ALGEBRAIC GEOMETRY AND THE MODULI OF KLEINIAN GROUPS

Abstract. here is the abstract

There are many analogies between the Riemann moduli space of curves and the moduli of Kleinian groups; however, the latter is in some ways much richer. In this section we provide basic definitions and a dictionary between the two languages.

We begin with the basic definitions, which can be motivated either by analogy with the classical Teichmüller theory for Riemann surfaces or by analogy with the deformation spaces of dynamical systems generated by rational maps. Either way, if G is a sufficiently nice[1] Kleinian group then we can define the space of quasiconformal deformations of G: it is the set of representations $\rho : G \to \mathrm{PSL}(2,\mathbb{C})$ such that

(1) ρ is faithful and $\rho(G)$ is discrete;
(2) ρ is type-preserving (i.e. $\rho(g)$ is parabolic iff g is parabolic); and
(3) there exists a quasiconformal map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\varphi G \varphi^{-1} = \rho(G)$.

This space of deformations is denoted by $\mathrm{QH}(G)$.

Recall that the Teichmüller space of $S = \Omega(G)/G$ is the set $T(S)$ of quasiconformal deformations of S, modulo isotopies of the surface. By the ending lamination theorem, one should expect that if $\Omega(G) \neq 0$ then the moduli of G should be given by the moduli of $\Omega(G)/G$. Indeed, one can show that $\mathrm{QH}(G)$ is a quotient space of $T(S)$ by a subgroup of $\mathrm{Mod}(S)$, namely the subgroup of mapping classes which can be realised by the extensions of actions of isotopies of $\mathbb{H}^3/G$ to the conformal boundary S. (For instance, if S is a compact genus two surface then there is a simple closed curve γ_∞ on the surface which splits the surface into two tori-with-deleted-discs; the Dehn twist along this curve can be realised by taking the surface and twisting it in ambient 3-space, so it is not detected in the hyperbolic manifold and hence is not seen by the group.) In particular we see that $\mathrm{QH}(G)$ is an intermediate space between the Teichmüller space and the Riemann moduli space $\mathcal{M}(S) = T(S)/\mathrm{Mod}(S)$.

We restrict now to Schottky groups. Fix G a rank r Schottky group; then $\mathrm{QH}(G)$ parameterises all rank r Schottky groups, and all such groups arise by holomorphic deformations of the coefficients of generators of G: a choice of generators for G is an element $Z_0 \in (\mathrm{PSL}(2,\mathbb{C}))^r$, and a holomorphic deformation of Z_0 is a holomorphic map $\mu : \Delta \to (\mathrm{PSL}(2,\mathbb{C}))^r$ such that $\mu(0) = Z_0$. By the λ-lemma every holomorphic deformation of generators induces a quasiconformal deformation of the group (though in general the space of deformations of generators is only a covering space of $\mathrm{QH}(G)$). In particular:

> *Every algebraic curve C of genus $g > 1$ arises as the surface at infinity uniformised by a Schottky group G of rank g, and there*

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[1] torsion free and geometrically finite with non-empty domain of discontinuity

is an open neighbourhood $U \subseteq \mathrm{QH}(G)$ of G with algebraic coordinates (entries of matrices generating the group) which is locally biconformal to the neighbourhood of that curve C in M_g.

Thus one advantage of studying Schottky groups as models for algebraic curves is the ease of construction: one can immediately write down curves of arbitrary genus (at least numerically, as we studied in earlier sections of this paper) which is a very hard thing to do with purely algebraic machinery [HM98, §6F].

Warning. The matrix entries do not provide a nice coordinate system in any geometric sense: they are very local and do not represent any geometric quantities: one natural choice of coordinates are the Fenchel–Nielsen coordinates, which can be generalised to the Kleinian group setting, but the relationship between these coordinates and the matrices generating the uniformisig groups are very subtle: this was alluded to from the algebraic side by Mumford [Mum99, p. 233] but this problem has a much longer history from the point of view of complex dynamics and geometric group theory surveyed in [EMS23].

One can compactify $\mathrm{QH}(G)$ in a natural way, by taking the closure $\overline{\mathrm{QH}(G)}$ in the character variety. This is a version of the usual Thurston compactification of Teichmüller space. The relationship between this compactification and the Deligne-Mumford compactification of M_g is very nice, and provides another reason for viewing Schottky groups as a natural model for algebraic curves. In order to explain this we must first understand some geometry.

Lemma 0.1. *For any sufficiently nice Kleinian group G, every point on the the boundary of $\mathrm{QH}(G)$ can be reached by a one-parameter holomorphic family of representations ρ_t (with $t \in [0, \infty)$) such that there is some word $W \in G$ with the property that as $t \to \infty$, $\operatorname{tr}^2 \rho_t(W) \to 4$ (i.e. the word W goes parabolic).* □

Let G be a sufficiently nice Kleinian group and let $\gamma \in G$ be a parabolic element, with fixed point ζ. Then there exists two open round discs U_1, U_2 in $\Omega(G)$ whose closures are tangent at ζ, and the quotients U_1/G and U_2/G are disjoint punctured discs in the quotient surface. Even better, in the ambient space $\mathbb{H}^3/G$ one can view these two punctured discs as 'compactifiable' in the sense that one can add a single point p to create a 'pinched tube' at infinity. The one-parameter family of Lemma 0.1 can be viewed very concretely as the following procedure: pick a closed geodesic wrapped around a tube on the surface $S = \Omega(G)/G$; this is represented by a loxodromic element $\gamma \in G \simeq \pi_1(S)$; now the elements γ_t as $t \to \infty$ are elements representing geodesics on $\Omega(G_t)/G_t$ becoming closer and closer to parabolic, i.e. they represent geodesics of ever-shortening length, until in the limit the tube wrapped by γ_t becomes pinched to a single point.

Let C be an algebraic curve of genus g. Natural deformations from $C \in M_g$ to curves $\tilde{C} \in \overline{M_g}$ with single nodes are obtained by taking loxodromic elements of C and deforming them to become parabolic elements.

This can be practically achieved with classical Schottky groups in a very natural way. Let C, C' be two of the round circles used to define some classical Schottky G and suppose that the loxodromic g pairing them is, in fact, hyperbolic. Write the matrix of g in terms of the data of C and C' and suppose that there is some holomorphic path in the parameter space of this data such that (i) at every point on the path the deformed pairing $\tilde{g}$ is still hyperbolic and the deformed group $\tilde{G}$ is still a Schottky group (this is a hard condition to check in general unless you deform C and C' in order to keep them far away from other defining curves, but this is not a particularly restricting thing to do) and (ii) in the limit, C and C' become tangent.

Example 0.2. Let X and Y be transformations given by the matrices

$$X = \begin{bmatrix} 1/t & 1/t - t \\ 1/t & 1/t \end{bmatrix} \text{ and } Y = \begin{bmatrix} 2-5i & 28i \\ -i & 2+5i \end{bmatrix}.$$

Here X and Y have been chosen to be hyperbolics, X with isometric circles at ± 1 of radius t and Y with isometric circles at $5 \pm 2i$ of radius 1. The group $\langle X, Y \rangle$ is a classical Schottky group if the isometric circles of X and Y don't collide, and one can check that this true for all $t \in (0,1)$. When $t \to 1$, X becomes parabolic, and the quotient surface degenerates to a torus with two paired punctures: i.e. a surface corresponding to a torus with a single node.

Warning. The numeric procedure for generating curves from surfaces given in previous sections breaks down for groups containing parabolics, since there are many more meromorphic functions on the surface than just the rational ones and so there is no longer a natural identification between the algebraic and meromorphic function fields.

One can iterate this process, pinching down different curves on the surface to obtain more and more nodes. The only condition is that curves which are pinched must not be isotopic to a puncture, or (equivalently) every component which arises must be hyperbolic. This is the same as the stability condition for $\overline{M_g}$. The maximally deformed surface is a surface formed as a union of thrice-punctured spheres (i.e. obtained from a compact surface by pinching a system of $3g-3$ non-intersecting and mutually non-isotopic closed curves). The corresponding groups are called maximally cusped groups [Ber70; Mas70] and correspond to graph curves [BE91].

> *Let $g > 2$; then the boundary strata of $\overline{M_g}$ can be obtained by geometrically natural deformations of Schottky groups, and can be modelled by cusp groups (that is, groups which are obtained from Schottky groups by sending a sequence of cyclic loxodromic subgroups to cyclic parabolic subgroups without changing the type of elements outside those subgroups).*

Example 0.3. We will give a one-parameter family of rank n classical Kleinian groups (for all $n > 2$) whose algebraic limit lies on the boundary of rank n Schottky space. The group will be generated by n elements, pairing the isometric circles shown in Figure 1. Define the three matrices

$$Y = \begin{bmatrix} -t^{-1} & -i(t+1)t^{-1} \\ i(1-t)t^{-1} & -t^{-1} \end{bmatrix}$$

$$X = \begin{bmatrix} -t^{-1} & i(t-1)(2t)^{-1} \\ 2t^{-1} & -t^{-1} \end{bmatrix}$$

$$Z = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and for each $k \in \mathbb{Z}$ set $X_k := Z^k X Z^{-k}$. The group G_t is then defined by

$$G_t := \langle Y, X_0, \dots, X_{n-2} \rangle.$$

The generators of G_t have been carefully chosen to not only have the correct isometric circles, but also to be hyperbolic for all t.

When $t = 1$, the generators all become parabolic and the quotient surface degenerates to a union of thrice-punctured spheres. This follows from a careful application of the full statement of the Poincaré polyhedron theorem, but the point is that

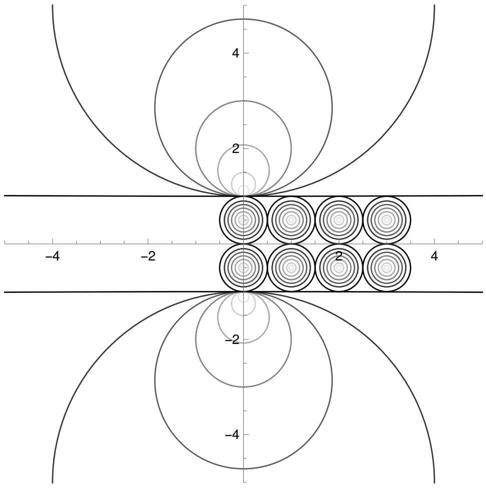

FIGURE 1. The isometric circles of the one-parameter family of groups described in Example 0.3 for $n = 5$; as $t \to 1$, the radius of the circles increases until in the limit they are tangent.

because the degeneration is highly symmetric[2] the meridian curves of the surface are pinched to cusps.

Warning. The boundary of $\overline{\mathrm{QH}(G)}$ is in general much more complicated (even locally) than the boundaries of either the Teichmüller space or the Deligne-Mumford compactification of the Riemann moduli space. The cusp groups obtained on the boundaries of Schottky space by pinching closed geodesics are dense in the boundary by a famous result of McMullen [McM91] but are only first category in the boundary: most groups are not cusp groups, for instance they might not be geometrically finite; they will be free, but will have limit set of positive measure in $\hat{\mathbb{C}}$. These groups correspond to pinching curves which are dense in the entire surface, and so are not so interesting from the algebraic point of view.

It is not immediately clear how to deal with marked points which are not nodal. It is a fundamental result in the geometric theory of Kleinian groups that for sufficiently nice groups every rank 1 cusp (i.e. puncture) is paired with another in the manner discussed above. One can either keep track of markings directly as extra data, or keep additional components of the surface around in order to connect to punctures corresponding to marked points. Masaaki [Yos97] takes this latter approach by using Fuchsian uniformisations and only keeping track of the algebraic data carried by one of the two components; in general one could add thrice-punctured spheres off any additional wanted marked points since these will not affect moduli.

Let us finally discuss the main downside of the study of algebraic moduli via Schottky groups: the boundary of Schottky space is very very complicated (it is cut out by infinite families of polynomials) and the construction of groups on the boundary with certain combinatorial properties is fairly difficult; given a sufficiently general Schottky group it is hard to actually write down a path to the boundary at

[2]More precisely, we reach the boundary upon deforming such that the points of tangency of the circles are mapped onto each other in cycles by the circle-pairing transformations [Mas87, p. IV.I.6]; if the circles become tangent in a non-symmetric way—i.e. the axes of the transformations which pair the circles are not parallel—then it is possible to deform the fundamental domain slightly such as to remove the tangency.

all (and even checking that a group is Schottky is hard as it is essentially equivalent to the discreteness problem for groups). One plausible method for doing this which is known to work in low-dimensional cases is via the work of Birman on braid groups, described in detail in [EMS22]. This line of thought is also followed to some extent by some numerical algebraic geometers [Bog12, Chapter 3].

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