Schottky groups over $\mathbb{Q}_p$

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February 28, 2023

Abstract

We recall the general theory of Schottky groups over $\mathbb{C}$, and place it in context within geometric group theory in order to make clear the analogies with the theory over non-Archimedean fields.

Contents

§1. Abstract CAT(0) theory

Let (X, d) be a CAT(0) metric space [4, Chapter II.1]. This means that the following conditions are satisfied:

1. **geodesicity**: for every pair of points $x, y \in X$ there exists an isometric embedding $f : [0, d(x, y)] \to X$ such that $f(0) = x$ and $f(d(x, y)) = y$.

2. CAT(0) **inequality** (Fig. 1): for every triangle $\Delta(p, q, r)$ in X, by the triangle inequality there exists a triangle $\bar{\Delta} = \Delta(\bar{p}, \bar{q}, \bar{r})$ in $\mathbb{E}^2$ with side lengths $d(\bar{p}, \bar{q}) = d(p, q)$, $d(\bar{q}, \bar{r}) = d(q, r)$, and $d(\bar{r}, \bar{p}) = d(r, p)$; the condition is that if $a \in [p, q]$ and $b \in [p, s]$ are such that $d(p, a) = d(p, b)$ and if $\bar{a}$ and $\bar{b}$ are chosen in $[\bar{p}, \bar{q}]$ and $[\bar{p}, \bar{r}]$ such that $d(\bar{p}, \bar{a}) = d(\bar{p}, \bar{b})$ then $d(a, b) \leq d(\bar{a}), d(\bar{b})$.

We will also assume that (X, d) is **complete**.

Two geodesic rays $c, c' : [0, \infty) \to X$ are called **asymptotic** if there exists δ such that $d(c(t), c'(t)) < \delta$ for all t. This is an equivalence class on rays, and the quotient space (the set of rays up to asyptoticity) is called the **visual boundary** $\partial(X)$. Given $x, y \in X$ and $\zeta, \xi \in \partial X$ we may define angles $\angle_x(\zeta, \xi)$ and $\angle_x(y, \zeta)$ in natural ways

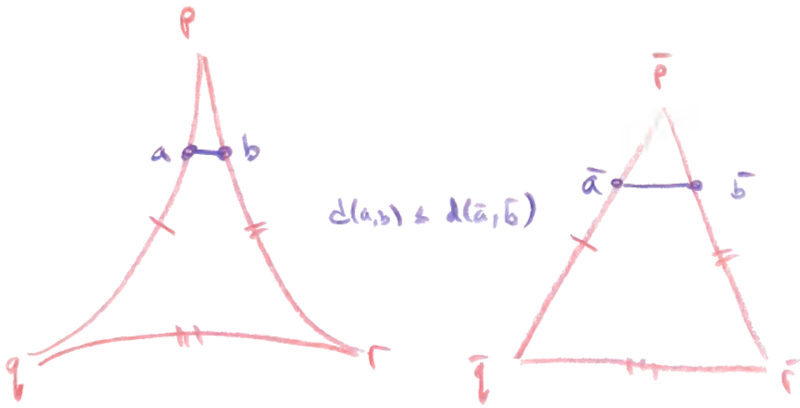

Figure 1: The CAT(0) inequality.

(via comparison). More precisely, if $c, c' : [0, T] \to X$ are geodesic rays then there is a limit

$$\angle(c, c') = \lim_{t \to 0} 2 \arcsin \frac{1}{2t} d(c(t), c'(t))$$

and this gives angles $\angle_{c(0)}(c(t), c'(t))$ for all t even when the domain is extended to allow $T = \infty$. We can now define the **sextant metric** on ∂X; if $\zeta, \xi \in \partial X$ then $\angle(\zeta, \xi) = \sup_{x \in X} \angle_x(\zeta, \xi)$. (Here on the right side we take appropriate representative rays for ζ and ξ based at each x.) This gives ∂X the structure of a complete metric space, and in fact it is CAT(1) (we have not appropriately defined this but one should think 'flat or spherical') [4, Chapter II.9].

Some examples:-

1.1 Example (Hyperbolic space)**.** Hyperbolic space $\mathbb{H}^n$ is CAT(0) (in fact strictly negatively curved) and $\partial \mathbb{H}^n = S^{n-1}$ (as a set). However with the sextant metric the boundary is discrete. This is because the angle between two points at infinity depends on the position of the observer, and the observer can be positioned to make this angle always π (Fig. 2).

1.2 Example (Euclidean space)**.** Euclidean space $\mathbb{E}^n$ is CAT(0) (zero curvature) and $\partial \mathbb{E}^n = S^{n-1}$. This time this is a homeomorphism, since the angle measured is independent of the position of observation.

1.3 Example (Trees)**.** Let T be an n-valent infinite tree. This is highly CAT(0) since all triangles are tripods. The space ∂T is the set of ends of T and the topology is a Cantor set.

1.4 Example (Euclidean buildings)**.** A **Euclidean building** is a building $\mathcal{B}$ whose apartments are Euclidean Coxeter complexes. The global metric on $\mathcal{B}$ comes from the local Euclidean metric on each apartment, and it is CAT(0) [1, Theorem 11.16]. The boundary $\partial \mathcal{B}$ can be given the structure of a **spherical building**, i.e. a building all of whose apartments are finite Coxeter complexes. As the terminology suggests these are CAT(1) spaces.

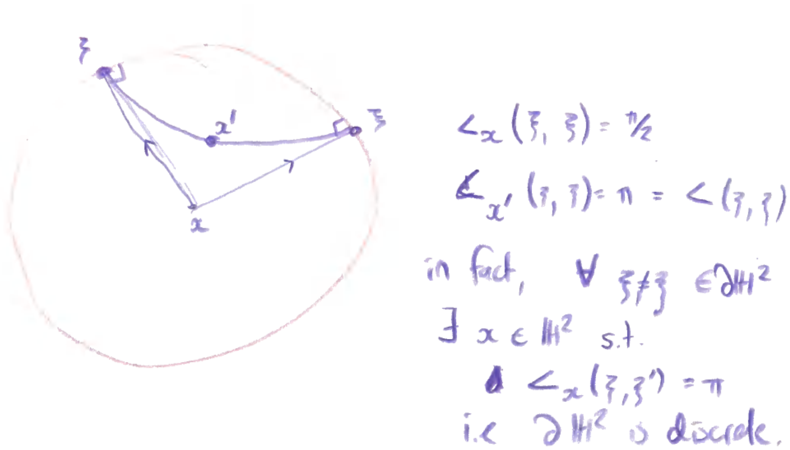

Figure 2: The sextant metric for $\mathbb{H}^2$ is discrete.

The differnce between $\mathbb{E}^n$ and $\mathbb{H}^n$ manifests itself in a very concrete fashion. Let f be a conformal map on S^{n-1}, and let $x \in B^n$ (the interior of S^{n-1}). From x we may draw two distinct geodesic rays to S^{n-1} and measure their angle θ; suppose their endpoints are ζ and ξ. Consider the images $f(\zeta)$ and $f(\xi)$. Now in $\mathbb{H}^n$ there is a unique point x' such that the rays $[x', f(\zeta)]$ and $[x', f(\xi)]$ meet at an angle θ. On the other hand, in $\mathbb{E}^n$ every point has this property. What we are saying is that conformal maps extend naturally from $\partial\mathbb{H}^n$ into $\mathbb{H}^n$ (define $f(x) := x'$) but that this property does not hold for $\mathbb{E}^n$. Clearly non-negative curvature is not enough to distinguish these cases, and Gromov gave a strengthening of the CAT(0) condition which can be used to gurantee existence of such an extension.

Let $\delta > 0$. A geodesic triangle is δ-**slim** if each of its sides is contained within a δ-neighbourhood of the union of the two other sides. If every geodesic triangle in X is δ-slim (for some UNIVERSAL δ) then X is called δ-**hyperbolic**. One can now show that if $\varkappa < 0$ then every CAT($\varkappa$) space is δ-hyperbolic (with δ depending only on $\varkappa$). If there exists some δ such that X is δ-hyperbolic then we simply say that X is a **hyperbolic space**.

The rough picture of the moduli theory comes from the following propiosition.

1.5 Proposition. *If X is a hyperbolic metric space, and X is quasi-isometric to some metric space Y, then ∂X and ∂Y are homeomorphic. The converse is not true [3].* ⅋⥃

The moduli space is usually the space of objects quasi-isometric to X; or the space of all quasi-isometry conjugates of holonomy groups $\pi_1(X)$; or the equivalent quasi-*conformal* things when passing to the boundary.

Given an isometry f of a hyperbolic space X, we may classify it into one of three types [10, §8], see Fig. 3.

1. f is **elliptic** if every orbit $\{f^n(x) : n \in \mathbb{Z}\}$ for $x \in X$ is bounded.
2. f is **parabolic** if every orbit $\{f^n(x) : n \in \mathbb{Z}\}$ is unbounded but there is a unique limit point of the orbit in ∂X.
3. f is **hyperbolic** if the map $\mathbb{Z} \to X$ given by $n \mapsto f^n(x)$ is a quasi-isometry for every $x \in X$, hence every orbit has two limit points in ∂X.

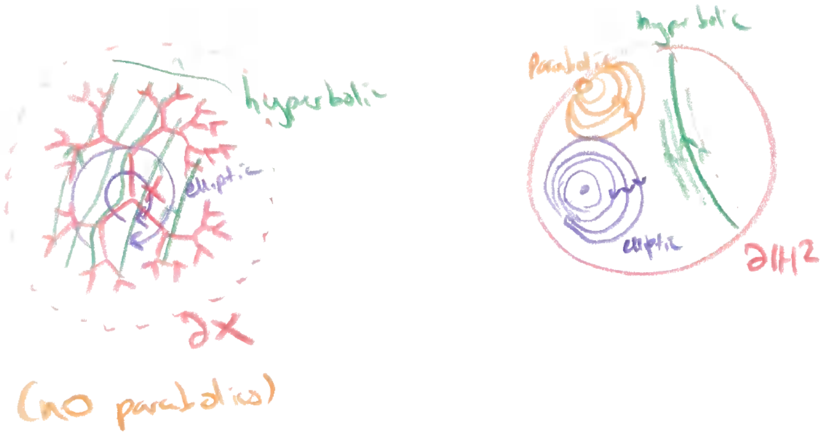

Figure 3: Isometries on the 3-valent tree (left) and $\mathbb{H}^2$ (right).

One obtains an action of f on ∂X by looking at the action on geodesic rays.

Remark. One can define a dual notion of **hyperbolic group**, a group for which one can choose a hyperbolic Cayley graph. This follows the general philosophy (known as far back as Dehn, and Stillwell in his introduction to Dehn's work [7] attributes it to Dyke and Poincaré) that if a group G acts properly on a space X then in the large scale G resembles X. In this interpretation, Gromov's **ideal hyperbolic boundary** ∂G is what is in modern language identified with the limit set of the group [10, §0.3(B')].

We claim that this action is somehow 'conformal', but the sextant metric as defined clearly does not capture this notion (it is the discrete metric on $\partial\mathbb{H}^n$). To fix this we just fix a basepoint $0 \in B^{n-1}$; the isometry group now preserves angles at 0 and so clearly acts isometrically with respect to the sextant metric.

§2. Review of the theory over $\mathbb{C}$

We recall that a **Kleinian group** is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$. The boundary $\partial\mathbb{H}^3$ is the sphere S^2, and if $\mathbb{H}^3$ is modelled as the upper half-plane $\{(z,t) \in \mathbb{C}\times\mathbb{R} : t > 0\}$ then $\partial\mathbb{H}^3 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with the usual conformal structure. We list some standard facts [2]:

1. $\mathrm{Isom}^+(\mathbb{H}^3)$ is naturally isomorphic to the group $\mathbb{M}$ of conformal motions of $\hat{\mathbb{C}}$ and the two group actions on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$ coincide.

2. The notions of elliptic, parabolic, and hyperbolic isometries can be detected in the conformal structure: a conformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is

   (a) **elliptic** if it has two fixed points and all orbits form round circles orthogonal to the great circle joining them;

   (b) **parabolic** if it has one fixed point and the orbits form a pencil of circles through it;

   (c) **hyperbolic** if it has two fixed points and the orbits form curves from one to the other (with one attracting and one repelling).[1]

[1] In the usual terminology of Kleinian groups, e.g. as defined in [11, §I.B], these elements are called **loxodromic**.

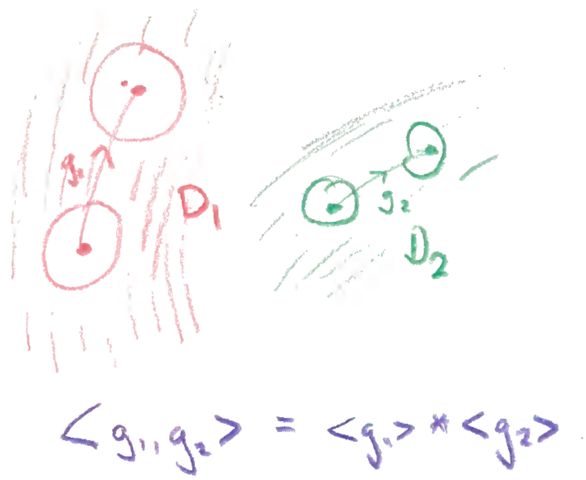

Figure 4: The Klein combination theorem for two disjoint circle pairings.

3. There is a standard isomorphism $\mathbb{M} \simeq \mathrm{PSL}(2, \mathbb{C})$ such that $\mathrm{tr}^2 f \in [0, 4)$ for elliptic elements f, $\mathrm{tr}^2 f = 4$ for parabolic elements f, and $\mathrm{tr}^2 f \notin [0, 4]$ for hyperbolic elements.

We now study Schottky groups, which are the simplest kinds of Kleinian groups. We will need some adjectives for group actions. Let G act as a group of homeomorphisms on a topological space X.

1. The action is **freely discontinuous** if for every point $x \in X$ there exists a neighbourhood V of x such that $GV \cap V = \varnothing$.
2. A set $D \subseteq X$ is **precisely invariant** under G if $GD \cap D = \varnothing$.
3. A **fundamental set** for the action of G on X is a subset $D \subseteq X$ containing exactly one representative from each G-orbit in X. (This is not to be confused with a **fundamental domain**, for which see [11, §II.G.1])

The following theorem appears as [11, §VII.A.13].

2.1 Theorem (Klein combination theorem)**.** *Let G_1 and G_2 be subgroups of* $\mathrm{Homeo}(X)$ *that act freely and discontinuously on some open subset $U \subseteq X$. Suppose that there is a fundamental set D_m for G_m ($m = 1, 2$) where $D_1 \cup D_2 = X$ and $D = D_1 \cap D_2 \neq \varnothing$. Then $G = \langle G_1, G_2 \rangle$ is the free product $G_1 * G_2$ and D is precisely invariant under the identity in G.*

2.2 Example. A **Schottky group** of rank r is defined by the data of r disjoint circles[2] $C_1, C_1', \ldots, C_r, C_r'$ in $\hat{\mathbb{C}}$ which bound a common exterior D, together with for each i a conformal map $g_i \in \mathbb{M}$ such that $g_i(C_i) = C_i'$ and g_i maps the exterior of C_i into the interior of C_i'; see an example for $r = 2$ in Fig. 4. By induction based on Theorem 2.1, the group $\langle g_1, \ldots, g_r \rangle$ is free on those r generators and maps the domain D entirely off itself. We also see that G is purely hyperbolic, since every word in G has two fixed points (do a ping pong argument).

[2]We mean *topological* circles, not round circles.

Figure 5: A surface arising from a rank 4 Fuchsian Schottky group.

A **classical Schottky group** is a Schottky group defined by the data of *round* circles. There exist Schottky groups which are not classical. A non-constructive proof of this was first given by Marden, but an explicit example was found by Yamamoto [16].

There are various alternative characterisations of Schottky groups, the main one being the following which we take from [11, §X.H.6].

2.3 Proposition. *Let G be a finitely generated, free, purely hyperbolic Kleinian group is a Schottky group.*

Remark. Some fairly high-power machinery is needed to reduce the number of adjectives here; for instance, Ahlfors' finiteness theorem.

We have already alluded to the following definition.

2.4 Definition. Let G be a Kleinian group. Then the set of accumulation points of orbits of points of $\hat{\mathbb{C}}$ under G is the **limit set** $\Lambda(G)$ and the complement $\hat{\mathbb{C}} \setminus \Lambda(G)$ is the **domain of discontinuity**.

2.5 Proposition. *Let G be a torsion-free[3] Kleinian group which is not virtually abelian.*

1. *$\Omega(G)$ is the maximal subset of $\hat{\mathbb{C}}$ on which G acts freely discontinuously [11, §II.E].*
2. *The quotient $\Omega(G)/G$ is a (possibly empty) Riemann surface [11, §II.F].*
3. *The quotient $\mathbb{H}^3/G$ is a hyperbolic 3-manifold [15].*

From considering the (closure of the) fundamental set of Example 2.2, we see the following.

2.6 Proposition. *If G is a Schottky group of rank r, then $\Omega(G)/G$ is a genus r compact surface and $\mathbb{H}^3/G$ is a genus r handlebody.*

See Fig. 5 for the genus 4 case.

This gives us motivation for a second alternative definition for Schottky groups:

2.7 Theorem. *A Kleinian group G is a Schottky group iff $\mathbb{H}^3/G$ is a handlebody.*

For the moduli theory (see [12, Example 5.28]):

[3]This assumption can be easily removed after slight modification to the conclusions.

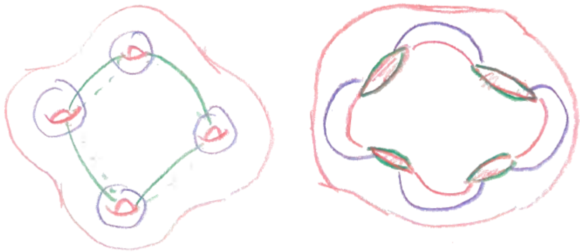

Figure 6: A surface arising from a rank 4 Fuchsian Schottky group.

2.8 Definition. The **Schottky space** of rank r is the set $\mathcal{Schottky}(r)$ of all faithful representations

$$\rho : F(r) \to \mathbb{M}$$

such that $\rho(F(r))$ is a Schottky group (where $F(r)$ is the free group on r symbols), modulo conjugacy in $\mathbb{M}$.

This space admits a natural complex structure in two ways. Firstly, as a subset of the representation variety of $F(r)$; and secondly, via the following result

2.9 Theorem. *Let G be an arbitrary Schottky group of rank r. Then $\mathcal{Schottky}(r)$ is equal to the set of quasiconformal conjugates of G, and is a holomorphic image of the genus r Teichmuller space.* ⩕⩮

It is not clear that the two metrics are equivalent, in fact we have conjectured only that they are quasi-isometric.

§3. The theory over $\mathbb{R}$

Let us now consider the Fuchsian case. A group is **Fuchsian** if it is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$; the conformal action on $\partial\mathbb{H}^2 = S^1$ is again as fractional linear transformations (with $\mathbb{R}$-coefficients) and so we can view a Fuchsian group as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$.

Warning. Just looking at matrix groups suggests that Fuchsian groups are naturally Kleinian groups, but one must remember that we are not working with matrix groups in isolation. Fuchsian groups act on $\mathbb{H}^2$, and Kleinian groups act on $\mathbb{H}^3$, so there is some subtlety involved.

3.1 Definition. A **Schottky group** over $\mathbb{H}^2$ of rank r is a purely hyperbolic Fuchsian group which is free of rank r.

Since $\mathbb{H}^2$ is simply connected, we can identify such a Schottky group G with $\pi_1(\mathbb{H}^2/G)$. Thus the quotient surface is a Riemann surface which topologically has r deleted discs. The moduli space $\mathcal{Schottky}_{\mathbb{R}}(r)$ containing G is equivalent to the usual Teichmüller space of genus 0 surfaces with r deleted discs.

Warning. In contrast to the complex case, Schottky groups do not uniformise compact surfaces (since the fundamental group of a compact surface is not free). The point is that they uniformise genus r surfaces cut 'in half' (compare Fig. 6 with the earlier Fig. 5).

§4. The theory over $\mathbb{Q}_p$

Let (K, v) be a discrete valuation field, and let $\mathcal{O}$ be its valuation ring; we will define Schottky groups in $G = \mathrm{PSL}(2, \mathbb{Q}_p)$ in analogy with the Archimedean cases in the previous sections. First, we must find a natural space on which G acts isometrically. For this we follow the standard reference by Serre [14].

Fix a two-dimensional vector space V over K. A **lattice** in V is a finitely-generated $\mathcal{O}$-submodule of V which spans V as a K-vector space. If $x \in K^*$ then xV is also a lattice, and we denote by X the set of lattices modulo this K^*-action.

Given two lattices L and M, by the structure theorem for modules over a PID we have that there exist compatible bases for L and M, i.e. a basis (e_1, e_2) for L and $n_1, n_2 \in \mathbb{Z}$ such that $(\pi^{n_1} e_1, \pi^{n_2} e_2)$ is a basis for M. The integer $|a - b|$ does not depend on the equivalence classes of L and M in X and so we define $d([L], [M]) = |a - b|$.

4.1 Theorem ([14, Chapter II, Theorem 1])**.** *The pair (X, d) is ($\mathbb{Z}$-valued) metric space, and the incidence structure defined by $[L] \sim [M] \iff d([L], [M]) = 1$ is a tree, the **Bruhat-Tits tree** T_K over (K, v). (This boils down to asking for there to exist representatives L, M such that $\pi L < M < L$.) This tree is regular with valance equal to one greater than the characteristic of the residue field.* ⅅ⥄

The tree has a natural 2-colouring, where we label vertices with the same colour if they are of even distance. The group $\mathrm{GL}(V)$ acts isometrically on the tree by virtue of acting on the lattices, and the group $\mathrm{SL}(V)$ acts as the subgroup of $\mathrm{GL}(V)$ which preserves this colouring.

Alternatively, we can throw the valuation v away and just look at the group $\mathrm{PSL}(2, K)$. The building of $\mathrm{PSL}(n, K)$ is equal to the complex of flags of proper non-zero subspaces of K^n; that is, in the case $n = 2$, the building is the projective space $\mathbb{P}^1 K$ [1, §6.5]. In fact the action is the usual action of $\mathrm{PSL}(2, K)$ on $\mathbb{P}^1 K$.

4.2 Proposition ([1, Exercise 6.114])**.** *The building $\mathbb{P}^1 K$ is the spherical building at infinity associated to T_K.* ⅅ⥄

Remark. In the cases of hyperbolic groups $\mathrm{Isom}^+(\mathbb{H}^n) \simeq \mathrm{PSL}(2, F)$, we have the most 'computable' action given by the action on the conformal boundary, and the action on the interior given by some kind of extension F'/F (either $\mathbb{C}/\mathbb{R}$ or $H/\mathbb{C}$—this latter we have not discussed but see [2, §4.1]). One should ask now how to realise the natural action which one has on the interior in the non-Archimedean case to the Archimedean case, for instance by looking at lattices of rank 2 in $\mathbb{R}^2$. One is tempted to reverse the non-Archimedean process and ask the following question: let B be the structure supported on $\bigcup_{\zeta, \xi \in \Lambda(G)} [\zeta, \xi]$; does the group action as a fractional linear map extend naturally to an action on this subset of $\mathbb{H}^n$, in such a way that the 'intersections' are in bijection with lattices in $\mathbb{R}^2$?

We quickly describe the rest of the theory. Let γ be an isometry of the n-valent tree T; its **translation length** is defined to be $\operatorname{trlen} \gamma = \inf_{x \in T} d(x, \gamma x)$ and the **axis** is $\{x \in T : d(x, \gamma x) = \operatorname{trlen} \gamma\}$. If the axis is non-empty then γ is called **semisimple**. In this case, if $\operatorname{trlen} \gamma = 0$ then γ is called **elliptic**, otherwise it is called **hyperbolic**.

4.3 Proposition. *Every isometry of a tree is semisimple. If an isometry γ is hyperbolic then it acts as a translation by* $\operatorname{trlen} \gamma$ *along its axis. Every elliptic isometry is finite order.* ⅅ⥄

Of course this implies that free groups are all purely hyperbolic. The converse is found as Theorem II.5 of Serre [14]:

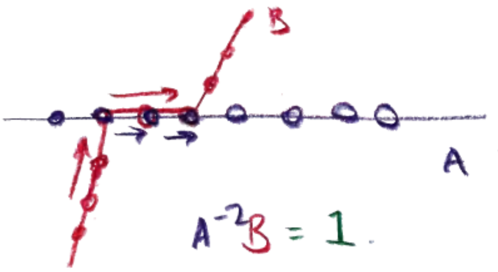

Figure 7: Long crossings of hyperbolic axes introduce relations.

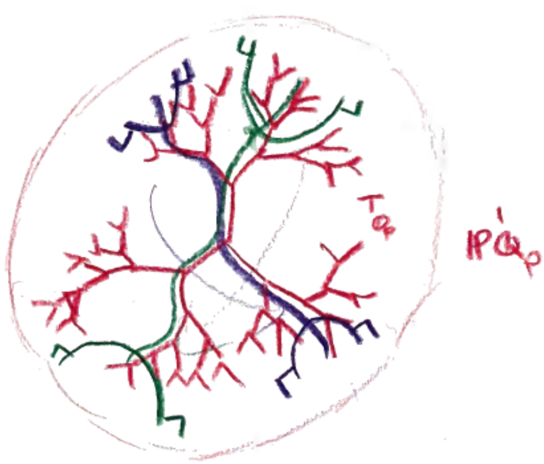

Figure 8: A fundamental domain for a non-Archimedian Schottky group.

4.4 Proposition. *Let (K, v) be locally compact (e.g. $K = \mathbb{Q}_p$). Let G be a discrete, torsion-free (so purely hyperbolic), subgroup of* $\mathrm{PSL}(2, K)$ *acting on the tree T. Then G is automatically free.*

Remark. In the case of G two-generated, all discrete subgroups of $\mathrm{PSL}(2, K)$ can be classified using the Maskit combination theorems [6].

We define a **Schottky group** of rank r over $\mathbb{Q}_p$ to be a discrete, torsion-free subgroup G of $\mathrm{PSL}(2, \mathbb{Q}_p)$. We observe that this gives us conditions on the axiss (axes) of the hyperbolic elements in the generating set: either they do not intersect, or the piece of the tree containing all of the intersections must be small. If the intersection is larger than either of the translation lengths then one can take products to obtain a transformation with a fixed point, as shown in Fig. 7. One can generalise this to the case of G a group of isometries of a possibly incomplete CAT(0) space, see [5].

We can apply (again) Theorem 2.1 to see what the quotient $T_{\mathbb{Q}_p}/G$ is. The picture is very similar to the case over $\mathbb{R}$, as one should expect from the philosophy that $\mathbb{H}^2$ is a 'thickened' tree. We show a fundamental domain for the genus two case in Fig. 8: one cuts out subtrees which cover the limit set of the full group (similar to

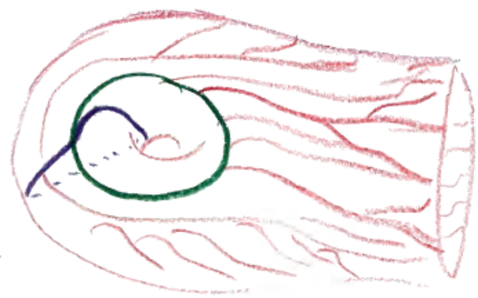

Figure 9: A quotient building(?) obtained from a non-Archimedian Schottky group of rank 2. The projections of the two axes are labelled following Fig. 8.

the construction of a Schottky domain by cutting out discs around components of the limit set) and the quotient looks like a complex which has cycles localised in a small piece with ends going off towards a circle at infinity, such that when it is thickened it looks like half of a genus two surface (Fig. 9). One can even generalise the point of view that a Schottky group is a group generated by transformations which pair circles; the quotient space at infinity $\mathbb{P}^1\mathbb{Q}_p$ is called a **Mumford curve** [9, 13].

4.5 Conjecture. *The moduli space $\mathcal{S}chottky_{\mathbb{Q}_p}(r)$ should look like a p-adic version of the Teichmüller space $\mathcal{S}chottky_{\mathbb{R}}(r)$; it will be the space of complexes quasi-isometric to the complex obtained from a single quotient.*

Remark. One might recall that the deformation theory of Schottky groups over $\mathbb{C}$ (more generally Kleinian groups) gives a natural diagram $\mathrm{Teich}(\Omega(G)/G) \to \mathrm{QH}(G) \to \mathcal{M}(\Omega(G)/G)$ where the full covering from the Teichmüller space to the Riemann moduli space is given by the mapping class group action and the intermediate covering comes from a natural subgroup of the mapping class group (this is surveyed in the expository article [8]). One is therefore tempted to ask for a similar picture in the p-adic case.

§5. Upper half-planes and Clifford algebras

Notation. We use $\mathbb{Y}$ (for Cayley) to denote the usual quaternion algebra $(-1,-1|\mathbb{R})$, since we use the standard notation $\mathbb{H}$ for hyperbolic spaces.

We have already seen that $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to the group of isometries of the hyperbolic plane, since it acts by fractional linear transformations (i.e. in the normal way as a group of projective transformations) on $\hat{\mathbb{R}} = \mathbb{P}^1\mathbb{R}$ and therefore extends uniquely to $\mathbb{H}^2$ which has $\hat{\mathbb{R}}$ as its boundary. One can also obtain this action directly: if we model $\mathbb{H}^2$ as the upper half-plane $H^2 = \{x + ti \in \mathbb{C} : t > 0\}$ then $\mathrm{PSL}(2,\mathbb{R})$ acts directly on it by fractional linear transformations as the hyperbolic isometry group. (We already saw this earlier when we remarked that isometries of $\mathbb{H}^2$ are just conformal maps preserving the disc.)

Similarly, $\mathrm{PSL}(2,\mathbb{C})$ can be made to act on $\mathbb{H}^3$ directly via fractional linear transformations: we model $\mathbb{H}^3$ as

$$H^3 = \{x + yi + tj \in \mathbb{Y} : t > 0\}$$

and set $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right].q \coloneqq (aq + b)(cq + d)^{-1}$ for $q \in H^3$. Again this preserves the set H^3 and acts as the group of hyperbolic isometries [2, §4.1].

One should immediately ask why, in this second example, we have distinguished the unit vector k: it does not seem like a particularly natural embedding of $\mathbb{R}^3 \hookrightarrow \mathbb{Y}$. The answer comes from the following pair of matrix representations,

$$^2\mathbb{R} = \mathbb{R} \oplus \mathbb{R} \ni (x, y) \mapsto \begin{bmatrix} x & y \\ -y & z \end{bmatrix} \in \mathbb{C}$$
$$^2\mathbb{C} = \mathbb{C} \oplus \mathbb{C} \ni (w, z) \mapsto \begin{bmatrix} w & z \\ -\bar{w} & \bar{z} \end{bmatrix} \in \mathbb{Y}$$

which exhibit H^2 and H^3 as $\mathbb{R}_{>0} \oplus \mathbb{R}$ and $\mathbb{R}_{>0} \oplus \mathbb{C}$ respectively.

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