Uniformisation, equivariance, and vanishing—Three kinds of functions hanging around your Riemann surface

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Abstract

Riemann surfaces have approximately three definitions: (GEOMETRIC) they are quotients of domains by group actions; (ANALYTIC) they are surfaces with complex charts that allow the definition of multivalued functions; and (ALGEBRAIC) they are complex varieties of dimension 1. The trifurcation of the theory into these three mountains happened very early on in their history, and often mathematicians tend to live on only one of the three peaks. We will explain the geometric uniformisation theory of Riemann surfaces from a very classical (pre-Thurston) viewpoint that requires minimal background (just elementary complex analysis and vague knowledge of what the hyperbolic plane looks like) and in the process we will climb high enough up the mountain that the other two summits are visible in the distance.

Ngauruhoe and Tongariro from Ruapehu (colourised). Whites Aviation (1947).
Peter Alsop, "Wonderland", p. 70. Potton & Burton (2020).

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§0. Elliptic integrals

We will take a point of view that is ultra-classical, but we make no attempt to be historically accurate. The history of early function theory is quite complicated, and is the subject of the excellent textbook by Bottazzini and Gray [12]. As well as the standard historical sources which you can look at by Poincaré [51] and Riemann [54], there is a nice modern monograph by Prasolov and Solovyev [52].

Elliptic integrals are named after arc lengths of ellipses, but to make things easy we look to an *a priori* more complicated curve for our motivation.

0.1 Example. Fix two points A, B in the plane, let $a = d(A, B)/2$. The **lemniscate of Bernoulli** is the locus of points P such that $d(A, P)d(P, B) = a^2$ [15, p. 18]; setting $a = 1$, one can show that the arc length from $(0, 0)$ to $(x(r), y(r))$ is given by the integral

$$\alpha(r) = \int_0^r \frac{dt}{\sqrt{1 - t^4}}.$$

This is the simplest example of an elliptic integral. Observe that the equivalent problem for the circle is to compute the length of the unit circle *centred at* $(1,0)$ cut out by a chord of length r; this is in fact one way to define $\arcsin r$ (hence why there is an 'arc' in 'arcsin'!) and motivates one to look at α^{-1} rather than α since it is plausible it might have quasi-trigonometric properties.

Consider the differential form

$$dz = \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}};$$

this is double-valued on $\mathbb{C}$, with branch points at ± 1 and $\pm 1/k$. It corresponds to the elliptic integral

$$\zeta(\xi) = \int_0^\xi dz = \int_0^\xi \frac{dw}{\sqrt{(1-w^2)(1-k^2w^2)}},$$

which of course depends on the choice of a path from 0 to $\xi \in \mathbb{C}$.

0.2 Theorem (Abel; 1827–1828). *The inverse function $\xi(\zeta)$ is doubly periodic, with periods K and L given by*

$$K = 2\int_0^1 dz \text{ and } L = \int_0^{1/k} dz.$$

In other words, ξ is determined by its values on the parallelogram with vertices $\{0, K, L, K+L\}$.

0.3 Definition. An **elliptic function** with periods K and L (two $\mathbb{R}$-independent complex numbers) is a meromorphic function $\ell e : \mathbb{C} \to \mathbb{C}$ such that $\ell(z) = \ell(z + mK + nL)$ for all $m, n \in \mathbb{Z}$.

We now compute, by the fundamental theorem of calculus,

$$\frac{d\xi}{d\zeta} = \frac{1}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}};$$

hence $x = \xi$ and $y = d\zeta/d\xi$ (there is a lot of calculus justification needed here of course, but we are living in the XIXth century and we don't care) parameterise the curve

$$(1-x^2)(1-k^2x^2) = y^2. \tag{0.4}$$

This is an elliptic curve, and its function field is generated over $\mathbb{C}$ by x and $y = \sqrt{(1-x^2)(1-k^2x^2)}$. On the other hand, both ξ and $d\zeta/d\xi$ are periodic with periods K and L and give well-defined functions on the torus $\mathbb{C}/\langle z \mapsto z+K, z \mapsto z+L\rangle$. We now have the following result of Poincaré:

0.5 Lemma. *If f, g are (sufficiently generic, i.e. the diagonal map $z \mapsto (f(z), g(z))$ is injective—this is true for x and y in Eq.* (0.4)*) elliptic functions with periods K, L and h is a third such function, then h is a rational function in f, g.*

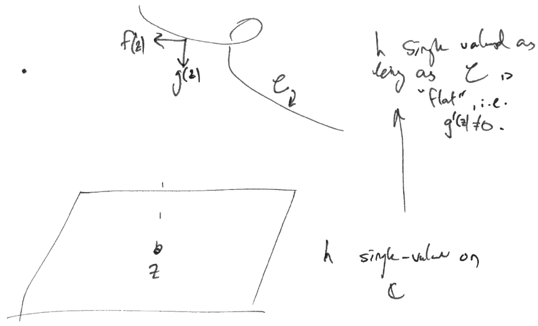

Figure 1: The lift of h to the Riemann surface of f and g.

Proof. Let $z \in P$ where P is the parallelogram of definition and let $g = \psi(f)$ be the Riemann surface of f and g (the proof that there exists such an algebraic relation ψ is entirely elementary but the proof is general to higher genus, we give it in Proposition 3.10); by the condition on f and g the function h is single-valued on this Riemann surface, except for finitely many poles where either h has a pole on $\mathbb{C}$ or g′ vanishes (see Fig. 1). Now we use that on a Riemann surface, every meromorphic function is rational. This is an argument due to Riemann via what we would call 'modern' methods; i.e. it is in this context that the Riemann-Roch theorem appears.

§1. Fuchsian groups and uniformisation

Motivated by the elliptic case, given a Riemann surface R we ask for the following:

1. A planar domain Ω and a group Γ of holomorphic functions on Ω such that $\Omega/\Gamma \simeq R$;
2. Even better, a subset $P \subseteq \Omega$ such that the images of P under Γ tile Ω and such that R is exactly P with the sides identified according to Γ;
3. A system of meromorphic functions $R \to \mathbb{C}$, equivalently functions $f : \Omega \to \mathbb{C}$ such that $\gamma f = f$ for all $\gamma \in \Gamma$;
4. A moduli space of Riemann surfaces with the same topological type as R;
5. An equation for the algebraic curve associated to R;
6. etc.

In this section we discuss questions (1) and (2), in limited detail. The interested reader might want to look at (i) the original papers of Poincaré, which are translated into English with commentary by John Stillwell [51]; (ii) the classic textbooks of Katok on Fuchsian groups [29] and Beardon on Möbius transformations [7]; (iii) the sections on uniformisation in the textbook of Farkas and Kra [21]; (iv) the modern undergraduate textbook by Bonahon [11].

Let's pause to state some basic definitions for convenience.

1.1 Definition. 1. A map $f : U \to \mathbb{R}^n$ (where $U \subseteq \mathbb{R}^n$ is open) is differentiable at x if there is an $n \times n$ matrix A and a map $\varepsilon : U \to \mathbb{R}$ with $\varepsilon(y) \to 0$ as $y \to x$ such that

$$f(y) = f(x) + A(y - x) + \varepsilon(y)|y - x|;$$

and such a differentiable map f is **conformal** at x if A is some positive scalar multiple of an orthogonal matrix. (By the Cauchy-Riemann equations, holomorphic functions are conformal.) Write $\mathrm{Conf}(U)$ for the group of conformal functions on U, and $\mathrm{Conf}^+(U)$ for the orientation-preserving ($\det A > 0$) half.

2. Might as well define a Riemann surface too: a **Riemann surface** is a Hausdorff topological space R with countably many connected components and which admits a chart $(U_\alpha, f_\alpha : U_\alpha \to C)_{\alpha \in A}$ with the usual axioms: (i) the U_α cover R; (ii) the f_α are homeomorphisms onto their images; (iii) for every $\alpha, \beta \in A$, $f_\beta f_\alpha^{-1}$ is holomorphic on its domain of definition.

   In particular, we allow the following things: (a) R might be disconnected; (b) R might have punctures or disc boundary components; (c) R might have infinitely many boundary components; (d) R might not be of finite genus... A Riemann surface is called **analytically finite** if it has finitely many components, finite genus, finitely many punctures, and no other boundary components. We will meet natural situations where the pathological cases occur, but there are various saneness theorems (with uniformly hard proofs) which guarantee that pathological Riemann surfaces occur only in pathologically natural situations. **For the remainder of this section, assume every Riemann surface is analytically finite.**

The fundamental result in the study of uniformisation by groups is the famous

1.2 Theorem (Riemann mapping theorem)**.** *If R is a simply connected Riemann surface, then R is biholomorphic to exactly one of the following:*

1. *The Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, if $\chi(R) = 2$;*
2. *The plane $\mathbb{C}$, if $\chi(R) = 0$;*
3. *The disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$, if $\chi(R) < 0$.*

(For a proof, see the theorem of Paragraph IV.6.1 of [21] or Chapter 6 of [2].)

Here, the Euler characteristic of a Riemann surface R of genus g with n punctures and deleted discs is

$$\chi(R) = 2 - 2g - 2n.$$

Note that each of the three cases corresponds to the three surface geometries: spherical, Euclidean, and hyperbolic. (We recall that the unit disc Δ is conformally equivalent to the upper half-plane $\mathbb{H}^2$ and both are models for the hyperbolic plane [7, Chapter 7].)

We now state a weak version of the Klein-Koebe-Poincaré uniformisation theorem (we exclude the torsion case, for simplicity; the more general theorem in the connected case can for example be found as Theorem IV.9.12 of [21]).

1.3 Theorem (Klein-Koebe-Poincaré uniformisation)**.** *Let R be a connected Riemann surface. Then:*

1. *If $\chi(R) = 2$, then R is the sphere (so admits a metric of constant sectional curvature 1)*
2. *If $\chi(R) = 0$ (so R is either of genus 1 with no punctures, or is a sphere with one puncture), then there is a group G of Euclidean motions on $\mathbb{C}$ such that $R = \mathbb{C}/G$ (so admits a metric of constant sectional curvature 0);*

3. *If $\chi(R) < 0$, then there exists a discrete group Γ of conformal mappings of $\mathbb{H}^2$ which acts as a group of hyperbolic isometries, and $R = \mathbb{H}^2/\Gamma$.*

1.4 Definition. A discrete group of isometries of $\mathbb{H}^2$ is called a **Fuchsian group**.

'Discrete' here means with respect to the compact-open topology, but we will see a more concrete topology in a moment.

1.5 Lemma. *There are natural isomorphisms between the three groups* $\mathrm{Isom}^+(\mathbb{H}^2)$, $\mathrm{PSL}(2,\mathbb{R})$*, and the group* $\mathrm{Conf}^+(\mathbb{H}^2)$ *of conformal maps on $\hat{\mathbb{C}}$ which preserve the upper half-plane.*

Here is the more concrete definition: one can define discreteness of subgroups of $\mathrm{SL}(2,\mathbb{C})$ by identifying the latter with a subset of $\mathbb{C}^4$ with the usual Euclidean norm; this topology descends to $\mathrm{PSL}(2,\mathbb{C})$, in fact the entire norm descends since $\mathrm{SL}(2,\mathbb{C})/\mathrm{PSL}(2,\mathbb{C}) = \{\pm 1\}$. Anyway, this lemma in fact is more strong and we get isomorphisms of topological groups. As a consequence, note that discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ are countable: given any radius R, the ball of radius R about I_2 contains only finitely many group elements. An alternative proof is found as [42, Corollary II.B.4].

Proof of Lemma 1.5. Define an action of $\mathrm{PSL}(2,\mathbb{R})$ on $\hat{\mathbb{C}}$ by fractional linear transformations, i.e.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.z := \frac{az+b}{cz+d};$$

then one can show by direct computation (use the determinant condition) that $\mathrm{PSL}(2,\mathbb{R})$ does indeed preserve $\mathbb{H}^2$. Fractional linear transformations with complex coefficients a, b, c, d which satisfy $|ad - bc| = 1$ are in natural bijection with conformal maps (this is a standard fact [2, Chapter 3] or [7, Chapter 4]), and fractional linear transformations which preserve $\mathbb{R}$ must necessarily have real coefficients (another computation). Getting that the conformal maps in fact give isometries of $\mathbb{H}^2$ requires annoying computations with the metric [7, Theorem 7.4.1]. ⅅ⥂

Oh, and we might as well state the converse to the uniformisation theorem (again in the torsion free case):

1.6 Lemma. *If $\Gamma \le \mathrm{PSL}(2,\mathbb{R})$ is Fuchsian, then $\mathbb{H}^2/\Gamma$ is a Riemann surface which admits a hyperbolic metric.*

Proof. The proof goes (very) locally, that is you pick a small region around every point $z \in \mathbb{H}^2/\Gamma$ and lift it back up to $\mathbb{H}^2$; in the torsion case you need to be careful about ramification points too. The details are in Beardon [7, Chapter 6]. ⅅ⥂

Let's do some examples.

1.7 Example (Compact Riemann surfaces of genus g)**.** One group G which gives the Riemann surface of genus g is generated by the $2g$ transformations depicted in Fig. 2. We need to check that the quotient is correct; let P be the hyperbolic $4n$-gon pictured. Each ϕ_i moves P off itself, such that $\phi_i P \cap P$ is exactly one side of P (namely, C_{2g+i}). Also, one can show by following the action of the ϕ_i on the edges that

$$[\phi_1, \phi_2][\phi_3, \phi_4] \cdots [\phi_{2g-1}, \phi_{2g}] = 1$$

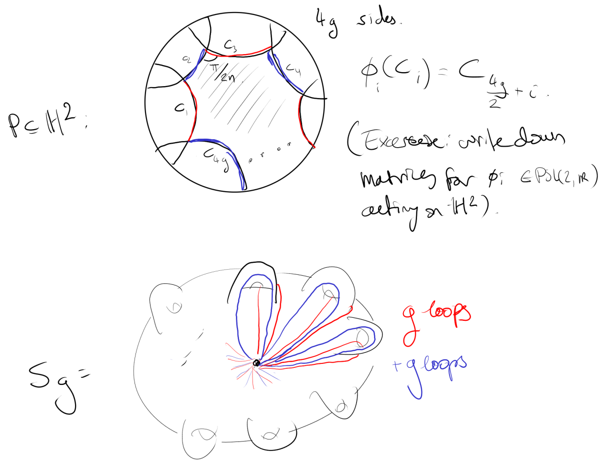

Figure 2: One complex structure on the genus g.

and no other combination of the ϕ_i will transform P back onto itself. Hence GP tiles $\mathbb{H}^2$ and the quotient $\mathbb{H}^2/G$ is equal to P/G where G is viewed as an automorphism of P which just permutes the edges; and the edge pairing set up by G gives the gluing depicted in the lower half of the figure.

(Of course, the space of all compact Riemann surfaces of genus g should have dimension $3g-3$, so where are all the other groups? They come from quasiconformal deformations of G, see the section on moduli below, but you can get at least some of them by varying the relative lengths of the circular arcs in the hyperbolic metric—this gives $2g$ dimensions a priori so you have to be more clever to get the rest. This direction leads to the **Fenchel-Nielsen coordinates** [20, §10.6] for Teichmüller space.)

This kind of ping-pong argument generalises, as was known to Poincaré:

1.8 Theorem (Poincaré (1883))**.** *Let P be an open convex polyhedron in $\mathbb{H}^n$ of full dimension, and suppose we have the following data in addition:-*

> *For every facet f of P (we include faces at infinity throughout), a facet f' of P and a group element $\phi_f \in \mathrm{Isom}^+(\mathbb{H}^n)$ such that (i) $\phi_f(f) = f'$, (ii) $\phi_{f'} = \phi_f^{-1}$, and (iii) $\phi_f(P) \cap P = f'$ (this is the data of a **facet-pairing system**).*

Let $G = \langle \phi_f : f \in P(n-1)$. The facet pairings induce an equivalence relation on $\overline{P}$ such that each element of the interior P has equivalence class a singleton. Write P^* for the quotient of $\overline{P}$ by the relation, let $\pi : \overline{P} \to P^*$ be the standard topological projection. The next assumption is that

(iv) For every $z \in P^*$, $|\pi^{-1}(z)| < \infty$.

For each codimension 2 face e, there is a sequence $(e = e_0, \ldots, e_m)$ of codimension 2 faces defined in the following way: pick a facet f_0 containing e_0; set $e_1 = \phi_{f_0}(e_0)$; set f_1 to be the facet containing e_1 which is not f_0'; look at the image of e_1 under ϕ_{f_1},...

and so on...; by (iv), this sequence has finite period k; set $h(e) = \phi_{f_k} \cdots \phi_{f_0}$*; let* $t(e)$ *be the order of* $h(e)$ *(possibly* $t(e) = \infty$*).*

(v) *Let* $\alpha(e)$ *be the dihedral angle of P around the codimension 2 face e; we require* $\sum \alpha(e_i)$ *to be a submultiple of* π *(if there is a facet f containing e such that* $f = f'$ *and* $\phi_f e = f$*) or* 2π *(otherwise).*

(vi) *Finally, we place a metric on* P^* *by analytic continuation, i.e. P has a natural metric from X, and if you walk along a path that crosses the edge of P then it continues to have a well-defined length after wrapping around to the other side; we require this metric to be complete.*

With all of these conditions, then (a) G is discrete; (b) G is generated by the ϕ_f *and a complete set of relations in G is given by* $h(e)^t = 1$ *for each e and* $\phi_{f'}\phi_f = 1$ *for each f; (c) the quotient space* P^* *is isometric to the quotient* $\mathbb{H}^n/G$*.*

For a complete discussion of this theorem, including the orbifold case, see Ratcliffe [53]. There is a natural combinatorial condition for condition (vii): the only place that completeness can go wrong is at cusps, and the required condition is that if you cut the polyhedron P around a cusp with a sphere then the restriction of the ϕ_f to the intersection (which is a lower-dimensional polyhedron) should set up a lower-dimensional facet pairing.

1.9 Definition. If $G \leq \mathrm{Isom}^+(\mathbb{H}^n)$ is the group obtained from a facet-pairing of a polyhedron $P \subseteq \mathbb{H}^n$ as above, then P is called a **fundamental polyhedron** for G; if there exists a fundamental polyhedron for G with finitely many facets, then G is called **geometrically finite**.

Remark. The Poincaré polyhedron theorem is essentially the only way to prove discreteness of a group. Of course the discreteness problem is hard and so the problem of finding a polyhedron which works is also hard.

Talking about hard group theoretic problems, one can read off all kinds of topological information about $\mathbb{H}^2/G$ from the group G, in theory. For instance, one has the following standard result from covering theory.

1.10 Proposition. *1. If* $p : X \to Y$ *is a regular covering map (i.e.* Aut p *acts transitively on fibres), with* $x_0 \in X$ *and* $y_0 = p(x_0)$*, then*

$$\mathrm{Aut}(p) \simeq \pi_1(Y, y_0)/p_*\pi_1(X, x_0).$$

2. If G acts freely discontinuously on a path connected and locally path connected Hausdorff space X, then $p : X \to X/G$ *is a regular covering map such that* $\mathrm{Aut}(p) = G$*.* ⅄⥱

Proof. See Bredon [13] (corollary III.6.9 and proposition III.7.2 resp.), or Lee [35] (corollary 12.8 and theorem 12.14 resp.). ⅄⥱

The point is that if G tiles $\mathbb{H}^2$ with copies of some connected open set Ω, then $G \simeq \pi_1(\mathbb{H}^2/G)/\pi_1(\Omega)$. Thus if we find a simply connected domain for G like those in the examples above then we can recover the homology type. In addition, we see that (in the simply connected domain case, again) every geodesic on the quotient lifts to a group element in the fundamental group; more precisely, the geodesic lifts to an arc in Ω preserved by the corresponding element of $\pi_1(\mathbb{H}^2/G)$ under the isomorphism.

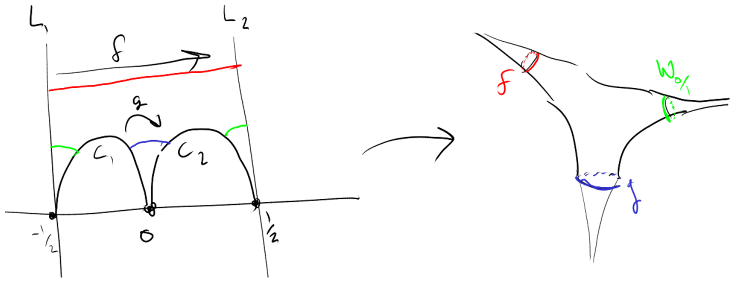

Figure 3: The (∞, ∞, ∞)-triangle group.

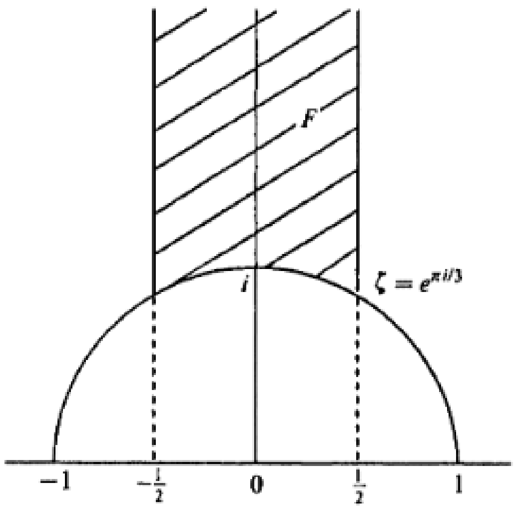

Figure 4: A fundamental polygon for $\mathrm{PSL}(2, \mathbb{Z})$, image from Miyake [47, fig. 4.1.1].

1.11 Example (The (∞, ∞, ∞)-triangle group)**.** Let P be the polyhedron bounded (in $\mathbb{H}^2$) by the two vertical lines $\Re z = \pm 1/2$ and the two circles of radius $1/4$ through $\pm 1/4$ (Fig. 3). Define a group G generated by the translations $f(z) = z + 1$ and $g(z) = \frac{z}{4z+1}$; these transformations pair up the circles, as shown in the figure. By Theorem 1.8, one can compute that the group is free on f and g. (The only nontrivial cycle relation is $(fg^{-1})^{\infty} = 1$; the transformation fg^{-1} is the hyperbolic translation which fixes $1/2$.) Note, this group is the orientation-preserving half of the group generated by the reflections across the sides of the triangle with vertices at $-1/2$, 0, and ∞. The quotient is a three-punctured sphere, which has trivial Teichmüller space [20, p. 278], so one might expect that the group cannot be deformed:- and indeed, deforming a triangle with three vertices at infinity just gives a projectively equivalent triangle.

1.12 Example ($\mathrm{PSL}(2, \mathbb{Z})$)**.** The prototypical arithmetic group is the **full modular group**, $\mathrm{PSL}(2, \mathbb{Z})$. Of course this group is discrete since $\mathbb{Z}^4$ is discrete in $\mathbb{R}^4$, but we can also use the standard fact that $\mathrm{PSL}(2,\mathbb{Z})$ has presentation

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \right\rangle$$

The corresponding tessellation is given in Fig. 4, and the quotient is a Riemann surface with one cusp, one cone point of angle $\pi/2$, and one of order $\pi/3$.

Any group of finite index in $\mathrm{PSL}(2,\mathbb{Z})$ is called a **modular group**. For example, the (∞,∞,∞)-triangle group of Example 1.11 is such a group. Here are some more—let $N \in \mathbb{N}$, and define

$$\Gamma(N) = \{A \in \mathrm{PSL}(2,\mathbb{Z}) : A \equiv I_2 \pmod N\}.$$

A group $G \leq \mathrm{PSL}(2,\mathbb{Z})$ is a **congruence subgroup** if $\Gamma(N) \leq G$ for some $N \in \mathbb{N}$.

§2. Kleinian groups

The best classical reference for Kleinian groups are Beardon [7] and Maskit [42]. For a more modern view from a post-Thurston point of view see Ratcliffe [53] (encyclopaedic, but from a very geometric point of view), Kapovich [28] (particularly from the point of view of ggt), and the very very modern and highly recommended Marden [41]. Of course as well as these books there is Thurston's famous paper [59], and his notes [58] (now published as of August 2022 by the AMS).

Poincaré recognised at the time of his seminal papers that it is necessary to deal with discrete subgroups of $\mathrm{PSL}(2,\mathbb{C})$ rather than just $\mathrm{PSL}(2,\mathbb{R})$. His motivation came from solving certain PDEs like those in Section 0 above, where the coefficients were allowed to deform into the complex plane. Our motivation will be that it would be nice to look at the full group of conformal maps on the Riemann sphere, rather than the subgroup of those which fix the upper half-plane. We have the following result from undergraduate complex analysis and classical geometry:-

2.1 Lemma. *There are natural isomorphisms between the following groups:*

1. *The group of* ***fractional linear transformations****, maps $\mathbb{C} \to \mathbb{C}$ of the form $z \mapsto \frac{az+b}{cz+d}$ for $|ad-bc| = 1$;*
2. *The group $\mathbb{M}$ of* ***Möbius transformations****, the orientation-preserving half of the group generated by all circle inversions on the Riemann sphere;*
3. *The group $\mathrm{Conf}^+(\hat{\mathbb{C}} = \mathrm{Aut}(\hat{\mathbb{C}})$ of conformal automorphisms of the Riemann sphere;*
4. *The group $\mathrm{PSL}(2,\mathbb{C})$.*

For a proof, see the references in the proof of Lemma 1.5 above.

Poincaré also made the following observation:-

2.2 Proposition (Poincaré extension)**.** *There is a natural isomorphism between the group of conformal automorphisms $\mathrm{Conf}^+(\hat{\mathbb{C}})$ and the group of hyperbolic isometries $\mathrm{Isom}^+(\mathbb{H}^3)$.*

Proof. Let ϕ be an (orientation-preserving) isometry of $\mathbb{H}^3$. Then ϕ extends naturally to a conformal homeomorphism (hence Möbius transformation) on the visual boundary $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$—this is a non-trivial observation but isn't too hard to check,

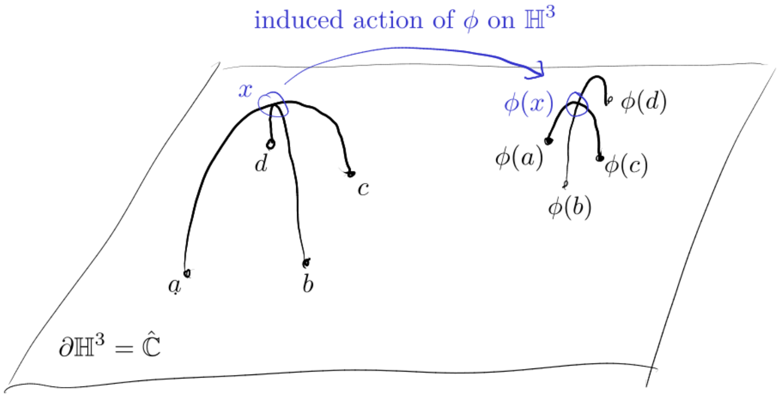

Figure 5: A conformal map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ extends uniquely to an hyperbolic isometry on $\mathbb{H}^3$: the map ϕ, in sending the points a, b, c, d to their images, induces a map on the hyperbolic segments $[a, c]$ and $[b, d]$; if the original segments intersect at some $x \in \mathbb{H}^3$, then their images intersect at some point which is defined to be $\phi(x)$. The difficulty is in checking that, for any $x \in \mathbb{H}^3$, different choices of segments intersecting at x induce the same image $\phi(x)$.

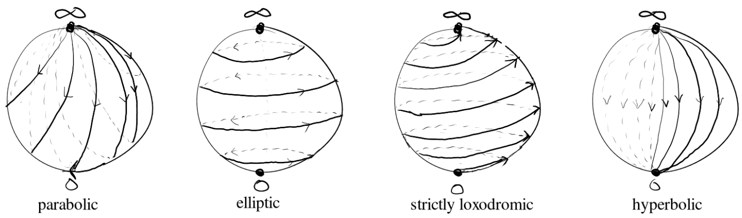

Figure 6: The conjugacy classes of Möbius transformations.

see [42, IV.A]—and in fact this map $\mathrm{Isom}(\mathbb{H}^3) \to \mathbb{M}$ is invertible (which is not a feature of all metric spaces of non-positive curvature). The inverse map is the so-called **Poincaré extension**, where to define the action of a conformal map ϕ on the boundary $\hat{\mathbb{C}}$ on a point $x \in \mathbb{H}^n$ one first finds a pair of intersecting geodesics through x, computes the images of the endpoints at infinity of these geodesics under ϕ, and uses these to draw two new geodesics which will intersect at the image of x under the extension of ϕ to $\mathbb{H}^3$; this process is depicted in Fig. 5 (see also the final paragraph of [42, A.13], or for more detail [7, §3.3] and [53, §4.4]).

In fact, the construction gives a natural geometry-respecting bijection between $\mathbb{M}$ and the group of isometries of $\mathbb{H}^3$ which is compatible with the compact-open topology on the isometry group.

Remark. The construction of a conformal map on the visual boundary generalises to all $CAT(0)$ spaces—see [14, Chapters II.8 and II.9]—but the converse requires strict negative curvature, more generally a δ-hyperbolic space, see [14, Chapter III.H].

We now give a dynamical classification of the elements of $\mathrm{PSL}(2, \mathbb{C})$; see Fig. 6.

2.3 Proposition. *Let $g \in \mathrm{PSL}(2, \mathbb{C})$ be a nontrivial element. Then g falls into exactly* *one of the following cases:-*

PARABOLIC. *g is conjugate to $z \mapsto z + 1$. In this case, g has exactly one fixed point in $\overline{\mathbb{H}^3}$, preserves horoballs based at its fixed point, and has* $\mathrm{tr}^2\, g = 4$.

ELLIPTIC. *g is conjugate to $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. In this case, g has exactly two fixed points in $\hat{\mathbb{C}}$, fixes pointwise the hyperbolic line joining them in $\mathbb{H}^3$, and acts as a rotation around that line;* $\mathrm{tr}^2\, g \in [0, 4)$.

LOXODROMIC. *g is conjugate to $z \mapsto \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$. In this case, g has exactly two fixed points in $\hat{\mathbb{C}}$, preserves the hyperbolic line joining them in $\mathbb{H}^3$, and acts as the composition of a translation around that line and a rotation around it;* $\mathrm{tr}^2\, g \in \mathbb{C} \setminus [0, 4]$.

*A loxodromic element is called **hyperbolic** if it has real trace, equivalently it has no rotational component. Otherwise it is called **strictly loxodromic**.*

By similar arguments to the dimension two case, the quotient of $\mathbb{H}^3$ by the action of a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is a hyperbolic orbifold, and all hyperbolic orbifolds are uniformised by such a group. On the other hand, the quotient of $\hat{\mathbb{C}}$ by such a group is not necessarily even Hausdorff: for instance consider for any parabolic element the projection of the fixed point to the quotient. The best we can do is the following:

2.4 Proposition. *Let $G \leq \mathrm{PSL}(2, \mathbb{C})$ be discrete. Let $\Lambda(G)$ be the closure of the set of fixed points of non-identity elements of G in $\hat{\mathbb{C}}$; this is the **limit set** of G. If the **domain of regularity** of G, defined by $\Omega(G) = \hat{\mathbb{C}} \setminus \Lambda(G)$, is non-empty, then $\Omega(G)/G$ is a (possibly disconnected, non-compact, orbifold) Riemann surface with the induced complex metric. Further, if $|\Lambda(G)| \geq 2$ (in which case it is perfect and the group is called **non-elementary**) then every component of the quotient has a hyperbolic metric.*

A discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ is called a **Kleinian group**. Historically, one would also require the group to have non-empty domain of regularity, but in modern times (post-1970s) we no longer require this.

The elementary groups are fully classified (they are the holonomy groups of Riemann surfaces with non-negative Euler characteristic). There is a strengthening of the uniformisation theorem for Kleinian groups, due to Bers:

2.5 Theorem (Bers' simultaneous uniformisation theorem)**.** *Let $S = \cup_{i=1}^{\infty} S_i$ be a countable union of hyperbolic Riemann surfaces. Then there exists a Kleinian group G such that $S = \Omega(G)/G$.*

For a proof using combination theorems, see [42, §VIII.B].

2.6 Example (Schottky groups)**.** Let $C_1, ..., C_n, C_1', ..., C_n'$ be $2n$ circles, all mutually disjoint and bounding a connected open subset D of $\hat{\mathbb{C}}$. For each i let $\phi_i \in \mathrm{PSL}(2, \mathbb{C})$ be a transformation which sends C_i to C_i' and which maps D into the interior of C_i'. Then $D \subseteq \Omega(G)$ (see Fig. 7); by the Poincareé polyhedron theorem (where is the polyhedron?!) the group $\langle \phi_i : i \in [n] \rangle$ is free and discrete, and $\Omega(G)/G$ is a compact connected Riemann surface of genus n. The group G is called a **classical Schottky group**.

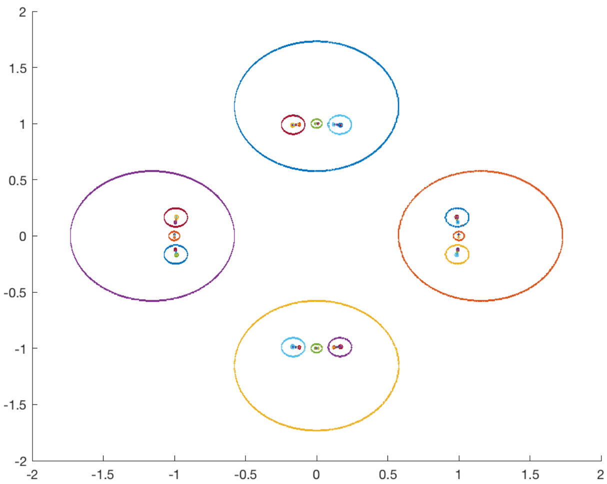

Figure 7: The limit set of a classical Schottky group on 4 circles.

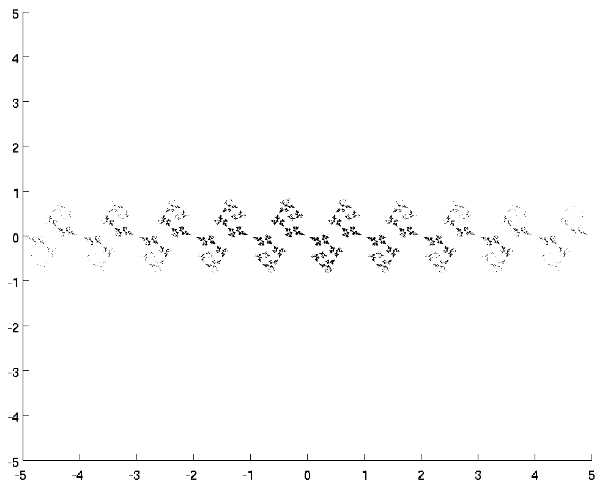

Figure 8: An example Riley limit set.

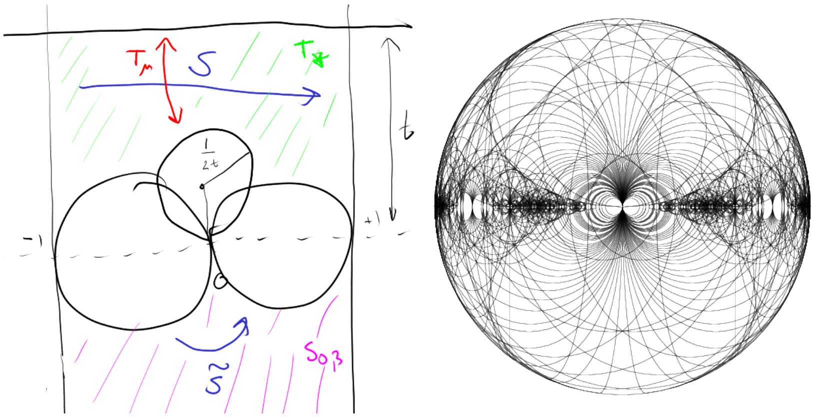

Figure 9: Maskit groups uniformise punctured torii; they are elements of the corresponding Bers slice.

2.7 Example (Riley group)**.** The group

$$\Gamma_{1+2i} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+2i & 1 \end{bmatrix} \right\rangle$$

is free, discrete, and has quotient a 4-punctured sphere (not so trivial to see this). It has limit set shown in Fig. 8.

2.8 Example (Punctured tori)**.** Suppose $\mu = r + ti \in \mathbb{C}$ and define

$$G_\mu := \left\langle S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \tilde{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, T_\mu = \begin{bmatrix} -i\mu & i \\ i & 0 \end{bmatrix} \right\rangle$$

See Fig. 9 for the fundamental polyhedron (left) and the images of the corresponding circles under the group (right).

If $\mu \gg 0$ then the group glues the top region up to a punctured torus and the bottom one up to a 3-times punctured sphere. The two regions respectively tile the upper half-plane and the lower half-plane, leaving $\hat{\mathbb{R}}$. Note the similarity to Fig. 3; in fact, the Maskit groups can be obtained via surgery from the (∞, ∞, ∞)-triangle groups. See the (highly recommended) book *Indra's Pearls* [50], the monograph of Akiyoshi, Sakuma, Wada, and Yamashita [5], the paper of Keen and Series [31], and Series' corresponding lecture notes [55]. **Warning**. These groups act separately on the top and bottom halves of their regular domain, but are *not* Fuchsian: in general, they do not preserve $\mathbb{R}$. In fact, their limit set (which separates the two 'half-planes' on which they act) is a quasiconformal deformation of the real line. They are **quasi-Fuchsian** groups.

2.9 Example (A circle-packing group)**.** Consider the group

$$\Gamma_{2i}^{5,\infty} = \left\langle \begin{bmatrix} e^{\pi i/5} & 1 \\ 0 & e^{-\pi i/5} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix} \right\rangle;$$

it has limit set dense in the circle-packing of Fig. 10. See the paper by Keen, Maskit, and Series [30] for a discussion of circle packings appearing on the boundary of deformation spaces.

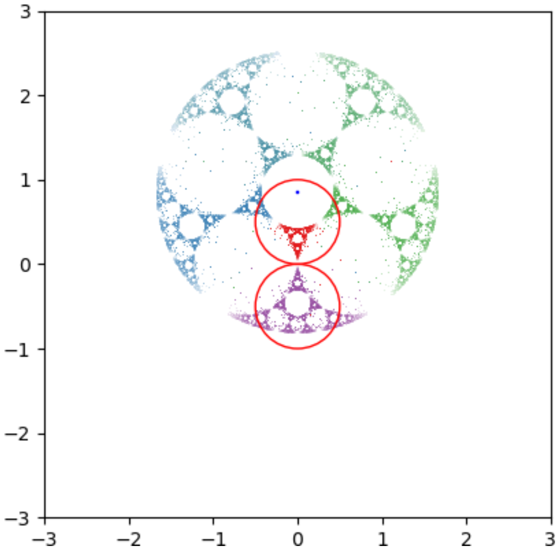

Figure 10: A circle packing limit set.

So far, all the groups we have seen have been all of finitely generated, geometrically finite, and analytically finite. Trivially, we have (geometrically finite) $\implies$ (analytically finite) [42, VI.E.1], and (geometrically finite) $\implies$ (finitely generated) [tautology!]. The following famous theorem gives another implication:

2.10 Theorem (Ahlfors' finiteness theorem)**.** *If G is a non-elementary finitely generated Kleinian group, then G is analytically finite.*

Historical remark. This theorem was originally stated by Ahlfors in [4] with corrections in [3], generalising similar results of Bers in the two-dimensional case; the proof uses Beltrami differentials and quasiconformal techniques. A modern account of Ahlfors' proof together with copious references to other proofs may be found in Section 8.14 of [28].

On the other hand, G analytically finite does not imply G finitely generated (easy but can't be bothered) [42, VIII.A.9] and finitely generated does not imply geometrically finite (hard) [25].

We give now the author's favourite examples of non-geometrically-finite groups.

2.11 Example (Bead groups)**.** Klein and Fricke in 1897 [24, 23] studied groups constructed in the following way, which we now call **bead groups**: pick a chain of n circles $C_1, \ldots, C_n$ such that C_i and C_{i+1} are tangent for each i (taken mod n) and such that there are no other intersection points; now let G be the orientation-preserving half of the group generated by the reflections in the C_i. This has limit set homeomorphic to a circle (but fractal, as shown in Fig. 11). These groups are geometrically infinite since they are not analytically finite.

2.12 Example (Atom groups)**.** Atom groups were studied by Accola in [1], see also [42, VIII.F.7]. Let C_1 and C_2 be the circles of an annulus, and choose two spirals in the annulus which wrap infinitely often around both circles; on each spiral pick a chain of mutually tangent circles as in Example 2.11 such that the chains don't

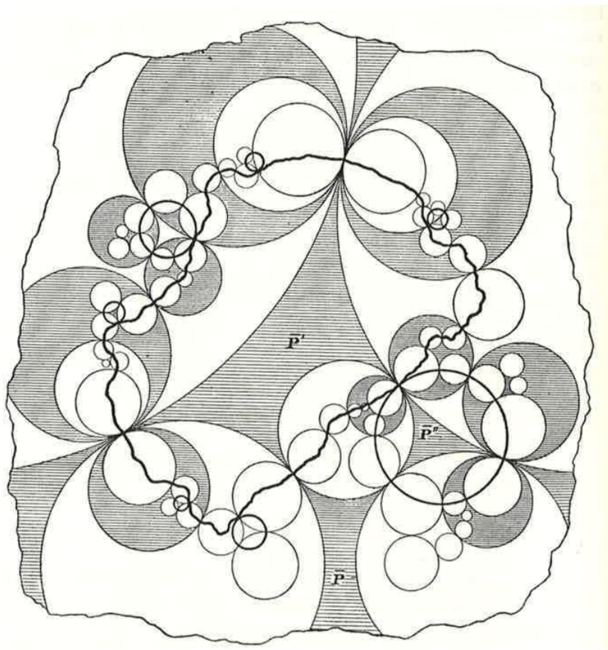

Figure 11: The limit set of a bead group is a quasicircle [24, Fig. 156 of Vol. 1].

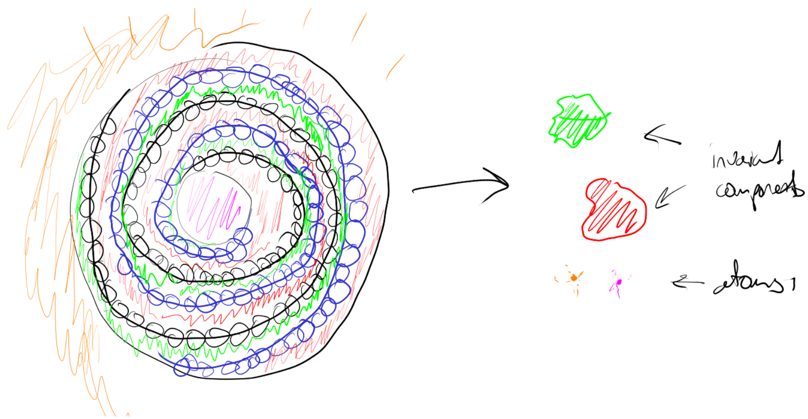

Figure 12: The data for an atom group, and its fundamental polyhedron.

intersect (Fig. 12), and let G again be the group generated by the reflections in the chained circles. There is a fundamental polyhedron for the group which intersects $\hat{\mathbb{C}}$ in the exterior discs of the annulus (orange and pink in the figure) and the two regions between the circle chains (red and green). Both orange and pink regions are mapped off themselves by every element of the group, in other words they have invariant stabilisers; thus in the quotient they carry their own metric, and the group cannot be analytically finite since the quotient contains discs. (These discs are called **atoms**.) This proves that the group does not have a finite generating set by Theorem 2.10. See also example 18 of [34].

Remark. We neglect a lot of very interesting groups, including knot and link complements [59], more complicated arithmetic groups [38], B-groups and degenerate groups [42, Chapter IX], and the groups which arise in the dynamics programme of Sullivan and McMullen (see e.g. [46]).

§3. Theta functions

There is no good reference explaining the relationship between theta functions (of which there are several kinds) and Riemann surfaces from the different points of view. Thus we have patched this section together from a variety of sources:- Farkas and Kra [21, Chapters II, III, and VI]; Mumford's *Lectures on curves and their Jacobians*, reproduced in [49]; Prasolov and Solovyev [52]; Poincaré [51]; Mumford's *Tata lectures* [48]; and Igusa [26]. We also restrict to the compact case; one should be able to extend some things to the boundary of the Deligne-Mumford compactification of the moduli space of compact Riemann surfaces, see the appendices to Imayoshi and Taniguchi [27]. This allows us to assume

Assumption. Every Fuchsian group which appears is geometrically finite.

We begin by stating this big theorem.

3.1 Theorem (Equivalence of algebra and conformal geometry)**.** *There is a bijective correspondence between the set of conformal equivalence classes of compact Riemann surfaces and the set of birational equivalence classes of algebraic function fields in one variable.*

Remark. This is *not true* in the non-compact case: $\exp(z)$ is meromorphic on $\mathbb{C}$, but the function field of $\mathbb{C}$ is $\mathbb{C}(z)$...

Proof. [21, §IV.11]

We shall give a stronger and more interesting point of view below, in Proposition 3.10 (which really should have been a theorem).

The **canonical homology basis** for the (topological) genus g Riemann surface is the one introduced in Example 1.7. We adopt the labelling of Fig. 13 (where we have kept the colours consistent with the earlier drawing).

3.2 Lemma. *Let R be a compact Riemann surface of genus g; then the space $\mathcal{H}^1(R)$ of holomorphic 1-forms on R is of dimension g, and has a canonical basis $(\omega_1, ..., \omega_g)$ such that*

$$\int_{a_i} \omega_j = \delta_{ij}$$

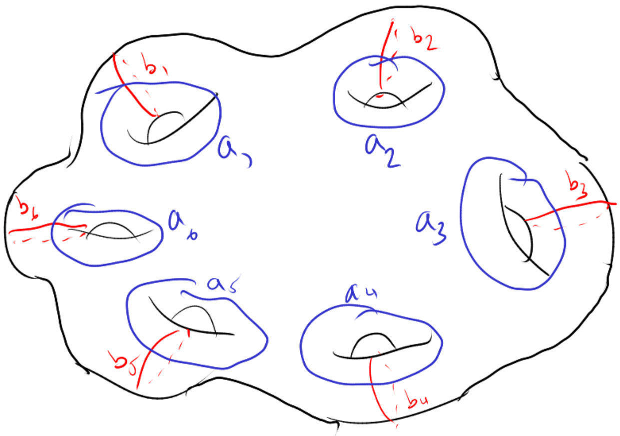

Figure 13: The canonical homology basis for a compact Riemann surface, here of genus six.

for all $1 \leq i, j \leq g$*—in other words, the matrix of integrals between the a's and the* ω*'s is the identity matrix* I*). Let* II *be the matrix of integrals on the b's, that is*

$$II_{ij} = \int_{b_i} \omega_j;$$

this matrix is symmetric with positive definite imaginary part.

Proof. [21, §III.2]

3.3 Definition. The **period lattice** $L(R)$ of a compact Riemann surface R is the column span of the $g \times 2g$ matrix $[I, II]$; it is a rank g lattice. The **Jacobian** of R, $J(R)$, is the quotient torus $\mathbb{C}^g/L(R)$. The **Abel-Jacobi map** is the map

$$\varphi : R \ni P \to \left(\int_0^P \omega_1, \ldots \int_P^0 \omega_g \right) \mod L \in \mathbb{C}^d/L.$$

Extend φ to non-prime divisors on R by linearity.

We list several results [21, Section III.6].

1. (Abel) If D is a divisor on R, then D is principal iff $\varphi(D) = 0 \mod L$ and $\deg D = 0$ (so $\ker \varphi$ is the group of principal divisors).
2. φ is an injective holomorphic mapping such that $\varphi(R)$ is a submanifold of $J(R)$; $\varphi(R) = J(R)$ iff $g = 1$.
3. (Jacobi) Every point of $J(D)$ is the image of an integral divisor of degree g. (in particular, φ is surjective).

Together, these imply the Abel-Jacobi theorem:

3.4 Theorem (Abel-Jacobi)**.** *Two divisors on R are linearly equivalent iff they have equal image under* φ*. Even stronger,* $J(R)$ *is isomorphic to the group of divisors of degree zero, modulo the subgroup of principal divisors.*

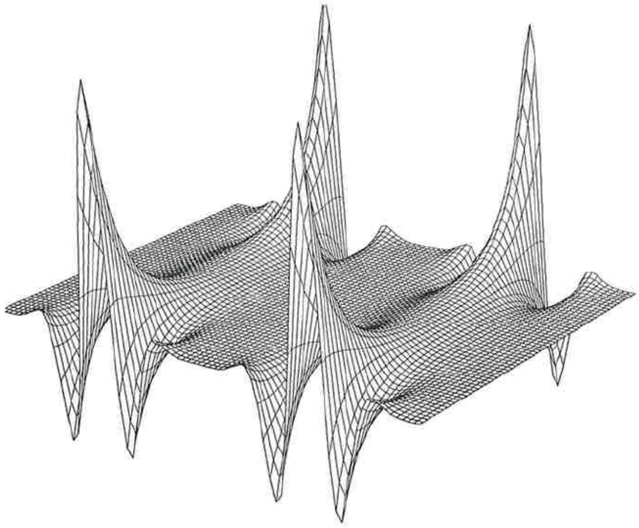

Figure 14: Frontispiece to [48].

Poincaré asked the following question:

3.5 Question. Do there exist functions $F : \mathbb{H}^2 \to \mathbb{H}^2$ such that F is automorphic with respect to a Fuchsian group G (that is, $F(g(z)) = F(z)$ for all $g \in G$, $z \in \mathbb{H}^2$)?

He proved that the answer is *yes*, essentially proving that there global meromorphic functions on hyperbolic Riemann surfaces (this also follows from Riemann-Roch, of course, and one can in fact prove Riemann-Roch itself via these methods). We reproduce the following quote as translated in [51, pp. 2–4]:

> For fifteen days I strove to prove that there could not be any functions like those I have since called Fuchsian functions... one evening, contrary to my custom, I drank black coffee and could not sleep. Ideas rose in crowds; I felt them collide until pairs interlocked, so to speak, making a stable combination. Bu the next morning I had established the existence of a class of Fuchsian functions... [He then describes further ideas coming to him on a geological expedition and while walking on a bluff.]

The construction goes via theta functions. Jacobi had already introduced such functions in 1838 for the elliptic case:

3.6 Definition. Let $\tau \in \mathbb{H}^2$. The **Jacobi theta function** is the function

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp\bigl(\pi i n^2 \tau + 2\pi i n z\bigr).$$

See Fig. 14

The Jacobi theta functions are entire functions which have the property that

$$\vartheta(z + a + b\tau; \tau) = \exp\bigl(-\pi i b^2 \tau - 2\pi i b z\bigr)\vartheta(z; \tau). \tag{3.7}$$

More generally, a theta function on $\mathbb{C}^g$ relative to a lattice L is an entire function satisfying

$$\vartheta(z + \omega) = \exp(Q_\omega z + c_\omega)\vartheta(z)$$

where Q_ω (a $\mathbb{C}$-linear form on $\mathbb{C}^g$) and c_ω (a complex constant) are allowed vary based on ω (not necessarily linearly). See [26, §II.2] for more details, but the point is that the elliptic functions of periods 1 and τ are ratios of these functions relative to the lattice $L = \mathbb{Z}\langle 1, \tau\rangle$ when $g = 1$ (for instance, see Theorem 5 in §III.8 of [26], but there will be a much better/more classical reference, perhaps it is found in [48] but one has not checked).

Poincaré's theta functions for an arbitrary Fuchsian group G are defined in the following way.

3.8 Definition. Let G be Fuchsian; a **theta-Fuchsian function** of degree m with respect to G is defined by a series

$$\vartheta(z) = \sum_{g\in G} H(gz)\left(\frac{dg(z)}{dz}\right)^m = \sum_{g\in G} H(gz)\left(c(g)z + d(g)\right)^{-2m}$$

where H is some rational function and where we write

$$\begin{bmatrix} a(g) & b(g) \\ c(g) & d(g) \end{bmatrix}$$

for a matrix representing g in $\mathrm{PSL}(2, \mathbb{C})$.

Compare this with the modern definition of an automorphic form, for instance as found in [47]:

3.9 Definition. Let G be a Fuchsian group and let $k \in \mathbb{Z}$. A **automorphic form** of weight k with respect to G is a meromorphic $f : \mathbb{H}^2 \to \mathbb{C}$ such that

$$(c(g) + d(g))^{-k} f(gz) = f(z)$$

Poincaré's theta-Fuchsian functions are automorphic forms of weight $2m$ (this is a short calculation), and so if ϑ_1 and ϑ_2 are two theta-Fuchsian functions of G then ϑ_1/ϑ_2 is automorphic with respect to G and hence gives a well-defined function on the quotient surface $\mathbb{H}^2/G$; and every automorphic function (take care of the terminological clash between automorphic *forms* and *functions*) is a quotient of two functions f/g where f and g are polynomials in theta-Fuchsian functions such that in every monomial which appears in f or g the sum of the degrees of the theta functions being multiplied is some fixed m. Write $\digamma(G)$ (one is always looking for an excuse to use digamma) for the set of Fuchsian functions with respect to G. Observe, $\digamma(G)$ is a field since automorphic functions are clearly a field.

Given a Riemann surface $\mathbb{H}^2/G$, we therefore have two natural sets of theta functions: the Poincaré theta functions $\mathbb{H}^2/G \to \mathbb{C}$, and the Jacobi theta functions $J(\mathbb{H}^2/G) \to \mathbb{C}$. We would like to know how they relate to each other!

Classically what we will now discuss is called **uniformisation** (for instance by Ford [22, §§90–91]), but is now more properly called **parameterisation**.

3.10 Proposition. *Fix a Fuchsian group G and two distinct $f, g \in \mathfrak{F}(G)$.*

1. *If $h \in \mathfrak{F}(G)$, then h is a rational function of f and g.*
2. *There is an algebraic relation $\psi(f, g) = 0$.*
3. *The curve in $\mathbb{C}^2$ cut out by the polynomial ψ is conformally equivalent to the Riemann surface $\mathbb{H}^2/G$.*

3.11 Example. There are many special cases which the reader will know.

- (Elliptic functions.) $\sin^2\theta + \cos^2\theta = 1$, and cos and sin parameterise the circle—the ellipse from which they arise as elliptic functions.
- (Congruence modular groups.) Recall from Example 1.12 the definition of $\Gamma(2)$. Every elliptic curve can be put into the Legendre form $y^2 = x(x-1)(x-\lambda)$; recall that elliptic curves are parameterised by $\tau \in \mathbb{H}^2$, in which case λ and the j-invariant are both functions of τ; the mapping class group of elliptic curves is $\mathrm{PSL}(2,\mathbb{Z}) = \Gamma(1)$, j is $\Gamma(1)$-automorphic as a function of τ and hence is $\Gamma(2)$-automorphic, also λ is $\Gamma(2)$-automorphic, and the two are related by the famous formula

  $$j(\tau) = \frac{4}{27}\frac{(1-\lambda+\lambda^2)^3}{\lambda^2(1-\lambda^2)}.$$

  In any case, the quotient $\mathbb{H}^2/\Gamma(2)$ should be parameterised by $\tau \mapsto (j, \lambda)$. Considering this and other similar examples very deeply leads to two major pieces of modern mathematics: monstrous moonshine and the Langlands programme.

We borrow the proof from Lehner's longer book [37, §6].

Sketch of proof of Proposition 3.10. Suppose f has n poles and g has m poles in a fundamental region of G. Define

$$\Phi(z) = \sum_{i=0}^{s}\sum_{j=0}^{t} a_{ij}(f(z))^i (g(z))^j$$

where $s, t \gg 0$ and where the a_{ij} are complex constants; Φ is automorphic on G. Count the poles of Φ with multiplicity; there are at most $ns + mt$ of them. There are $(s+1)(t+1) = \nu + 1$ constants a_{ij}; choose ν points $z_1, ..., z_\nu \in R$ which are distinct from each other and from the poles of f and g, and write down the system of linear equations $\Phi(z_1) = \cdots = \Phi(z_\nu) = 0$. These are ν equations in $\nu + 1$ variables so there exists a solution for the a_{ij} such that Φ has at least ν zeros in R. We write $s, t \gg 0$; in fact, choose them big enough that $\nu = (s+1)(t+1) - 1 > ns + mt$. Then Φ has more zeros than poles in R; and hence (since automorphic functions have the same number of zeros as poles) we have $\Phi \equiv 0$.

Now we use that $\mathfrak{F}(G)$ is isomorphic to a function field, which we stated as Theorem 3.1 (so it seems it did deserve to be called a theorem), to see that the curve $\psi(x, y) = 0$ in $\mathbb{C}^2$ is indeed the Riemann surface we started with.

As well as [37] and Poincaré's original paper, one can look at Mumford [48, §I.4] for the toroidal (genus 1) case, Lehner's shorter book [36, Chapter 3] for a classical point of view via Fuchsian groups, Ford's book [22, Chapter IX] for an ultra-classical point of view via Riemann surfaces of multivalued functions, and (as we mentioned by Theorem 3.1) Farkas and Kra [21, Chapter III] for a synthetic analytic point of view.

Before we talk about the Jacobi theta functions, we briefly go back to Section 0 and finish off the point of view of differentials. Define the **Schwartzian derivative** of some automorphic $f \in \mathfrak{F}(G)$ by the rule

$$\mathsf{D}(f)_z = \frac{2f'(z)f'''(z) - 3(f''(z))^2}{2(w'(z))^2}.$$

Then $\mathsf{D}(f)_z/(w'(z))^2$ is also automorphic. In particular, with the awful notation of Section 0, we have the following result.

3.12 Proposition. *Let $w(z) \in \mathfrak{F}(G)$ be a nonconstant automorphic function, and let z be the inverse function of w. Then*

$$z = \frac{\eta_1(w)}{\eta_2(w)}$$

where η_1, η_2 are functions which satisfy a linear differential equation

$$\frac{d^2\eta}{dw^2} = A(w)\eta$$

where A is an algebraic function of w.

Proof. For the details, see [37, §V.6E] (though the theorem is due to Poincaré, Stillwell writes that his proof is 'extremely condensed and unmotivated' [51, pp. 24–25]). The point is that

$$\eta_1(z) = z(w')^{1/2} \text{ and } \eta_2(z) = (w')^{1/2}$$

work: $w' = \eta_1^2 z^{-1} = \eta_2^2$, then differentiate to compute $\mathsf{D}(z)_w$ and do some algebra.

Finally we return to the Jacobi theta functions. Recall that these are the theta functions whose quotients are the automorphic functions on the Jacobian $J(R)$. More generally, we define for $z \in \mathbb{C}^g$ and $\tau \in \mathfrak{S}_g$ (the space of complex $g \times g$ matrices with positive definite imaginary part)

$$\vartheta(z;\tau) = \sum_{N\in\mathbb{Z}^g}^{\infty} \exp\bigl(\pi i N^t \tau N + 2\pi i N^t z\bigr).$$

(We view N as a column matrix, so for instance $N^t z \in \mathbb{C}$.) Similarly to the one-dimensional case in Eq. (3.7), for $a, b \in \mathbb{Z}^g$ we have

$$\vartheta(z + Ia + \tau b;\tau) = \exp\bigl(-\pi i b^t \tau b - 2\pi i b^t z\bigr)\vartheta(z;\tau).$$

We define more theta functions, for $a, b \in \mathbb{Z}^g$, by

$$\vartheta_{a,b}(z;\tau) = \exp\left(\pi i b^t \tau b + 2\pi i b^t z + 2\pi i b^t a\right)\vartheta(z + Ia + \tau b;\tau) = \exp\bigl(2\pi i b^t a\bigr)\vartheta(z;\tau);$$

these functions are highly symmetric and admit a lot of relations; in fact,

3.13 Lemma. *Up to sign, there are exactly 2^{2g} different theta functions of the form $\vartheta_{a,b}(z;\tau)$ for fixed τ. Of these, $2^{g-1}(2^g+1)$ are even, and $2^{g-1}(2^g-1)$ are odd.*

(see the corollary to the proposition of §VI.I.5 of [21]).

In analogy with the elliptic case, we want τ to be a period matrix. That is, set $\tau = II$ where II is the matrix defined in Lemma 3.2; the 2g columns of $[I, II]$ are linearly independent over $\mathbb{R}$, so every $e \in \mathbb{C}^g$ can be writen in the form $Ia + \tau b$, and we can ask about the function $\vartheta_{a,b}(z;\tau)$, or the composition $\vartheta_{a,b} \circ \varphi$ with the map $\varphi : R \to J(R)$. For instance:

3.14 Proposition.

1. *If $\varphi_{a,b} \circ \varphi$ is not identically zero on R, then it has exactly g zeros.*

2. *Let $P_1 \cdots P_g$ be the divisor of zeros of $\vartheta_{a,b} \circ \varphi$; then $\varphi(P_1 \cdots P_g) = -IIb - Ia - K$, where and K is some vector depending only on the choice of canonical homology basis and basepoint of $J(M)$.*

(See [21, §VI.2.4] for detailed discussion and proof.)

This study naturally leads to the

3.15 Schottky problem. Which complex tori $\mathbb{C}^g$ arise as Jacobian varieties?

and the

3.16 Torelli theorem. *The principal divisor of ϑ on $J(R)$ and the torus structure on $J(R)$ determine the conformal type of R.*

See the appendix to [49] and Section III.12 of [21].

§4. Moduli

Remark. Much of the exposition here comes from the expository paper [18] because I like the way I wrote it there. It takes a more modern point of view than these notes, which may be of interest.

We would like to construct a moduli space of Kleinian groups. Such a moduli space should have the following properties:

1. The underlying hyperbolic manifolds should move in a continuous way (i.e. should naturally have the geometric convergence topology).

2. The (complex structures of the) boundary Riemann surfaces should move continuously (i.e. should have a Teichmüller structure).

3. The matrices of the group as a subgroup of the Lie group $\mathrm{PSL}(2,\mathbb{C})$ should move holomorphically in the entries.

4. The limit sets of the Kleinian group as subsets of $\hat{\mathbb{C}}$ should move holomorphically.

The λ-lemma, which we state below (Theorem 4.7), tells us that in order for (3) and (4) to hold we need to allow quasiconformal deformation of the groups. We therefore pause to explain what this means; a good reference for this point of view is the modern textbook of Astala, Iwaniec, and Martin [6]. The goal is to relate all of this to the classical theory of Teichmüller space and Riemann moduli space, as described in the textbook of Imayoshi and Taniguchi which takes the analytical view

[27] or the popular modern book of Farb and Margalit which takes the geometric group theory view [20].

We will be vague about differentiability requirements for quasiconformal maps; though technically we should give a definition in the language of Sobolev spaces, this would take us too far afield (but see [6, Theorem 2.5.4]).

4.1 Definition. Let Ω and Ω' be planar domains, and let $f : \Omega \to \Omega'$ be a sufficiently nice homeomorphism. Then f is called K-**quasiconformal** if there exists a bounded measurable function μ satisfying

$$\|\mu\|_\infty \leq \frac{K-1}{K+1} < 1$$

such that for almost every $z \in \Omega$,

$$\frac{\partial f}{\partial \bar{z}}(z) = \mu(z)\frac{\partial f}{\partial z}(z).$$

The PDE displayed in the definition is the **Beltrami equation**, μ is called the **complex dilitation**, and f is conformal iff $K = 1$. The idea is essentially that $\mu(z)$ measures the distortion of circles under the action of the differential df_z, see the nice motivation in §1.4 of [27].

We can now start to study moduli, beginning with the Fuchsian theory.

4.2 Definition. Fix a Riemann surface R (not necessarily compact, but with only punctures as boundary components for simplicity). Consider the set $\tilde{T}$ of pairs (S, f) where S is a Riemann surface and $f : R \to S$ is a surjective quasiconformal mapping. The **Teichmüller space** of R, denoted $\mathrm{Teich}(R)$, is the space of equivalence classes of $\tilde{T}$ under the relation $(S_1, f_1) \sim (S_2, f_2)$ iff $f_2 f_1^{-1}$ is homotopic to a conformal map on R. There is a natural metric on the Teichmüller space, coming from the identification of the space with a certain space of quadratic differentials, but that is for another time.[1]

The **mapping class group** of R, denoted $\mathrm{Mod}(R)$, is the group $\pi_0(\mathrm{Homeo}^+(R))$; this is the group of orientation-preserving diffeomorphisms modulo isotopy. (Observe that these homeomorphisms are allowed to permute the punctures.) There is a natural action of $\mathrm{Mod}(R)$ on $\mathrm{Teich}(R)$; if $\phi \in \mathrm{Mod}(R)$ and $(S, f) \in \mathrm{Teich}(R)$ then define

$$\phi \cdot (S, f) := (S, f \circ \phi^{-1}).$$

The mapping class group acts discontinuously as a group of isometries on the Teichmüller space [27, Theorem 6.18], and the quotient $\mathrm{Teich}(R)/\mathrm{Mod}(R)$ is called the *Riemann moduli space* and denoted by $\mathcal{M}(R)$.

As always, we should quote Thurston here [60, p. 259]:

> Informally, in Teichmüller space, we pay attention not just to what metric a surface is wearing, but also to how it is worn. In moduli space, all surfaces wearing the same metric are equivalent. The importance of the distinction will be clear to anybody who, after putting a pajama suit on an infant, has found one leg to be twisted.

[1] The interested reader can find a very nice intuitive discussion of this in [20].

4.3 Example. It is a standard result that the Teichmüller space of the punctured torus is $\mathbb{H}^2$ and the mapping class group is $\mathrm{PSL}(2, \mathbb{Z})$. It is somewhat harder to see that the Teichmüller space of the four-punctured sphere is also $\mathbb{H}^2$ and the mapping class group is $\mathrm{PSL}(2, \mathbb{Z}) \ltimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$. (The factor of a Klein 4-group is generated by the two **hyperelliptic involutions** on the torus. We survey the relation with braid groups in [19].) In general the computation of mapping class groups is very hard, even in the compact case.

We can now state the definition of the Teichmüller space of Fuchsian groups.

4.4 Definition. The **(reduced) Teichmüller space** of a Fuchsian group Γ is the set $\mathrm{Teich}^{\#}(\Gamma)$ of quasiconformal mappings $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which fix $\{0, 1, \infty\}$ pointwise and such that $f \Gamma f^{-1}$ is Fuchsian.

4.5 Lemma. *If R is compact and uniformised by the Fuchsian group G, then there is a natural identification* $\mathrm{Teich}(R) = \mathrm{Teich}^{\#}(G)$.

Proof. [27, §5.1]

This definition does not give very much insight. The key ideas in this direction are primarily due to Sullivan and his collaborators, who made a clear analogy between moduli spaces of Kleinian groups and moduli spaces of dynamical systems. More precisely, we need the λ-lemma, which states that holomorphic deformations of the matrices in a group will give quasiconformal deformations of the quotient surface. We first make clear the kinds of deformations needed.

4.6 Definition. Let $A \subseteq \hat{\mathbb{C}}$. A **holomorphic motion** of A is a map $\Phi : \Delta \times A \to \hat{\mathbb{C}}$ (where Δ is the unit disc in $\mathbb{C}$) such that

1. For each $a \in A$, the map $\Delta \ni \lambda \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is holomorphic;
2. For each $\lambda \in \Delta$, the map $A \ni a \mapsto \Phi(\lambda, a) \in \hat{\mathbb{C}}$ is injective;
3. The mapping $A \ni a \mapsto \Phi(0, a) \in \hat{\mathbb{C}}$ is the identity on A.

See the schematic in Fig. 15.

The original version of the following result was due to Mañé, Sad, and Sullivan [39], but we need an extended version due to Słodkowski [56, 57] which was made equivariant by Earle, Kra, and Krushkal' [17]:

4.7 Theorem (Equivariant extended λ-lemma)**.** *Let $A \subseteq \hat{\mathbb{C}}$ have at least three points, and let Γ be a group of conformal motions preserving A. Let $\Phi : \Delta \times A \to \hat{\mathbb{C}}$ be a holomorphic motion on A, and suppose that for each $\gamma \in \Gamma$ and each $\lambda \in \Delta$ there is a conformal map $\theta_\lambda(\gamma)$ such that*

$$\Phi(\lambda, \gamma(z)) = \theta_\lambda(\gamma)(\Phi(\lambda, z)) \tag{4.8}$$

for all $z \in A$. Then Φ can be extended to a holomorphic motion $\Phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that (4.8) holds for all $z \in \hat{\mathbb{C}}$, and such that

1. *For each $\lambda \in \Delta$, the map $\tilde{\Phi}_\lambda$ defined by $\hat{\mathbb{C}} \ni a \mapsto \tilde{\Phi}(\lambda, a) \in \hat{\mathbb{C}}$ is a quasiconformal homeomorphism with maximal dilatation satisfying*

$$K(\tilde{\Phi}_\lambda) \leq \frac{1 + |\lambda|}{1 - |\lambda|};$$

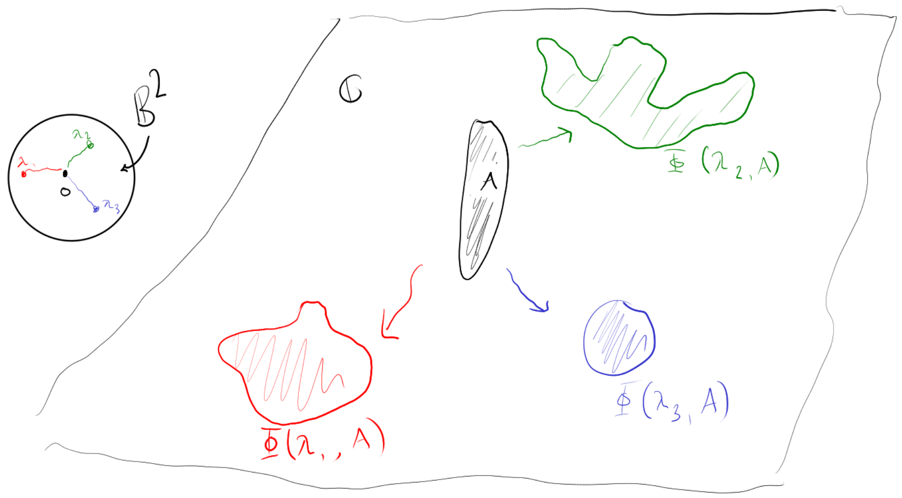

Figure 15: Images of $A \subset \mathbb{C}$ under a holomorphic motion Φ.

2. *$\tilde{\Phi}$ is jointly continuous in $\Delta \times \hat{\mathbb{C}}$; and*

3. *For all $\lambda_1, \lambda_2 \in \Delta$, $\tilde{\Phi}_{\lambda_1} \tilde{\Phi}_{\lambda_2}^{-1}$ is quasiconformal with*

$$\log K\left(\tilde{\Phi}_{\lambda_1} \tilde{\Phi}_{\lambda_2}^{-1}\right) \leq \rho(\lambda_1, \lambda_2)$$

(where ρ is the hyperbolic metric on Δ).

A holomorphic motion of the coefficients of a matrix in a Kleinian group will give a holomorphic motion of the limit set as long as the fixed points of the group don't collide (since the latter is the closure of a set of points depending algebraically on the matrix) By the λ-lemmas, a holomorphic motion of the limit set of a Kleinian group (induced by a holomorphic motion of the matrix coefficients) extends quasiconformally to the entire Riemann sphere; in particular, the ordinary set. This motivates:

4.9 Definition. The **quasiconformal deformation space** of a Kleinian group Γ, denoted by $\mathrm{QH}(\Gamma)$, is the space of representations $\theta : \Gamma \to \mathrm{PSL}(2, \mathbb{C})$ (up to conjugacy) such that

1. θ is faithful and $\theta\Gamma$ is discrete;

2. θ is type-preserving, that is if $\gamma \in \Gamma$ is parabolic (resp. elliptic of order n) then $\theta\gamma$ is parabolic (resp. elliptic of order n); and

3. the groups $\theta\Gamma$ are all quasiconformally conjugate (i.e. there exists some quasiconformal $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ depending on θ such that, as functions, $\theta\Gamma = f\Gamma f^{-1}$).

The space is equipped with a natural metric defined in roughly the same way as the classical Teichmüller metric (the distance between two deformations is defined to be the log of the maximal dilatation of the composition of the two quasiconformal homeomorphisms). In the special case that Γ is Fuchsian, we call $\mathrm{QH}(\Gamma)$ a **quasi-Fuchsian moduli space**.

Figure 16: The Maskit embedding of punctured tori groups, from [50, p. 288].

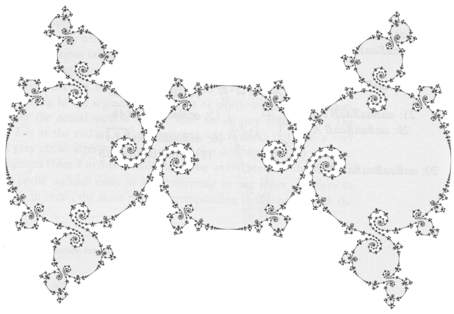

Figure 17: The limit set of a punctured torus group, from [50, p. 264].

This definition was studied first by Bers [8, 9], Kra [33, 32] and Maskit [43, 44]; modern textbooks and monographs which discuss this theory include [45] (essentially the whole book), [41, Chapter 5], and [28, Chapter 8]

The exact relationship between $\mathrm{QH}(\Gamma)$ and $\mathrm{Teich}(\Omega(\Gamma)/\Gamma)$ is subtle to define (see the references following the theorem), so we give only a rough version here. This result is attibuted by Bers [9, §2.4] to Bers and Greenberg [10] and Marden [40].

4.10 Theorem. *Let Γ be a finitely generated non-elementary Kleinian group with non-empty domain of regularity and let $S = \Omega(\Gamma)/\Gamma$. Then there is a well-defined holomorphic surjection $p : \mathrm{Teich}(S) \to \mathrm{QH}(\Gamma)$ together with a discrete subgroup $\widehat{\mathrm{Mod}}(S) \leq \mathrm{Mod}(S)$ and a natural bijection $\mathrm{QH}(\Gamma) \simeq \mathrm{Teich}(S)/\widehat{\mathrm{Mod}}(S)$ such that the two projection maps agree.*

In the case that Γ is geometrically finite, the group $\widehat{\mathrm{Mod}}(S)$ is the subgroup generated by Dehn twists along simple closed curves which bound compression discs. The point is that we are now allowing conjugacy by hyperbolic isometries; so the moduli space can no longer detect deformations that are trivial on the interior of the hyperbolic manifold.

4.11 Example (Maskit groups)**.** Recall the groups G_μ of Example 2.8 which gave

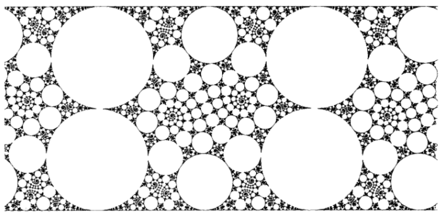

Figure 18: The limit set of a cusp group on the boundary of the Maskit embedding of punctured torus groups, from [31, Fig. 2].

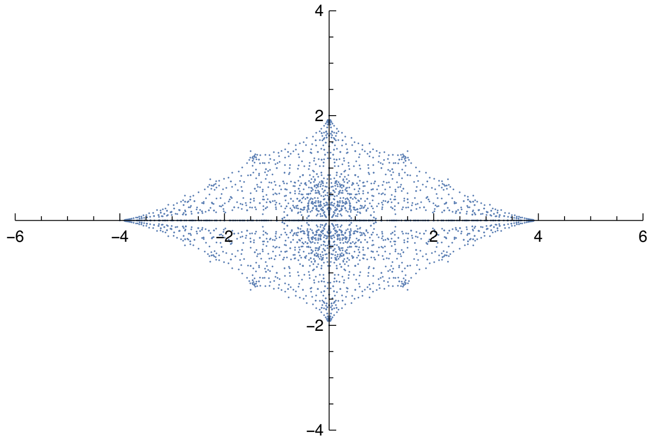

Figure 19: The (parabolic) Riley slice.

punctured tori for some values of μ. The **Maskit embedding** is the set of all $\mu \in \mathbb{C}$ such that G_μ glues the upper half-plane up to a punctured torus and the lower half-plane up to a 3-times punctured sphere; this is pictured in Fig. 16, and forms the easiest example of a quasiconformal deformation space. See the references in the earlier example for a detailed discussion of the moduli structure. We give examples of limit sets, one on the interior of the deformation space Fig. 17 and one on the boundary Fig. 18. The idea is that continuous deformation in the moduli space results from and results in continuous deformation of the limit sets.

4.12 Example (Riley groups)**.** A more complicated example is the **Riley slice** of Schottky space (Fig. 19), which is the quasiconformal deformation space of four-times punctured spheres. (It is more complicated because the corresponding surface is not in two halves, more precisely it is a quasi-Fuchsian group whose limit set does not fill the entire topological circle that it is contained in and so there are additional phenomena that occur which come from hyperbolic geometry and knot theory. For a full discussion see [18].)

4.13 Example (Schottky space)**.** More generally, a **Schottky group** (on n circles) is an element of the quasiconformal deformation space of a classical Schottky group on n circles; by the theory above, this deformation space is a lift of the usual moduli space $\mathcal{M}_{g,0}$. The boundary contains various **cusp groups**, these are groups where loxodromic elements which represent non-boundary-parallel simple closed curves on the surface are pinched to become parabolic. (A group is called **maximally cusped** if this process cannot be continued, in which case it represents a disjoint union of three-times punctured spheres.) There are other groups on the boundary, including **B-groups** and groups with dense limit set; these correspond to 'ideal' surfaces on the boundary of the Riemann moduli space, or **virtual ends** of the hyperbolic 3-manifold which is really being degenerated.

The boundaries of deformation spaces are very intricate and not very well understood in general. For a survey of some results see [41, Chapter 5]; suffice it to say that most modern techniques for their study are based on Thurston's theory of the geometry of hyperbolic 3-manifolds via objects like foliations and laminations. (This is not surprising, since we have already seen quadratic differentials appear and these are somehow dual to foliations.) For the classical study of the boundary, which is complicated even in the Fuchsian case, see the seminal papers of Bers [8] and Maskit [43]; some of the conjectures contained in there have been resolved, but many have not.

We end with some references to algebraic points of view; primarily, see [28] but there is also a nice discussion of the 'algebraicity' of the various conditions of Definition 4.9 in [27], and we should also mention the original work of Culler and Shallen [16] which is particularly worth a read for algebraic geometers.

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