MATHS 782

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§1. Surfaces

1. Let $\mathcal{R}$ be a parallelogram in $\mathbb{R}^2$. Let f and g be affine translations which map edges to their opposite edges, and let $G = \langle f, g\rangle$ so that $G\mathcal{R}$ is a tiling of $\mathbb{R}^2$. Prove directly (i.e. without appealing to the Poincaré polyhedron theorem or ping pong lemma) that:

   (a) there is a homeomorphism between $\mathbb{T} = \mathcal{R}/\sim$ and $\mathbb{R}^2/G$, where $\sim$ is the equivalence relation "$x \sim y$ if and only if either $x = y$ or x and y both lie on the boundary of $\mathcal{R}$ and $x = f(y)$ or $x = g(y)$".

   (b) a presentation for G is $\langle f, g : [f, g] = 1\rangle$.

2. We consider tilings.[1]

   (a) Show that the only edge-to-edge tilings of $\mathbb{R}^2$ by regular polygons (where each tile is congruent to every other tile) are the obvious ones (by triangles, hexagons, and squares).

   (b) Show that in $\mathbb{H}^2$ every regular n-gon with $n > 4$ tiles the plane, and compute the internal angle sum of each n-gon.

   (c) We return to the Euclidean plane, and we consider edge-to-edge tilings with regular polygons where the congruence classes of the tiles are drawn from some finite set (i.e. we allow tiles of possibly different shapes).

      i. Let v be a vertex of the tiling of valence s. Suppose that the incident tiles are an n_1-gon, an n_2-gon, ..., an n_s-gon. Show that the equation

$$\frac{n_1 - 2}{n_1} + \cdots + \frac{n_s - 2}{n_s} = 2$$

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[1] See §2.1 of B. Grünbaum and G.C. Shephard, *Tilings and Patterns.* W.H. Freeman and Co., 1987.

holds, and that there are exactly 17 integer solutions for the $n_1, ..., n_s$ (here are three for free: $\{6,6,6\}$, $\{4,4,4,4\}$ and $\{3,3,3,3,3,3\}$). We call the particular cyclic ordering of $n_1, ..., n_s$ at v the **type** of v (defined up to cyclic order of course), and there are 21 possible types (some of the 17 integer solutions appear in nontrivially different orders).

ii. An **Archimedian tiling** is an edge-to-edge tilings with regular polygons where the congruence classes of the tiles are drawn from some finite set and where all vertices are the same species. Show that (i) if you have a vertex drawn from ten of the 21 types then the resulting arrangement cannot be extended to an Archimedian tiling of the plane; (ii) prove that all of the remaining eleven can be extended (warning: it is not enough to draw a picture!).

3. Let $Q \in \mathbb{R}^2$ and consider the circle C in $\mathbb{R}^2$ with centre P and radius R. The **reflection**[2] Q' of Q in the circle is the unique point such that

$$||Q - P|| \cdot ||Q' - P|| = R^2.$$

(a) Show that the reflection is well-defined and gives a homeomorphism from the plane minus P to itself.

(b) Show that the reflection of a circle D in C is either a circle (if D does not pass through P) or a line (if D passes through P).

(c) Prove that if two circles D, D' intersect at some angle θ then their reflections in C also intersect at the angle θ.

(d) Show that any two non-intersecting circles lie in a family of disjoint circles which fill the plane and whose centres lie on the line joining the two centres of the starting circles.

(e) Show that any two non-intersecting circles can be moved by reflection in a suitable third circle to become a pair of concentric circles. [Hint: take a circle centred at one of the limiting points of the pencil from the previous part.]

(f) Prove *Steiner's porism*: suppose you have two non-concentric circles C_1, C_2, one inside the other, and suppose there exists a finite chain $D_1, ..., D_n$ of circles such that each D_i is tangent to C_1 and C_2, and to D_{i+1} and D_{i-1} (subscripts taken mod n, so $D_{n+1} = D_1$. Then if D_1' is any circle tangent to both C_1 and C_2 it can be extended to a circle chain which is mutually tangent to both C_1 and C_2 and is cyclic.

[2] See H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*. Mathematical Association of America, 1967.

4. Let $M \in \mathrm{PSL}(2,\mathbb{R})$.
   (a) Show that M has at most two eigenvectors when acting on $\mathbb{C}^2$ and that w.l.o.g. we may assume that neither lies in the subspace $\{(x,0)^t : x \in \mathbb{C}\}$.
   (b) Show that if M has a fixed point at $z \neq \infty$ then $(z,1)^t$ is an eigenvector.
   (c) Show that M has a unique eigenvector $(z,1)^t$ if and only if $(\operatorname{tr} M)^2 = 4$.
   (d) Deduce also that if M has two eigenvectors $(z,1)^t$ and $(w,1)^t$ then
      - $w, z \in \mathbb{R}$ if and only if $\operatorname{tr}^2 M < 4$
      - $w, z \in \mathbb{C} \setminus \mathbb{R}$ if and only if $\operatorname{tr}^2 M > 4$.
   (e) Show that every element of $\mathrm{PSL}(2,\mathbb{R})$ is conjugate to $z \longmapsto z+1$ (if it is parabolic), or $z \longmapsto \lambda z$ for some $\lambda \in \mathbb{R}^*$ (if it is hyperbolic), or an element of $O(2)$ (if it is elliptic).
   (f) Show that if a subgroup of $\mathrm{PSL}(2,\mathbb{R})$ contains two parabolic elements f, g with a shared fixed point then it is either indiscrete or there exists some h such that $h^n = f$ and $h^m = g$.
5. Show that if G is a free group then G cannot be the holonomy group of a compact surface.
6. Results used in the proof of the classification of surfaces:
   (a) If X is the triangulated surface, T is a maximal tree in the edges of the triangulation, and Γ is the dual to T, then Γ is connected.
   (b) If G is a connected graph, then $\chi(G) \leq 1$ and this is sharp if and only if G is a tree.
   (c) (Not used in the proof, but deduce it:) if T triangulates the compact surface X then $\chi(X) = 2 - 2g$.
7. Show that $\mathrm{SL}(2,\mathbb{Z})$ is generated by the linear transformations $S(z) = z+1$, $T(z) = -1/z$. Hint: use row and column operations and the Euclidean algorithm. Hence observe that $\mathrm{PSL}(2,\mathbb{Z})$ tiles $\mathbb{H}^2$ by the hyperbolic quadrilateral with vertices ∞, i, $(1+i\sqrt{3})/2$, $1+i$. By similar methods to the lecture (i.e. considering cycles in the dual graph to the tiling) deduce that $\mathrm{PSL}(2,\mathbb{Z}) = \langle S, T : S^2 = (ST)^3 = 1 \rangle$.