Connectedness of Hilbert schemes

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Contents

Fix an algebraically closed field k of characteristic zero.

§1. Recalling the Hilbert scheme

Let us very quickly recall the definition of the Hilbert scheme, following Harris and Morrison [4, §1B]. It is the scheme $\mathcal{H}_r^p$ which parameterises subschemes $X \subseteq \mathbb{P}^r$ with Hilbert polynomial $p_X = p$, and is a fine moduli space for the contravariant functor $\mathbf{Hilb}_r^p : \mathsf{Sch}^{\mathrm{op}} \to \mathsf{Set}$ which sends B to the set of proper flat families $\mathcal{X} \to B$ fitting into the diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{i} & \mathbb{P}^r \times B & \xrightarrow{\pi} & \mathbb{P}^r \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\pi} & & \\ & & B & & \end{array}$$

where each fibre of $\mathcal{X}$ has Hilbert polynomial p.

As we saw on Tuesday, the functor $\mathbf{Hilb}_r^p$ is representable by a projective scheme $\mathcal{H}_r^p$ (Grothendieck's theorem). We also know that the tangent space to $X \in \mathcal{H}_r^p$ is the space $H^0(X, \mathcal{N}_{X/\mathbb{P}^r})$ of global sections of the normal sheaf [4, §1C].

Notation. If $\mathfrak{a} \trianglelefteq k[x_0, ..., x_r]$ is an ideal with Hilbert polynomial p then write $[\mathfrak{a}]$ for the corresponding point in $\mathcal{H}_r^p$.

We will try to keep in mind the running example of the scheme parameterising twisted cubics; see [10], as well as [4, pp. 14–16]. Recall that a twisted cubic is the intersection of three cubic hypersurfaces and so is represented by the ideal

$$\mathfrak{a} = (YW - X^2, WZ - XY, XZ - Y^2) \in A = k[W, X, Y, Z].$$

One can show that the Hilbert function of $\mathfrak{a}$ is

$$h(t) = 3\binom{t+1}{3} - 2\binom{t}{3}$$

and that for $t \geq 0$ this agrees with the polynomial $p(t) = 3t + 1$. The corresponding Hilbert scheme $\mathcal{H}_3^p$ has two irreducible components, one 12-dimensional parameterising twisted cubics and one 15-dimensional parameterising unions of plane cubics and isolated points; the intersection consists of nodal plane cubics with embedded points at the node (detecting the infinitesimal direction of the deformation of a smooth cubic which lead to the node). Anyway the detailed study of this example will be left to a later lecture. We are primarily interested in analysing this example in terms of the natural $\mathrm{PGL}(4,k)$ action.

First observe that $\mathrm{PGL}(r+1,k)$ acts on the Hilbert scheme. Each component has a single open orbit, therefore there can be only finitely many other orbits. In general components and singular loci are unions of group orbits.

Let G be the Borel subgroup $G \leq \mathrm{PGL}(r+1,k)$ of upper triangular matrices. (General definition of **Borel** can be found in [11, §10.5] but we only need this one.) Let G act on the Hilbert scheme $\mathcal{H}$ containing a subscheme $X \subseteq \mathbb{P}^r$, by the Borel fixed point theorem there exists an element $\mathcal{H}$ which is fixed by G. In particular every subscheme of $\mathbb{P}^r$ has a flat specialisation which is fixed by G (of course not necessarily pointwise).

1.1 Lemma ([8, Proposition 2.3])**.** *An ideal $\mathfrak{a} \trianglelefteq k[x_0, \dots, x_n]$ is fixed by G iff both $\mathfrak{a}$ is a monomial ideal and for all $x^u \in \mathfrak{a}$ and all $x_i \mid x^u$, $(x_j/x_i)x^u \in \mathfrak{a}$ for all $j < i$.* ⅍⥂

1.2 Example. There are exactly three types of Borel-fixed points in the Hilbert scheme of twisted cubics:

1. a spacial double line;
2. a planar triple line with embedded point in the same plane;
3. a planar triple line with embedded point not lying in that plane.

Orbits (1) and (2) lie in the component with generic point the twisted cubic and (2) and (3) lie in the component with generic point a planar cusped cubic with embedded point. Hence since every component has a fixed point these must be a complete list of components.

Remark. Before continuing we must explain point (1) in the above example since it may be confusing. The **degree** of an n-dimensional subscheme $X \subseteq \mathbb{P}^r$ is defined to be $r!$ times the leading coefficient of the Hilbert polynomial p_X. One can show (see e.g. [3, §III.3]) that this is equal to the length of the intersection of X with a general plane of dimension $r-n$. In the case that X is reduced this is the same as the number of points of the intersection (which agrees with the classical definition of degree). Recall that the **length** of a zero-dimensional local ring $(R, \mathfrak{m})$ is the length ℓ of the shortest chain

$$0 = \mathfrak{m}_1 \trianglelefteq \mathfrak{m}_2 \trianglelefteq \cdots \leq \mathfrak{m}_\ell = \mathfrak{m} \trianglelefteq R$$

such that each module $\mathfrak{m}_i/\mathfrak{m}_{i-1} \simeq R/\mathfrak{m}$.

A spacial double line has ideal $(F,G)^2$ where F and G are linear in three variables (for the sake of argument we work in $\mathbb{A}^3$). The intersection of this line with a plane $H = 0$ is then defined by the ideal $\mathfrak{a} = (F^2, FG, G^2, H)$. The maximal ideal of $A/\mathfrak{a}$ (A the polynomial algebra) is the image of $\mathfrak{m} = (F, G, H)$, i.e. $\mathfrak{m}/\mathfrak{a} \trianglelefteq A/\mathfrak{a}$. We have $(F, G, H)/(F^2, FG, G^2, H) = (F, G)/(F, G)^2$, which has length 3: $0 \trianglelefteq (F)/(F, G)^2 \trianglelefteq (F, G)/(F, G)^2 \trianglelefteq A/\mathfrak{a}$.

§2. Gröbner bases

2.1 Theorem (Hilbert basis theorem)**.** *If R is a Noetherian ring then $R[X]$ is Noetherian.*

Proof. Suppose R is Noetherian, and let $\mathfrak{a}$ be an ideal in $R[X]$; we will show that $\mathfrak{a}$ is finitely generated.

We consider the sets of all the leading coefficients of each degree of polynomial in $\mathfrak{a}$. Form the sets $\mathfrak{b}_n \subseteq R$ for $n \in \mathbb{Z}_{\geq 0}$, defined as follows:

$$\mathfrak{b}_n := \{r \in R : \exists_{f \in \mathfrak{a}(n)} \text{ such that } \mathrm{LC}(f) = r\} \tag{2.2}$$

(here, $\mathfrak{a}(n)$ is the nth graded part of $\mathfrak{a}$ and $\mathrm{LC}(f)$ is the leading coefficent of f).

Exercise. Each $\mathfrak{b}_n$ is an ideal of R.

We now use the Noetherian property of R twice. First, we have ascending chain of ideals $\mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \cdots$ in R and by the ascending chain condition there exists some $N \in \mathbb{Z}_{\geq 0}$ such that for all $n \geq N$, $\mathfrak{b}_n = \mathfrak{b}_N$. Second, for each ideal $\mathfrak{b}_n$ such that $n \leq N$, by the Noetherian property there is a finite set of elements generating $\mathfrak{b}_n$. Let B_n be a finite set of generators for each $\mathfrak{b}_n$ ($0 \leq n \leq N$). For each n define a choice function $Ч_n : B_n \to \mathfrak{a}(n)$ which sends a leading coefficient $r \in \mathfrak{b}_n$ to some degree n polynomial with that leading coefficient. We can choose $Ч_0 : B_0 \to \mathfrak{a}(0)$ to be the identity map. Let $B = \bigcup_{n=0}^{N} Ч_n$. This set is finite.

We now show that the set B is sufficient to generate the entirety of $\mathfrak{a}$. Let $g \in \mathfrak{a}$ be arbitrary.

Firstly, note that if g is constant then g is generated by B. (Indeed, if $g = 0$ this is trivial; and otherwise, $g \in \mathfrak{a}(0)$, which is equal to its own ideal of leading coefficients $\mathfrak{b}_0$, which is generated by $Ч_0(\mathfrak{b}_0) \subseteq B$.)

Now we proceed by induction on the degree of g. Pick k to the smallest integer such that $\mathrm{LC}(g) \in \mathfrak{b}_k$; so $k \leq N$. There exist elements $c_1, \ldots, c_\ell \in B_k$ such that $\mathrm{LC}(g) = c_1 + \cdots + c_\ell$ and so $\mathrm{LC}(g) = \mathrm{LC}(Ч_k(c_1) + \cdots + Ч_k(c_\ell))$. Let $h = Ч_k(c_1) + \cdots + Ч_k(c_\ell)$, so h is a sum of elements of B. Further, $\partial(g - h) < \partial g$ (since the leading coefficients cancel) and so by the inductive hypothesis $(g - h)$ is generated by B. We therefore may conclude that $g = h + (g - h)$ is generated by B. 𓅃𓍿𓎡

In the proof, the important property of the generating set B was that the set $\mathrm{LC}(B) = \bigcup B_n$ generates the ideal generated by the $\mathfrak{b}_n$, which is in particular the ideal generated by all leading coefficients of elements of $\mathfrak{a}$. We are lead to consider the following definition in somewhat more generality.

2.3 Definition. Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A **monomial order** $>$ on A is any well-ordering (i.e. total order with d.c.c.) on the set of monomials of A which is compatible with multiplication, i.e. if f, g, h are monomials then $f > g$ implies $fh > gh$. We write $\mathrm{in}_>(S)$ for the set of leading terms of the elements of $S \subseteq A$ with respect to the order $>$.

A **Gröbner basis** with respect to $>$ for an ideal $\mathfrak{a} \trianglelefteq A$ is a finite subset $B \subset \mathfrak{a}$ such that (i) $\mathfrak{a}$ is generated by B and (ii) $\mathrm{in}_>(\mathfrak{a})$ is generated by $\mathrm{in}_>(B)$.

Remark. Observe that an adaptation of the final part of the above proof of the Hilbert basis theorem shows that condition (i) in the definition of a Gröbner basis is redundant (c.f. [1, §1.3, Exercise 3]).

2.4 Example (Lexicographic order)**.** We will usually work with the **lexicographic order**: if $\alpha, \beta \in \mathbb{Z}^n_{\geq 0}$ are the exponent vectors of two monomials $X^\alpha, X^\beta \in A$ then we say $X^\alpha > X^\beta$ iff the left-most non-zero component of $\alpha - \beta$ is positive. (Note this depends on having an order on the variables of A. We will usually choose the order $X > Y > Z$ or $X_1 > X_2 > \cdots$.)

2.5 Example. Let C be the twisted cubic in $\mathbb{A}^3_k$ that is the intersection of the three quadric surfaces defined by the vanishing of

$$f = Y - X^2, g = Z - XY, h = XZ - Y^2 \in A = k[X, Y, Z].$$

The leading coefficients under the lexicographic order are $-X^2$, XY, and XZ respectively. In particular we see that every element of the ideal generated by the leading coefficients of f, g, and h is divisible by X. But $k \coloneqq Zg + Yh = Z^2 - Y^3$ has leading coefficient Y^3, i.e. $Y^2 \in \mathrm{in}(f, g, h)$ is not in $(\mathrm{in}(f), \mathrm{in}(g), \mathrm{in}(h))$. In particular $\{f, g, h\}$ is not a Gröbner basis for (f, g, h). However, k is in some sense the only obstruction: one can show that $\{f, g, h, k\}$ *is* a Gröbner basis using Buchberger's criterion. (Roughly speaking Buchberger's algorithm for constructing a Gröbner basis goes via constructing obstructions in this way—here k is the so-called S-polynomial $S(g, h)$.)

Given $A = k[x_1, \dots, x_n]$ as above and a vector $w \in \mathbb{Z}^n$ (called a **weight**) we can define a partial order $>_w$ on the monomials of A by $x^a >_w x^b$ iff $w \cdot a > w \cdot b$; this is called a **weight order**. The **lexicographic product order** of a sequence $w_1, \dots, w_k$ of weights is the monomial order defined by $x^a > x^b$ if $x^a >_{w_i} x^b$ for the first i such that x^a and x^b are comparable with respect to $>_{w_i}$. In particular the lexicographic order is the product of $e_1, \dots, e_n$. We extend the notation in to weight orders: given a weight w define $\mathrm{in}_w(f)$ to be the sum of all terms of f maximal with respect to $>_w$.

§3. Connectedness of the Hilbert scheme

From now on we always use the same lexicographic order we introduced at the end of the previous section. We also fix a global polynomial algebra $A = k[x_0, \dots, x_n]$ (with the usual grading).

3.1 Definition. A **lexicographic ideal** is a monomial ideal $\mathfrak{a}$ such that the dth graded part $\mathfrak{a}(d)$ is spanned by the first $\dim_k \mathfrak{a}(d)$ monomials in the lexicographic order.

3.2 Theorem (Macaulay (1927), [6])**.** *For every graded ideal $\mathfrak{a} \trianglelefteq A$, there exists a unique lexicographic ideal $\mathfrak{a}_{\mathrm{lex.}}$ with the same Hilbert polynomial.* ⅅ⥃

We will, in this section, prove the following theorem of Hartshorne [5]:

3.3 Theorem. *For any p and r, the Hilbert scheme $\mathcal{H}^p_r$ is connected.*

The basic scheme(!) for showing connectedness is to give a path in the Hilbert scheme from any ideal to the unique lexicographic ideal. This can be done in different ways, but we will follow the proof of Peeva and Stillman [9, 7]. This proof goes via showing that for each X there is a chain of curves on $\mathcal{H}^p_r$, $C_1, \dots, C_p$, such that (i) $X \in C_1$, (ii) each $C_i \cap C_{i+1} \neq \varnothing$, and (iii) $X_{\mathrm{lex.}} \in C_p$ (i.e. $X_{\mathrm{lex.}}$ the scheme corresponding to the unique lexicographic ideal).

§3.1. Gröbner deformations

The point is to define for every homogeneous ideal $\mathfrak{a}$ a natural 1-parameter family over $\mathbb{A}^1$ such that the generic fibre of the family is $\mathfrak{a}$ but the special fibre is $\mathrm{in}_w(\mathfrak{a})$, for some weight vector $w \in \mathbb{Z}_{\geq 0}$. In order to do this we construct the ideal

$$\tilde{\mathfrak{a}} := (\tilde{f} \,:\, f \in \mathfrak{a}) \trianglelefteq A[t]$$

where

$$\widetilde{\sum_u f_u x^u} := t^{\max_u w \cdot u} \sum_u f_u t^{-w \cdot u} x^u$$

(this is cooked up so that $\mathrm{in}_w f$ is exactly $\tilde{(f)}|_{t=0}$).

3.4 Lemma ([2, Exercise 15.25])**.** *If $\{f_1, ..., f_r\}$ is a Groebner basis for $\mathfrak{a}$, then $\tilde{\mathfrak{a}} = (\tilde{f}_1, ..., \tilde{f}_r)$. In fact this is even a Gröbner basis with respect to $\gg$ defined by $x_i \gg t$ for all i and $>$ in the x_i's.*

The ideal $\tilde{\mathfrak{a}}$ defines a subscheme of $\mathbb{P}^n \times \mathbb{A}^1$, and we get a diagram

$$\begin{array}{ccc} \operatorname{Proj} A[t]/\tilde{\mathfrak{a}} & \hookrightarrow & \mathbb{P}^n \times \mathbb{A}^1 \\ \downarrow & & \\ \mathbb{A}^1 & & \end{array}$$

(where the grading on $A[t]$ is the same as the grading on A but with $\deg t := 0$) induced by the inclusion $k[t] \subseteq A[t]$ (note that this is indeed an inclusion since no polynomial in t alone is killed by the quotient). This is a flat family with generic fibre over $t \neq 0$ equal to $\operatorname{Proj} A/\mathfrak{a}$ and special fibre over $t = 0$ equal to $\operatorname{Proj} A/\mathrm{in}_w(\mathfrak{a})$ [2, Theorem 15.17].

3.5 Example. Let $F = XZ - Y^2$ be the conic tangent to $X = 0$ and $Z = 0$ with axis $Y = 0$. The initial term is XZ. Pick the weight $w = (1, 0, 0)$, and then $\tilde{F} = t(t^{-1}XZ - t^{-0}Y^2) = XZ$, so the limit is $\mathbf{Z}(X) \cup \mathbf{Z}(Z)$.

One should think of $(1, 0, 0)$ as encoding the repulsion of $\mathbf{Z}(F)$ from the point $Y = Z = 0$.

3.6 Example. Take again the twisted cubic defined by the vanishing of

$$f = WY - X^2, g = WZ - XY, h = XZ - WY^2 \in A = k[X, Y, Z].$$

We saw that a Gröbner basis for this ideal is

$$\{f = WY - X^2, g = WZ - XY, h = XZ - Y^2, k = WZ^2 - Y^3\}.$$

Pick the weight $(0, 1, 2, 10)$. Then we have

$$\tilde{f} = WY - X^2, \tilde{g} = WZ - t^7XY, \tilde{h} = XZ - t^7Y^2, \tilde{k} = WZ^2 - t^{14}Y^3.$$

In the limit we have $WY = X^2, WZ = 0, XZ = 0$, and $WZ^2 = 0$. If $W = 1$ then $Y = X^2$ and $Z = 0$ (so a union of a conic and a line). If $W = 0$ then we have $X^2 = 0$ and $XZ = 0$ (so around $Z = 0$ we have an infinitesimal point).

§3.2. First degenerating to a Borel-fixed point

3.7 Proposition. *Let $\mathfrak{a} \trianglelefteq A$ be a fixed homogenous ideal and let $w \in \mathbb{Z}^{n+1}$ be a general weight. Then there is an open set $U \subseteq \mathrm{GL}(n+1,k)$ such that $\mathrm{in}_w(g\mathfrak{a})$ is constant for $g \in U$, and this initial ideal is fixed by the action of G if $w_0 > \cdots > w_n$.*

Proof. This is the combination of [2, Theorems 15.18 and 15.20]. The ideal $\mathrm{gin}_w :=(\mathrm{in}_w(g\mathfrak{a}))$ is called the **generic initial ideal** of $\mathfrak{a}$ with respect to w. ⅅ⥄

For any $\mathfrak{a}$ cutting out a scheme $[\mathfrak{a}] \in \mathcal{H}_r^p$ there exists a path in $U \subseteq \mathrm{GL}(n+1,k)$ from 1 to g and hence a path in $\mathcal{H}_r^p$ from $[\mathfrak{a}]$ to the scheme $[g\mathfrak{a}]$. More precisely we have a one-parameter family $\mathbb{A}^1 \to U \to \mathcal{H}_r^p$ and in particular we have that $[g\mathfrak{a}]$ is in the same connected component as $[\mathfrak{a}]$. Now we can take the Gröbner degeneration from $g\mathfrak{a}$ to $\mathrm{in}_w(g\mathfrak{a})$ for weight w, and this is Borel-fixed. In total then we have a path from $[\mathfrak{a}]$ to a Borel-fixed point.

§3.3. Then degenerating to the lexicographic ideal

Let us supose we sit at a Borel fixed ideal $\mathfrak{a}$ which is not lexicographic. We recall that the lexicographic ideal has the property that the list of monomials of degree d in lexicographic order starts at the maximal monomial and then heads down without gaps. We will write an ideal $\mathfrak{b}$ which is 'closer' to this than $\mathfrak{a}$, in the sense of **??**.

To do this, let d be the smallest degree in which $\mathfrak{a}(d)$ is not the top segment of monomials in the lexicographic order. Let $m \in \mathfrak{a}_{\text{lex.}}(d) \setminus \mathfrak{a}(d)$ be the largest monomial in lexicographic order which is 'missing', and let f be the largest monomial in $\mathfrak{a}(d)$ whch is below m. (This is the setup of Definition 4.1 of [9].)

3.8 Lemma. *Every monomial ideal $\mathfrak{a}$ has a unique minimal (finite) set of monomial generators* $\mathrm{mg}(\mathfrak{a})$ *[8, Lemma 1.2]. If f is below m in a graded piece of a Borel-fixed ideal as above, then f is a minimal generator [9, Lemma 2.3].* ⅅ⥄

We now replace f in the generating set of $\mathfrak{a}$ by $f - m$. We also need to replace certain other generators by some modifications of $f - m$ to ensure that f is gone from $\mathfrak{a}$; the actual ideal $\mathfrak{b}$ which we end up with is the ideal

$$\mathfrak{b} = \mathrm{gin}(\mathrm{in}(N))$$

where N is the binomial ideal

$$(\{gu - nu\}, \mathrm{mg}\,\mathfrak{a} \setminus \{gu\})\,;$$

here the gu range over certain modifications of f and nu are certain modifications of m, defined combinatorially in [9, Construction 4.3].

3.9 Lemma. *Recall that if $\mathfrak{a} \trianglelefteq A$ then its saturation is*

$$\overline{a} := \{f \in A \,:\, \exists_{u\in\mathbb{Z}}\forall_i x_i^u f \in \mathfrak{a}\}.$$

1. *The saturation of a Borel-fixed ideal is Borel-fixed.*
2. *There are only finitely many saturated Borel-fixed ideals with a given Hilbert polynomial.* ⅅ⥄

Replace $\mathfrak{b}$ with the saturation of its generic initial ideal. This is Borel-fixed by (1) of Lemma 3.9 and is closer to $\mathfrak{a}_{\text{lex.}}$ then $\mathfrak{a}$. This process can be iterated and terminates after finitely many steps by (2) of Lemma 3.9; each step is also a Groebner deformation, which gives us the desired connectedness [9, Proposition 4.13].

3.10 Example. The lexicographic ideal in $\mathcal{H}_3^{3t+1}$ defines a planar triple line with coplanar embedded point.

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