drawn by Riley in the 1980s with the aid of a computer; his drawing, which shows one quadrant of the boundary, is reproduced in Fig. 16, and the full boundary is approximated as the boundary of the plotted points in Fig. 17. (The method by which this plot was obtained will be explained later on.) Observe that the exterior of the slice (that is, the region filled out in blue) is contained within the interior of the cardioid shape, and has clear vertices at the four special points we have just considered.

To round off the discussion, let us mention that the boundary of the slice does have the main property which we indicated above via our informal deformation argument: groups Γ_ρ such that the quotient $\Omega(\Gamma_\rho)/\Gamma_\rho$ is a disjoint union of 3-punctured spheres (called **cusp groups**) do lie on the boundary of the Riley slice. This was shown by Bers and Maskit in a series of two papers in the 1960s [5, 17] (though of course not explicitly in the context we study here). In the early 1990s it was shown by McMullen [20] that the cusps groups are actually *dense* on the boundary of the Riley slice (again, in a more general context). He won a Fields Medal in 1998, in part for this result. See also Theorem 2.1 below.

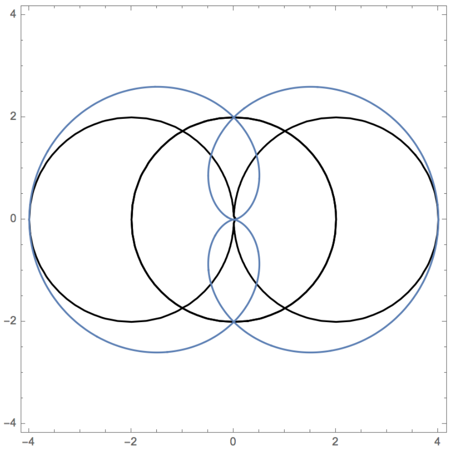


Figure 15: Three circles bounding the region of ℂ in which the Klein combination theorem works for us.

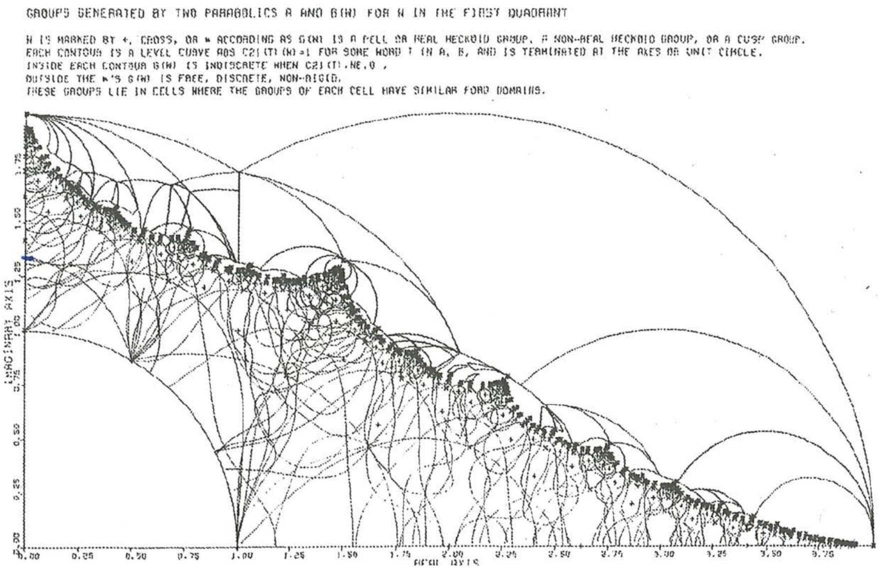


Figure 16: A quadrant of the Riley slice boundary as drawn by Riley. This figure is reproduced from [1, p. VIII].

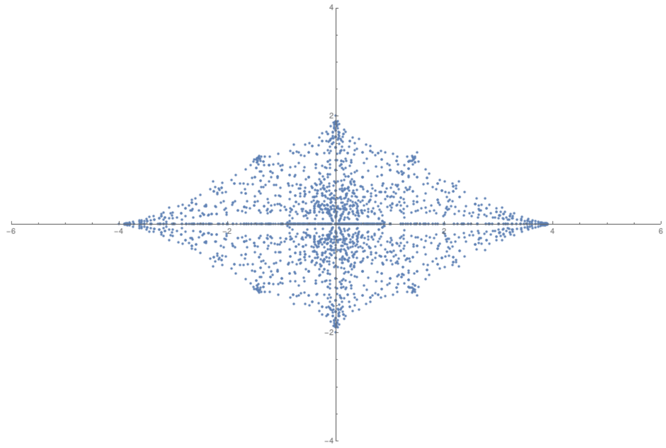


Figure 17: A plot of the boundary of the Riley slice.

§1.4. *Groups generated by two elliptic elements*

Observe that if we instead consider groups generated by two elliptic elements, then the isometric circles intersect transversely rather than at tangent points. We may conjugate any such group to the group generated by the two elements

$$f = \begin{bmatrix} \zeta & 0 \\ 0 & \overline{\zeta} \end{bmatrix}, g = \begin{bmatrix} \xi & 1 \\ \mu & \overline{\xi} \end{bmatrix}$$

where $|\zeta| = |\xi| = 1$ and $\mu \in \mathbb{C} \setminus \{0\}$. The isometric circles of $g^{\pm 1}$ are

$$S_1 = S\left(|\mu|^{-1}, -\frac{1}{\xi\mu}\right) \qquad S_2 = S\left(|\mu|^{-1}, -\frac{\xi}{\mu}\right)$$

§1.5. *Once-punctured torus groups*

We shall now briefly study the Kleinian groups Γ such that $\Omega(\Gamma)/\Gamma$ is a once-punctured torus; the study of such groups was initiated by Jørgensen; see the lecture notes by Series [25] or the book of Akiyoshi, Sakuma, Wada, and Yamashita [1]

1.12 Definition. Let T be a topological once-punctured torus. A **marked punctured torus group** is the image Γ of a discrete faithful representation $\rho : \pi_1(T) \to \mathrm{PSL}(2,\mathbb{C})$ such that if $\omega \in \pi_1(T)$ is represented by a simple loop about the puncture, then $\rho\omega \in \Gamma$ is parabolic.

It is clear that such a group is necessarily of the form $\Gamma = \langle \alpha, \beta \ : \ \mathrm{tr}^2[\alpha,\beta] = 4\rangle$, where the generators α and β correspond to the 'standard' generating loops of $\pi_1(T)$, such that $[\alpha,\beta]$ is a loop about the puncture.

We can produce some once-punctured torus groups by performing surgery on Riley cusp groups. Suppose $\Gamma_\rho = \langle f, g\rangle$ is such a group, and let S be one of the 3-punctured sphere components, say corresponding to the component containing the puncture produced by A. The surgery we perform is the deletion of a neighbourhood of the two cusps belonging to A and the gluing together of the produced boundary curves.

The group produced is the extension group

$$\langle \Gamma, h : hfh^{-1} = g \rangle.$$

We can justify this geometrically: this new element h will correspond to the longitude of the new 'handle'.

The technical justification comes from the second Maskit combination theorem. In order to state this theorem we shall need some terminology.

1.13 Definition. Suppose G is a Kleinian group, that J, J_1, and J_2 are geometrically finite subgroups of G_0, and that $B, B_1, B_2 \subseteq \hat{\mathbb{C}}$ are closed topological discs. We say variously that

- B is **precisely invariant** under J in G_0 if $J = \mathrm{Stab}_{G_0} \hat{\mathbb{C}}$ and $gB \cap B = \emptyset$ for all $g \in G_0 \setminus J$.
- (B_1, B_2) are **precisely invariant** under (J_1, J_2) in G_0 if, for each m, B_m is precisely invariant under J_m and for all $g \in G_0$, $gB_m \cap B_{3-m} = \emptyset$.
- B is a (J, G_0)-**block** if it is J-invariant and the following conditions are satisfied:
  1. $B \cap \Omega(G_0) = B \cap \Omega(J)$;
  2. $B \cap \Omega(J)$ is precisely invariant under J in G_0; and
  3. For every puncture on $\Omega(J)/J$, there is a punctured-disc neighbourhood U of the puncture such that either U is contained in the projection of B, or U is disjoint from the projection of B.
- B_1 and B_2 are **jointly** t**-blocked** for some $t \in \mathrm{PSL}(2, \mathbb{C}) \setminus G_0$:
  1. For $m \in \{1, 2\}$, B_m is a (J_m, G_0)-block;
  2. For $m \in \{1, 2\}$, $(B_1 \cap \Omega(G_0), B_2 \cap \Omega(G_0))$ is precisely invariant under (J_1, J_2), ;
  3. t maps the interior of B_1 onto the interior of B_2; and
  4. $tJ_1t^{-1} = J_2$.
- If B_1, B_2 are jointly t-blocked, then a fundamental set D_0 for G_0 is called **maximal** if $D_0 \cap B_m$ is a fundamental set for the action of J_m on B_m for each m, and if $t(D \cap \partial B_1) = D \cap \partial B_2$.

1.14 Theorem (Maskit (1988))**.** *Let J_1 and J_2 be geometrically finite subgroups of a Kleinian group G_0 , and let $G_1 = \langle t \rangle$ be infinite cyclic. Let B_1 and B_2 be jointly t-blocked closed topological discs, and let A be the common exterior of B_1 and B_2.*

Let D_0 be a maximal fundamental set for G_0, set $G = \langle G_0, t \rangle$, and set $D = D_0 \cap (A \cup \partial B_1)$. Then we may conclude the following:

(i) $G = G_0 *_t$.

(ii) *G is discrete.*

(viii) *D is a fundamental domain for G.*

The full theorem gives more detailed information about the cusps and the shape of the limit set of G; for the full statement and proof, see section VII.E of Maskit [16].

Remark. There is a strong case for giving this theorem a more descriptive name: perhaps one might call the second combination theorem the 'Maskit puncture-pair gluing theorem' and the first combination theorem the 'Maskit handle gluing theorem'.

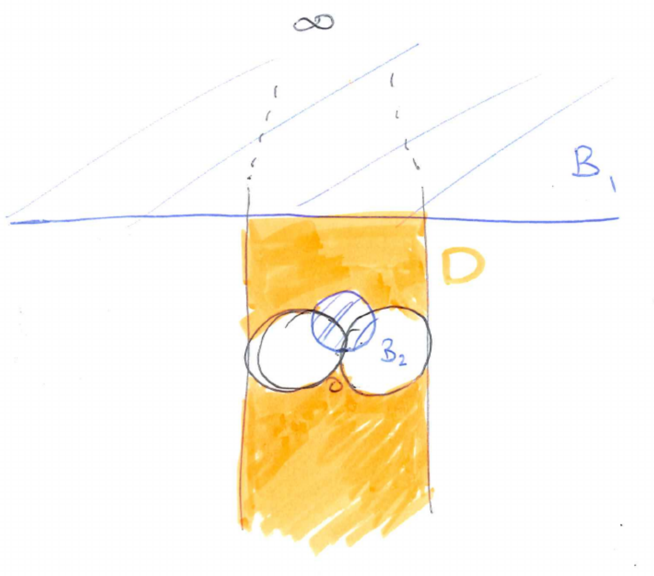


Figure 18: The fundamental domain D constructed for a punctured torus group by the second Maskit combination theorem.

In our case, the extension element $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ must satisfy the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \rho & 1 \end{bmatrix};$$

solving the above equation for h shows that $a = 0$ and $c^2 = -\rho$, so after normalisation h is the transformation

$$h(z) = \frac{i\rho^{-1/2}}{i\rho^{1/2}z + d} = \frac{-\rho^{-1}}{-z + i\rho^{-1/2}d}.$$

Consider for simplicity the case $\rho = 4$, so the transformation reduces to

$$h(z) = \frac{-1/4}{-z + i1/2d} = \frac{1}{4z - 2id}.$$

Take $J_1 = \langle f \rangle$ and $J_2 = \langle g \rangle$. The relevant blocks will then be small discs tangent to ∞ and to 0 (the blue discs B_1 and B_2 in Fig. 18); D_0 can be taken to be the closure of the isometric circle exterior fundamental domain; and so D is the region of Fig. 18 in orange.

1.15 Exercise. Apply the Poincaré polyhedron theorem to the domain D of Fig. 18 and check that we in fact obtain a 3-punctured sphere and a 1-punctured torus as quotient surface.

§2. The thick-thin decomposition

In this lecture, we shall formalise the notion of a 'cusp' which we used informally in the previous lecture, using Thurston's "thick-thin decomposition". The goal of the next few lectures is to prove the following theorem, and to give the analogue for 2-elliptic groups:

2.1 Theorem. *Let Γ be a Kleinian group generated by two parabolic elements; let G be a deformation*[1] *of Γ. Then the Riemann surface $\Omega(G)/G$ is either a 4-punctured sphere, or a disjoint union of two 3-punctured spheres.*

(This theorem may be found in [18].)

§2.1. The Kazhdan-Margulis lemma

In this section, we shall follow the proof of the Kazhdan-Margulis lemma given in chapter D of Benedetto-Petronio [4]; this proof is essentially the same as that given by Thurston [26] (section 5.10). An alternative proof which is more group-theoretic is found in section 12.6 of Ratcliffe [24]. The theory has an essentially arithmetic flavour, and this viewpoint is pursued in section 1.3 of Maclauchlan-Reid [15] and in section 12.8 of Ratcliffe [24].

2.2 Theorem (Kazhdan-Margulis lemma)**.** *There exists some $\varepsilon > 0$ such that, for every $x \in \mathbb{H}^3$ and every Kleinian group Γ, the subgroup $\Gamma_\varepsilon(x) \leq \Gamma$ generated by*

$$F_\varepsilon(x) := \{\gamma \in \Gamma : d(gx, x) \leq \varepsilon\}$$

is essentially nilpotent.

Notation. The supremum of all ε satisfying the conclusion of the Kazhdan-Margulis lemma is called the **third Margulis constant**. We denote it by μ_3.

Remark. One may make the following improvements: For all $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that for every connected, simply connected, complete, Riemannian n-manifold M with sectional curvatures k satisfying $-1 \leq k \leq 0$, and for every $x \in M$, and for every freely discontinuous $\Gamma \leq \mathrm{Isom}(M)$, then the subgroup of Γ generated by

$$\{\gamma \in \Gamma : d(gx, x) \leq \varepsilon_n\}$$

is essentially nilpotent.

The supremum of the set of all the ε_n satisfying the above conclusion is called the nth **Margulis constant**.

Our main interest is actually the following corollary:

2.3 Corollary. *There exists some $\varepsilon > 0$ such that, for every $x \in \mathbb{H}^3$ and every Kleinian group Γ, the subgroup $\Gamma_\varepsilon(x) \leq \Gamma$ is elementary.*

Proof of the corollary. In fact, we show that every solvable subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is elementary.

Let $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$ be solvable; the **solvability degree** of Γ is the smallest natural number k such that $\Gamma^{(k)} = 1$. We prove that Γ is elementary by induction on k. If $k = 0$ then $\Gamma = 1$ is trivially elementary, so assume that $k > 0$ and that all subgroups of $\mathrm{PSL}(2, \mathbb{C})$ with solvability degree less than k are elementary. In particular, $\Gamma^{(1)} = [\Gamma, \Gamma]$ is elementary, as it has solvability degree $k - 1$.[2] We split into two cases.

$\Gamma^{(1)}$ **is *not* of elliptic type.** The union F of all the finite orbits of $\Gamma^{(1)}$ in this case is exactly the limit set. Let $\gamma \in \Gamma$ and $\sigma \in \Gamma^{(1)}$, so $\gamma^{-1}\sigma\gamma \in \Gamma^{(1)}$ (as derived subgroups are normal); hence $\gamma^{-1}\sigma\gamma F = F$ so $\sigma\gamma F = \gamma F$. Thus γF is left invariant by every $\sigma \in \Gamma^{-1}$ and thus by cardinality must equal F; hence F is left invariant by Γ, so Γ is elementary.

[1] We shall define this in the next lecture.

[2] We avoid the notation Γ' for the derived subgroup here, as later in the proof of the Kazhdan-Margulis lemma we shall be defining 'primed' groups in a completely different way.

$\Gamma^{(1)}$ **is of elliptic type.** The second case is not such a familiar argument. If $\Gamma^{(1)}$ is elliptic, then let F be the set of all points in $\mathbb{B}^3$ fixed by $\Gamma^{(1)}$. By the classification of elementary groups, this is either a point or a hyperbolic line. Suppose $x \in F$, $\gamma \in \Gamma$, and $\sigma \in \Gamma^{(1)}$; then $\gamma^{-1}\sigma\gamma x = x$, so $\sigma\gamma x = \gamma x$, and thus $\gamma x \in F$ since σ was arbitrary. In particular, $\Gamma F \subseteq F$. Let $\bar{\Gamma}$ be the set of hyperbolic isometries of F obtained by restricting elements of Γ, and let $\rho : \Gamma \to \bar{\Gamma}$ be the restriction epimorphism. Since $\Gamma^{(1)} \leq \ker\rho$, ρ induces an epimorphism $\Gamma/\Gamma^{(1)} \to \bar{\Gamma}$. But note that $\Gamma/\Gamma^{(1)}$ is the abelianisation of Γ, and so $\bar{\Gamma}$ is abelian and hence elementary. In particular, $\bar{\Gamma}$ has a finite orbit in F, so Γ has a finite orbit on F and thus is elementary.

This completes the proof of the corollary. ∎

We now prove the Kazhdan-Margulis lemma. The idea is to study elements of Γ which are both ε-small at x (in that they move x a distance most ε), and which have ε-small derivatives at x. The second notion is slightly harder to make precise, since the derivatives of elements of γ are not isometries of $\mathbb{H}^3$ — one must transfer images of $d_x g$ 'back' to $T_x\mathbb{H}^3$ from $T_{gx}\mathbb{H}^3$ via parallel transport.

These ideas are encapsulated in the following definition. For $x \in \mathbb{H}^3$ and $g \in \mathrm{PSL}(2,\mathbb{C})$, define

$$\|g\|_x \coloneqq \max\{d(x,gx), \alpha_x(I, P_{gx,x} \circ d_x g)\}$$

where $d(\cdot,\cdot)$ is the usual hyperbolic metric, $P_{z,y}$ is the parallel transport from $T_y\mathbb{H}^3$ to $T_z\mathbb{H}^3$ along the unique geodesic segment $[y,z]$ (c.f. chapter 4 of Lee [12]), and $\alpha_x(\cdot,\cdot)$ is the angular metric on $\mathrm{PSL}(2,\mathbb{C})$ defined by

$$\alpha_x(A,B) \coloneqq \max\{\theta(Aw, Bw) : w \in T_x\mathbb{H}^3\}.$$

In order to show nilpotency, we must be able to bound the norms of commutators. The next lemma is essentially a 'differential' bound; it says that, if we restrict ourselves to ε'-neighbourhoods (for some sufficiently small ε') of the identity in $\mathrm{PSL}(2,\mathbb{C})$, then the commutator is on the order of $1/2\varepsilon'$.

2.4 Proposition. *There exists a constant ε' such that, for all $g, h \in \mathrm{PSL}(2,\mathbb{C})$, if*

$$\|g\|_x \leq \varepsilon' \text{ and } \|h\|_{\eta,x} \leq \varepsilon'$$

then

$$\|[g,h]\|_x \leq \frac{1}{2}\max\{\|g\|_x, \|h\|_x\}.$$

2.5 Lemma. *The statement of Proposition 2.4 for arbitrary $x \in \mathbb{H}^3$ follows from the statement of Proposition 2.4 for $x = 0 \in \mathbb{B}^3$.*

Proof. To see this, we first note that for $k \in \mathrm{PSL}(2,\mathbb{C})$ and $y, z \in \mathbb{H}^3$, we have

$$P_{ky,kz} = d_z k \circ P_{y,z} \circ (d_y k)^{-1}.$$

We deduce from this equality that for all $f \in \mathrm{PSL}(2,\mathbb{C})$,

$$
\begin{aligned}
&\alpha_x(I, P_{f^{-1}gfx,x} \circ d_x(f^{-1}gf)) \\
&\quad = \max_{v \in T_x\mathbb{H}^3} \theta(v, (P_{f^{-1}gfx,x} \circ d_x(f^{-1}gf))v) \\
&\quad = \max_{v \in T_x\mathbb{H}^3} \theta(v, (d_{fx}f^{-1} \circ P_{gfx,fx} \circ (d_{gfx}f^{-1})^{-1} \\
&\qquad\quad \circ d_{gfx}f^{-1} \circ d_{fx}g \circ d_x f)v) && \text{applying (*) with } k = f^{-1} \text{ and the chain rule} \\
&\quad = \max_{v \in T_x\mathbb{H}^3} \theta(v, ((d_x f)^{-1} \circ P_{gfx,fx} \circ d_{fx}g)(d_x f(v))) && \text{simplifying} \\
&\quad = \max_{v \in T_x\mathbb{H}^3} \theta((d_x f)v, (P_{gfx,fx} \circ d_{fx}g)(d_x f(v))) && d_x f \text{ is orthogonal} \\
&\quad = \max_{w \in T_{fx}\mathbb{H}^3} \theta(w, (P_{gfx,fx} \circ d_{fx}g)w) && \text{differential is pushforward} \\
&\quad = \alpha_{fx}(I, P_{g(fx),fx} \circ d_{fx}g).
\end{aligned}
\tag{2.6}
$$

It is easy to see that for all such f we have $d(x, f^{-1}gf(x)) = d(fx, gfx)$ as well, and so we may conclude that for all isometries f

$$\|g\|_x = \|g\|_{fx}.$$

Noting also that $[f^{-1}gf, f^{-1}hf] = f^{-1}[g,h]f$, and choosing any isometry f sending x to 0, the statement of the lemma for x follows from the statement of the lemma for 0 and the observations just made. This proves the claim. ∎

We shall view the tangent bundle $T\mathbb{B}^3$ as canonically identified with $\mathbb{B}^3 \times \mathbb{R}^3$ such that each tangent space has as basis the parallel transport of the basis $\{e_1, e_2, e_3\}$; this means that, if $f : \mathbb{B}^3 \to \mathbb{B}^3$ is smooth, we may view $d_0 f : \{0\} \times \mathbb{R}^3 \to \{f(0)\} \times \mathbb{R}^3$ as a linear map $\mathbb{R}^3 \to \mathbb{R}^3$ and ask that it is a homothety with respect to the transported basis.

2.7 Lemma. *If $x_0 \in \mathbb{B}^3$, then there is a unique element $f^{(x_0)} \in \mathrm{PSL}(2,\mathbb{C})$ such that $f^{(x_0)}(0) = x_0$ and such that $d_0 f^{(x_0)}$ is a homothety $T_0\mathbb{B}^3 \to T_{x_0}\mathbb{B}^3$.*

Proof. **Uniqueness.** Suppose f_1, f_2 are two such elements. Then $f_1^{-1}f_2$ fixes 0, and $d_0 f_1^{-1} f_2 = d_{x_0} f_1^{-1} \circ d_0 f_2 = (d_0 f_1)^{-1} \circ d_0 f_2 = \lambda^{-1}\mu I$ for some $\lambda, \mu > 0$. But an element fixing 0 is elliptic so $\lambda^{-1}\mu = 1$, thus $f_1 = f_2$.

Existence. Take $f^{(0)}$ to be the identity; if $x_0 \neq 0$, then

$$f^{(x_0)}(x) = \frac{(1 - \|x_0\|^2)x + (1 + \|x\|^2 + 2\langle x_0, x | x_0, x\rangle)x_0}{1 + 2\langle x_0, x | x_0, x\rangle + \|x_0\|^2\|x\|^2}.$$

∎

Proof of Proposition 2.4. By Lemma 2.5, we may assume $x = 0$ and so for the remainder of the proof, we use $\|\cdot\|$ for the norm $\|\cdot\|_0$.

Using the maps of Lemma 2.7, we construct a map

$$
\begin{aligned}
\phi : O(3) \times \mathbb{B}^3 &\to \mathrm{PSL}(2,\mathbb{C}) \\
(A, x) &\longmapsto f^{(x)} \circ A.
\end{aligned}
$$

This is a bicontinuous group isomorphism by construction; using this, we may endow PSL(2, $\mathbb{C}$) with a Lie group structure. *This is not the usual Lie group structure on* PSL(2, $\mathbb{C}$) *from* GL(2, $\mathbb{C}$)*; it is the Lie group structure coming from the hyperboloid model of* $\mathbb{H}^3$. We now compare this Lie structure to the normed structure (PSL(2, $\mathbb{C}$), $||\cdot||$).

Observe that given any $x_0 \in \mathbb{B}^3$, there is a unique isometric homothety $f : T_0\mathbb{B}^3 \to T_{x_0}\mathbb{B}^3$ (there is a single parameter associated with a homothety, and it is determined by the action on the norm). Since geodesics through 0 in $\mathbb{B}^3$ are Euclidean, P_{0,x_0} is a homothety. Further, since P_{0,x_0} is an isometry, it must coincide with $d_0 f^{(x_0)}$ which is also an isometric homthety.

This observation allows us to make the following computation:

$$d_0\phi(A, x_0) = d_0(d^{x_0}\circ A) = d_{A0}f^{(x_0)}\circ d_0 A = P_{0,x_0}\circ A$$

and hence

$$\max_{v\in T_0\mathbb{B}^3} \theta(v, P_{x_0,0}\circ d_0\phi(A, x_0)v) = \max_{v\in T_0\mathbb{B}^3} \theta(v, P_{0,x_0}^{-1}\circ P_{0,x_0}\circ Av) = \max_{v\in T_0\mathbb{B}^3} \theta(v, Av);$$

in particular,

$$\begin{aligned} ||\phi(A, x_0)|| &= \max\{d(0, \phi(A, x_0)0), \max\{\theta(v, P_{x_0,0}\circ d_0\phi(A, x_0)v) : v \in T_0\mathbb{H}^3\}\} \\ &= \max\{d(0, x_0), \max_{v\in T_0\mathbb{B}^3} \theta(v, Av)\}\} \\ &= \max\{d(0, x_0), \alpha_0(I, A)\}. \end{aligned}$$

In particular, $||\cdot||$ is comparable with the norm on $O(3) \times \mathbb{B}^3$.

Recall now that for any Lie group G, the differential of $G \times G \ni (g, h) \longmapsto [g, h] \in G$ at $(1, 1)$ is 0 (e.g. this follows immediately from the easy exercise 7-2 of [13]). Let us recall now the statement we are trying to prove, and let $g, h \in$ PSL(2, $\mathbb{C}$); considering this differential, after mapping back through ϕ there exists $\varepsilon' > 0$ such that $||[g, h]|| \leq \frac{1}{2}\max\{||g||, ||h||\}$ whenever $||g||, ||h|| < \varepsilon'$. ∎

Define now the group $\Gamma'_\varepsilon(x)$ (for $\varepsilon > 0$ and $x \in \mathbb{B}^3$) generated by the set

$$F'_\varepsilon(x) = \{\gamma \in \Gamma : ||\gamma||_x \leq \varepsilon\};$$

this is the subgroup of elements of $\Gamma_\varepsilon(x)$ which also have 'small derivative' in the sense described above. As mentioned above, we wish to check that for sufficiently small ε, the group $\Gamma'_\varepsilon(x)$ is nilpotent; this result is an easy corollary of Proposition 2.4.

2.8 Corollary. *The group* $\Gamma'_{\varepsilon'}(x)$ *is nilpotent.*

Remark. This is in fact a special case of the **Zassenhaus theorem**, see e.g. theorem 4.52 of Kapovich [9].

Proof. Since Γ is discrete, there exists $\delta > 0$ such that $\Gamma'_\delta(x) = 1$. Choose $m \in \mathbb{N}$ such that $\varepsilon'(1/2)^m \leq \delta$. If $g_1, ..., g_m \in F'_{\varepsilon'}(x)$, then

$$||[g_1, [g_2, ..., [g_{m-1}, g_m]...]]||_x \leq \varepsilon' 2^{-m} \leq \delta;$$

hence $[g_1, [g_2, ..., [g_{m-1}, g_m]...]] \in \Gamma'_\delta(x) = 1$. This shows that the m-fold commutators of the generators of $\Gamma'_{\varepsilon'}(x)$ are trivial. Now observe that for any group G, if $f, g, h \in G$ then $[f, gh] = [f, g][g, [f, h]][f, h]$, so every m-nesting of commutators of elements of $\Gamma'_{\varepsilon'}(x)$ is a product of $\geq m$-nested commutators of elements of $F'_{\varepsilon'}(x)$. ∎

We now find some ε such that the group G generated by elements of

$$\Gamma_\varepsilon(x) \cap F'_{\varepsilon'}(x) = \{\gamma \in \Gamma_\varepsilon(x) : \|\gamma\|_x \le \varepsilon'\}$$

is of finite index in $\Gamma_\varepsilon(x)$; since this group is contained in $\Gamma'_{\varepsilon'}(x)$ it will be nilpotent and we will be done.

To be more precise, we shall prove the following proposition and deduce the Kazhdan-Margulis lemma as an easy corollary. Recall that the group $\Gamma_\varepsilon(x)$ is generated by $F_\varepsilon(x) := \{\gamma \in \Gamma : d(x, \gamma x) \le \varepsilon\}$. Since Γ is discrete, this set is finite.; in the following, let .

2.9 Proposition. *There exist $k > 0$ and $m \in \mathbb{N}$ such that if $\varepsilon = \varepsilon'/k$, and if the finitely many elements of $F_\varepsilon(x)$ are labelled $\{\gamma_1, ..., \gamma_h\}$, then for all $\gamma \in \Gamma_\varepsilon(x)$ there exists $\tilde{\gamma} = \gamma_{j_1} \cdots \gamma_{j_l}$ (l dependent on γ) with $l \le m$ such that $\gamma G = \tilde{\gamma} G$.*

2.10 Corollary (Kazhdan-Margulis). *The group G is of finite index in $\Gamma_\varepsilon(x)$.*

Proof. $[\Gamma_\varepsilon(x) : G] \le$ (the number of choices of at most m elements of $\{\gamma_1, ..., \gamma_h\}$ with repetition) (count the cosets). ∎

We shall begin by finding the correct m.

2.11 Lemma. *There exists $m \in \mathbb{N}$ such that for all $S \subseteq O(T_x\mathbb{H}^3)$ such that $|S| \ge m$, there exist $A, B \in S$ such that $A \ne B$ and $\alpha_x(A, B) \le \varepsilon'/2$.*

Proof. The metric α_x induces the usual (compact) topology on $O(T_x\mathbb{H}^3) \simeq O(3)$. Suppose that for each $n \in \mathbb{N}$ we may find $S_n \subseteq O(3)$ with $|S_n| \ge n$ such that for every pair of distinct points $A, B \in S_n$, $\alpha_x(A, B) > \varepsilon'/2$. Let $\mathcal{U}$ be the open cover of $O(3)$ consisting of balls $B(\varepsilon'/4, A)$ for $A \in O(3)$; by compactness, finitely many of the balls are enough to cover $O(3)$, say $B_1 = B(\varepsilon'/4, A_1), ..., B_r = B(\varepsilon'/4, A_r)$. Consider S_{r+1}. Observe that each element of S_r must lie in a different B_i; and lo! we have contradicted the pigeon-hole principle. ∎

Our main estimate for the proof of Proposition 2.9 is the following.

2.12 Lemma. *There exists a constant $k > 0$ such that if $\varepsilon = \varepsilon'/k$, if the elements of $F_\varepsilon(x)$ are labelled $\{\gamma_1, ..., \gamma_h\}$, and if*

$$\eta = \gamma_{i_1} \cdots \gamma_{i_p}$$
$$\nu = \gamma_{j_1} \cdots \gamma_{j_q}$$

are any nontrivial elements satisfying the conditions

- *$p + q \le m + 1$, and*
- *$\alpha_x(P_{\eta\nu x, x} \circ d_x(\eta\nu), P_{\nu x, x} \circ d_x\nu) \le \varepsilon'/2$*

then $\left\|\nu^{-1}\eta\nu\right\| \le \varepsilon'$.

We deduce the proposition first, and then prove the lemma.

Proof of Proposition 2.9. Let $k > 0$ be the universal constant whose existence is postulated in Lemma 2.12, and let $\varepsilon = \varepsilon'/k$.

Let $\gamma \in \Gamma_\varepsilon(x)$, and suppose $\gamma_{i_1} \cdots \gamma_{i_l}$ is a minimal length word for γ in the generators $F_\varepsilon(x)$; we may assume that $l \le m + 1$ (or else there is nothing to prove). For each $s \in \{0, ..., m\}$, define

$$\theta_s = \gamma_{i_{l-s}} \gamma_{i_{l-s+1}} \cdots \gamma_{i_l}$$

and construct the set

$$\left\{P_{\theta_s x,x} \circ d_x\theta_s \,:\, s \in \{0, ..., m\}\right\}.$$

By Lemma 2.11, there exist $s, t \in \{0, ..., m\}$, with $s < t$, such that

$$\alpha_x(P_{\theta_s x,x} \circ d_x\theta_s, P_{\theta_t x,x} \circ d_x\theta_t) \leq \varepsilon'/2.$$

Ergo, if we define

$$\mu := \gamma_{i_1} \cdots \gamma_{i_{l-t-1}} = \gamma\theta_t^{-1}$$
$$\eta := \gamma_{i_{l-t}} \cdots \gamma_{i_{l-s-1}} = \theta_t\theta_s^{-1}$$
$$\nu := \gamma_{i_{l-s}} \cdots \gamma_{i_l} = \theta_s$$

then η and ν satisfy the cleverly constructed conditions of Lemma 2.12 and so

$$\left\|\nu^{-1}\eta\nu\right\|_x \leq \varepsilon' \;\Rightarrow\; \nu^{-1}\eta\nu \in G$$

(it is in the defining generating set for G). Then:

$$\gamma G = \mu\eta\nu G = \mu\eta\nu(\nu^{-1}\eta\nu)^{-1}G = \mu\nu G$$

and $\mu\nu$ is a word of strictly lower length than γ. Either the minimal length of $\mu\nu$ is at most m and we are done, or it is not and we may iterate the same argument but with $\mu\nu$ instead of γ; we will be done in finitely many steps, which proves the proposition. ∎

To complete the proof of the Kazhdan-Margulis lemma it therefore suffices to prove Lemma 2.12. We need one final geometric lemma on curvature, which we shall not prove:

2.13 Lemma. *If $D \subseteq M$ is a precompact domain with smooth boundary in an oriented Riemann surface M, then*

$$\theta(v, P_{\partial D}v) = \int_D \kappa(x)\, dm(x)$$

where $\kappa(x)$ is the sectional curvature of M at x and $dm(x)$ is the area form at x. ∎

Proof of Lemma 2.12. We wish to find k to bound $\left\|\nu^{-1}\eta\nu\right\|_x$ by ε'; there are two estimates needed. Suppose in the following that $\{\gamma_1, ..., \gamma_h\}$ are the elements of $F_{\varepsilon'/k}(x)$ for some $k > 0$.

Bounding of $d(x, \nu^{-1}\eta\nu x)$. We compute:

$$d(x, \nu^{-1}\eta\nu x) = d(\nu x, \eta\nu x) \leq d(x, \nu x) + d(x, \eta\nu x)$$

Bounding the two terms on the right, we obtain

$$\begin{aligned} d(x, \nu x) &= d(x, \gamma_{j_1} \cdots \gamma_{j_q} x) \\ &\leq d(x, \gamma_{j_1} x) + d(\gamma_{j_1} x, \gamma_{j_1} \cdots \gamma_{j_q} x) \\ &\leq \varepsilon'/k + d(x, \gamma_{j_2} \cdots \gamma_{j_q} x) \\ &\leq \cdots && \text{(induction)} \\ &\leq q\varepsilon'/k; \\ d(x, \eta\nu x) &\leq (p+q)\varepsilon'/k && \text{(same argument).} \end{aligned} \tag{2.14}$$

Thus $d(x, \nu^{-1}\eta\nu x) \leq (p+2q)\varepsilon'/k$; by assumption, $p + q \leq m + 1$ and so $p + 2q \leq 2(m+1)$, and so $d(x, \nu^{-1}\eta\nu x) \leq 2(m+1)\varepsilon'/k$. Hence if $k \geq 2(m+1)$, then $d(x, \nu^{-1}\eta\nu x) \leq \varepsilon'$.

Bounding of $\alpha_x(I, P_{\nu^{-1}\eta\nu x,x} \circ d_x \nu^{-1}\eta\nu)$**.** By the same argument as Eq. (2.6), we have that

$$\alpha_x(I, P_{\nu^{-1}\eta\nu x,x} \circ d_x \nu^{-1}\eta\nu) = \alpha_{\nu x}(I, P_{\eta(\nu x),\nu x} \circ d_{\nu x}\eta).$$

We concentrate on bounding the form of the quantity on the right.

By hypothesis, we have that

$$\begin{aligned} \varepsilon'/2 &\geq \alpha_x(P_{\eta\nu x,x} \circ d_x(\eta\nu), P_{\nu x,x} \circ d_x \nu) \\ &= \alpha_x(P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_x(\eta\nu), d_x\nu) \\ &= \alpha_x(P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta \circ d_x \nu, d_x\nu) \\ &= \alpha_{\nu x}(P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta, I). \end{aligned}$$

Consider the triangle Δ with vertices $x, \nu x, \eta\nu x$. By Lemma 2.13,

$$\theta(v, P_{\eta\nu x,\nu x} \circ P_{x,\eta\nu x,x} \circ P_{\nu x,x}) = \theta(v, P_{\partial\Delta} v) = -\operatorname{Area}\Delta$$

and so the parallel transport

$$\varphi := P_{\eta\nu x,\nu x} \circ P_{x,\eta\nu x,x} \circ P_{\nu x,x})$$

is a rotation of angle Area Δ in the hyperbolic plane spanned by Δ.

If $v \in T_{\nu x}\mathbb{H}^3$, then

$$\begin{aligned} \theta(P_{\eta\nu x,\nu x} \circ d_{\nu x}\eta v, v) &= \theta(\varphi \circ P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta v, v) \\ &\leq \theta(\varphi \circ P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta v, P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta v) \\ &\qquad + \theta(P_{x,\nu x} \circ P_{\eta\nu x,x} \circ d_{\nu x}\eta v, v) \\ &\leq \operatorname{Area}(\Delta) + \varepsilon'/2. \end{aligned}$$

Recall now that there exists λ such that for any triangle Δ in $\mathbb{H}^2$ with side lengths a, b, c, $\operatorname{Area}(\Delta) \leq \lambda \max\{a, b, c\}$. (Indeed, this is true for Euclidean triangles, and there is bounded comparison between Euclidean and hyperbolic areas of triangles with non-ideal vertices in the ball model.) In particular,

$$\operatorname{Area}(\Delta) \geq \lambda \max\{d(x, \nu x), d(x, \eta\nu x), d(\nu x, \eta\nu x)\} \leq \lambda(d(x, \nu x) + d(x, \eta\nu x))$$

(by the triangle inequality); and in Eq. (2.14), we bounded the final quantity by $\lambda 2(m+1)\varepsilon'/k$. Hence

$$\theta(P_{\eta\nu x,\nu x} \circ d_{\nu x}\eta v, v) \leq \lambda 2(m+1)\varepsilon'/k + \varepsilon'/2$$

and so if $k \geq 4\lambda(m+1)$ then

$$\alpha_x(I, P_{\nu^{-1}\eta\nu x,x} \circ d_x \nu^{-1}\eta\nu) = \alpha_{\nu x}(I, P_{\eta(\nu x),\nu x} \circ d_{\nu x}\eta) = \max_{v \in T_{\nu x}\mathbb{H}^3} \theta(P_{\eta\nu x,\nu x} \circ d_{\nu x}\eta v, v) \leq \varepsilon'.$$

Picking k to be the maximum of the two bounds obtained completes the proof of the lemma, and hence of the Kazhdan-Margulis lemma. ∎

§2.2. The thick-thin decomposition

There are now two philosophical approaches to take: we can work with "thin parts" of a manifold M and then study the lifts of these parts to $\mathbb{H}^3$ where our groups act as deck transformations (this is the Thurstonesque approach, and is taken in section 4.5 of his book [27]), or we can work with a group Γ acting in $\mathbb{H}^3$ and move to the thin parts of the manifold $\mathbb{H}^3/\Gamma$ via projection (this second approach is more group-theoretic in flavour). Because our interests are primarily with homotopy classes of geodesics, it is preferrable to work in the second setting — that is, our basic objects will lie in $\mathbb{H}^3$, not in our manifolds. (We stress that the difference between the approaches is only philosophical and the two approaches are not only equivalent but technically easy to convert between.) In order to carry out this study, we follow section 4.13 of [9].

For the remainder of this section, we fix an elliptic-free Kleinian group Γ, denote by M the manifold $\mathbb{H}^3/\Gamma$, and let μ_3 be the third Margulis constant.

2.15 Definition. For $\varepsilon < \mu_3$, let Γ_ε denote the ε–**Margulis set**

$$\{x \in \mathbb{H}^3 \,:\, \exists_{\gamma\in\Gamma} d(x,\gamma x) \leq \varepsilon\}.$$

The idea is illustrated by Fig. 19.

2.16 Lemma. *1. The set Γ_ε is Γ-invariant.*

2. If U is a connected component of Γ_ε, then $\mathrm{Stab}_\Gamma U$ is elementary.

Proof. 1. Suppose $x \in \Gamma_\varepsilon$. Then $d(x,\gamma_1 x) \leq \varepsilon$ for some $\gamma_1 \in \Gamma$. Let $\gamma \in \Gamma$; then

$$d(\gamma x,(\gamma\gamma_1\gamma^{-1})\gamma x) = d(\gamma x,\gamma\gamma_1 x) = d(x,\gamma_1 x) \leq \varepsilon$$

so $\gamma\gamma_1\gamma^{-1}$ works as the 'ε-small translator' for γx. In particular, $\gamma x \in \Gamma_\varepsilon$.

2. We wish to apply Corollary 2.3; we show that if $x \in U$ and $\gamma \in \mathrm{Stab}_\Gamma U$ then γ is a product of elements in $F_\varepsilon(x)$.

∎

§3. Deformation spaces

In this lecture, we shall formalise the notion of a deformation space.

Also we will study Thurston's notes, chapter 10.

§4. The enumeration of simple closed curves

Let Γ be a finitely generated Kleinian group, generated by elements $e_1, ..., e_l$. We consider the simple closed curves on $\Omega(\Gamma)/\Gamma$. These correspond to paths in the cover $\Omega(\Gamma)$: if $\alpha \in \pi_1(\Omega(\Gamma)/\Gamma)$ is such a curve based at x, then it lifts to a curve based at a lift $\tilde{x}$ and terminating at $\alpha\tilde{x}$ (treating α as an element of the deck transformation group) which misses the other lattice points (Fig. 20)

Suppose that $\alpha \in \pi_1(\Omega(\Gamma)/\Gamma)$ is a lift of some element $\bar{\alpha} \in \Gamma$; write this element in terms of the generators of Γ, say as $\bar{\alpha} = e_{j_k}^{r_k} \cdots e_{j_2}^{r_2} e_{j_1}^{r_1}$. This provides a factorisation of the loop α in terms of lifts of the e_i which, in $\Omega(\Gamma)$, corresponds to writing the curve from $\tilde{x}$ to $\alpha\tilde{x}$ as a concetenation of the curves joining $\tilde{x}$, $e_{j_1}^{r_1}\tilde{x}$, $e_{j_2}^{r_2}e_{j_1}^{r_1}\tilde{x}$, etc. In sufficiently nice situations, we can use this property to obtain such a factorisation from a picture of the curve overlaid on a tiling like Fig. 20. For a nice example, we will consider the Euclidean torus T obtained from the group $G = \langle g, h\rangle$, where $g, h \in \mathrm{Isom}_{\mathrm{Euc.}}(\hat{\mathbb{C}})$ are given by $g = (z \longmapsto z+1)$ and $h = (z \longmapsto z+i)$. A fundamental domain for this action is the interior of the unit square. Pick $\tilde{x} = 0 \in \hat{\mathbb{C}}$; then g pairs the two horizontal edges and h the two vertical edges. Given any path from $\tilde{x}$ to any other point, not passing through any lattice point (i.e. any point of $\mathbb{Z}+\mathbb{Z}i$), we can homotope it to a path largely following the lattice lines but with small deformations

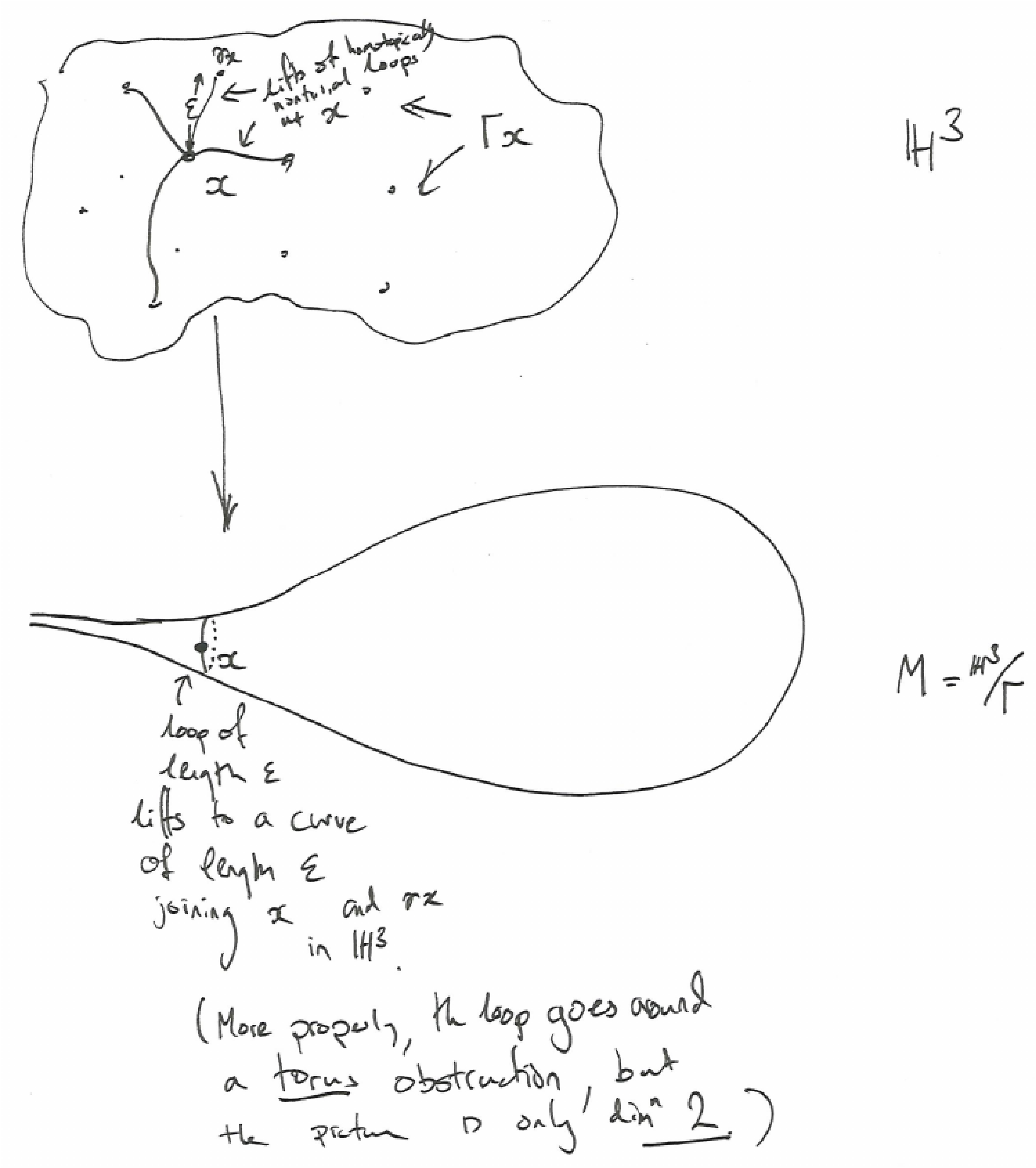


Figure 19: The loops based at a point $x \in \mathbb{H}^3/\Gamma$ and their lifts.

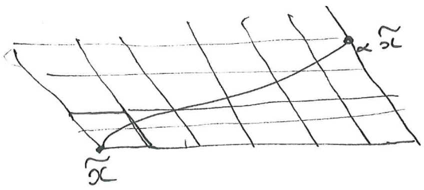


Figure 20: The lift of some $\alpha \in \pi_1(\Omega(\Gamma)/\Gamma)$.

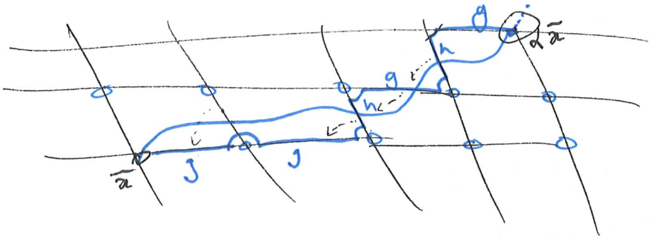


Figure 21: Deforming curves on T to their 'lattice' homotopy representatives in the cover.

around the lattice points. An example is shown in Fig. 21. Observe that the homotopy class of the lifts of $g \in G$ to $\pi_1(T)$ is represented by horizontal line segments, and similarly the homotopy class of the lifts of h is represented by vertical line segments.

We now move to the case of interest. Denote by $\mathcal{R}$ the Riley slice of Section 1.3; to fix notation, the 2-parabolic group corresponding to $\rho \in \mathbb{C}$ will be denoted by G_ρ, and the generators will be labelled

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } Y = \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix}.$$

Recall that a fundamental domain for G_ρ for $|\rho|$ sufficiently large is given by $\hat{\mathbb{C}}$ with two pairs of tangent deleted discs (Fig. 22). Since the properties of lifted curves are purely topological, and all 4-times punctured spheres are topologically equivalent, it is enough to consider the groups where this picture is qualitatively correct (i.e. the isometric circles of Y are contained in the open strip bounded by the vertical lines $z \in \mathbb{C} : \operatorname{Re} z = \pm 1/2$).

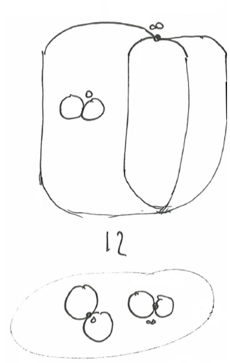


Figure 22: Two views of a fundamental domain for G_ρ with $|\rho|$ large enough.

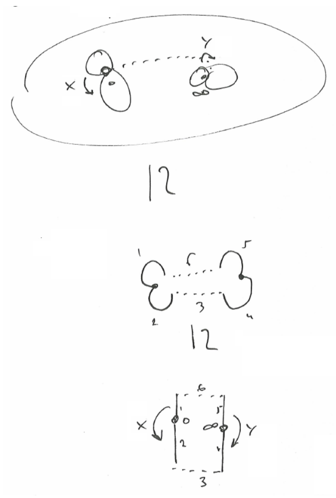


Figure 23: Cutting along the dotted line in the fundamental domain for G_ρ to obtain a fundamental hexagon for an extension group.

To produce a tessellation of the plane, we 'flatten' the domain. Geometrically, we cut along a curve joining 0 and ∞ that is contained within the fundamental domain; algebraically, this corresponds to adjoining to G_ρ a new element T which will glue together the two sides of this cut. The process of performing this cut geometrically is depicted in Fig. 23; observe that we obtain a hexagon. (In fact, we have a hyperbolic hexagon, where the indicated sides are paired by X and Y and where the dotted sides are paired by the adjoined transformation T.) This tiling gives a cover D of $\Omega(G_\rho)$, and a word in this tiling will project to a word in We then perform the homotopy to the retraction as above, remembering that we are working in a cover D of $\Omega(G_\rho)$ and so the Z factors in any expansion are killed upon taking the quotient.

We now make the following observations, which we prove following the papers [10, §2] and [11, §1]:

4.1 Lemma. *Let γ be any simple closed non-boundary-parallel loop on the four-times punctured sphere. (*Non-boundary-parallel *in this case is equivalent to 'separates pairs of punctures'.) Then γ is homotopic to a curve represented in D by a line with slope an element of $\hat{\mathbb{Q}}$. Conversely, every such line projects to such a curve. This correspondence is such that the curve in $\pi_1(S)$ induced by the generator of $\pi_1(W)$ corresponds to $\infty \in \hat{\mathbb{Q}}$.*

§5. The Keen–Series theory of pleating rays

§5.1. Fuchsian groups

A reference for results in this subsection is Chapter 8 of [3].

Recall, a Kleinian group G is **Fuchsian** if it is conjugate in $\mathrm{PSL}(2,\mathbb{C})$ to a subgroup of $\mathrm{PSL}(2,\mathbb{R})$. This is equivalent to any of the following conditions:

- There exists a disc $\Delta \subseteq \Omega(G)$ left invariant by G;
- There is a circle $C \subseteq \hat{\mathbb{C}}$ such that $\Lambda(G) \subseteq C$.

In any case, G acts as a hyperbolic isometry group on the natural hyperbolic metric of Δ; the circle C is then the sphere at infinity of this hyperbolic metric. Whenever G is said to be Fuchsian, it will always carry the data of the choice of Δ along with it implicitly.

We say that G is Fuchsian **of the first type** if, for every $x \in \overline{\Delta}$, every $\xi \in \partial\Delta$ is an accumulation point of Gx. Otherwise, we say that G is Fuchsian **of the second type**. Suppose G is of the second type; then $\partial\Delta$ is the disjoint union of $\Lambda(\Delta)$ together with a countable collection of mutually disjoint open arcs $\{\sigma_i\}_{i\in I}$ of S. For each $i \in I$, set H_i to be the hyperbolic half-plane in Δ bounded by σ_i and the hyperbolic geodesic with the same endpoints as σ_i; then the **Nielsen region** of G is the set

$$N(G) = \bigcap_{i\in I} H_i.$$

If G is Fuchsian of the first type, acting on the invariant disc Δ, then $N(G)$ is defined to be Δ itself.

5.1 Proposition. *$N(G)$ is the smallest non-empty G-invariant convex open subset of Δ.* ∎

We say that $h \in G$ is a **boundary hyperbolic element** if it leaves invariant one of the intervals σ_i. These are studied in Sections 10.3 and 10.4 of [3]; we recall the main results here. For the sake of language, if S is a hyperbolic surface then a **cylinder** on S is a boundary component corresponding to a deleted disc.

5.2 Proposition. *A finitely generated Fuchsian group G has finitely many conjugacy classes of maximal hyperbolic boundary elements*[3]*; and these conjugacy classes are in bijective correspondence with the cylinders of Δ/G.* ∎

§5.2. Pleating rays of $\mathcal{R}$

Fix some $\rho \in \mathcal{R}$. Recall that $\gamma(p/q)$ denotes the unique geodesic corresponding to $p/q \in \mathbb{Q}$, and that $W_{p/q}$ denotes the Farey word which represents that geodesic. The goal of this section is to show that the curve $\gamma(p/q)$ splits $\Omega(G_\rho)/G_\rho$ into two components, each a sphere with a deleted disc and two punctures. To this end, let $\mathcal{U}_{p/q}$ denote the set of subgroups of G_ρ generated by two parabolics, say u_1 and u_2, such that u_1u_2 lifts to $\gamma(p/q)$.

Observe that the groups of $\mathcal{U}_{p/q}$ are characterised by a purely topological property. Thus we

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[3] Recall that in a group G an element g is **maximal** if, whenever there is some $h \in G$ and $n \in \mathbb{Z}$ with $h^n = g$, the element h actually *is* g.

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