On the MathRepo page "Farey polynomials"

What is there, and why is it like that?

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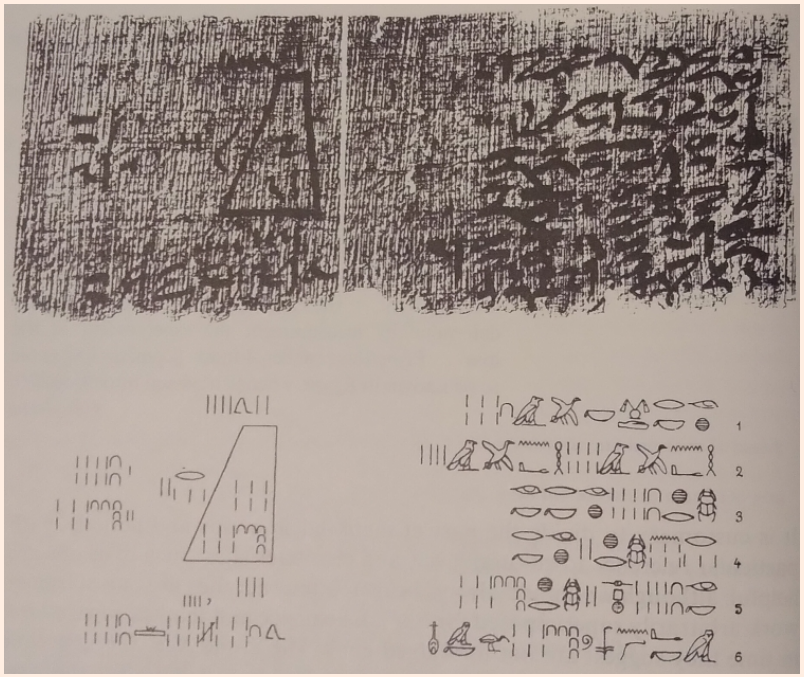


The Moscow papyrus, XII dyn. (c.1900 BC)
Boyer, "A history of mathematics", Wiley (1968). p.10

The relation of the MathRepo page to the paper

> Although these notes, in conformity with custom, come after the poem, the reader is advised to consult them first and then study the poem with their help, rereading them of course as he goes through its text, and perhaps, after having done with the poem, consulting them a third time so as to complete the picture.
>
> —
>
> "Charles Kinbote"

How intimate should the relationship between the paper and the MathRepo page be? This is a question which depends on the goals of both the paper and the creation of such a repository page, but I would argue that the relationship should not be so intimate that you cannot read one without the other.

What are the goals of the page?

- Provide a venue for computations for a technical paper prerequisite to our upcoming classification of all the arithmetic subgroups of $SL(2, \mathbb{C})$ generated by two torsion elements;
- Advertise some related objects which seem to be of independent interest in combinatorics, dynamics, & number theory.

These goals are better satisfied by an informal note than a published paper, but such a note doesn't replace the need for proper expository prose with more motivation, background, and technical detail. Since this paper is one of a series of related papers with a common goal, we wrote a separate expository paper which provides such an overview. It is important to think about how different texts that one produces will relate to each other, so that they all have a clear purpose (both to author and reader) and complement each other.

bibliographic information and abstract

historical background & motivation

necessary background in classical number theory

practical definition which requires no background in geometric topology(!)

conjectures

information on software used to produce tables

bibliography

author contact details

Background

- **Problem.** To understand the motivation and content of the paper entirely needs a background in several fields. How should one optimise the amount of detail one gives?

Let us ask a more precise question.

- **Q.** Should the page be entirely self-contained?
- **A1.** Yes, to the extent that it should be understandable by the average graduate student in an adjacent field and actually contain mathematical detail needed to at least understand the broad statement of the main theorem.
- **A2.** But no, in the sense that it should not contain all of the detail of the corresponding paper.

Background

This leads us to our solution of the problem.

- Give a *short* technical motivation for experts with (1) full references to a more detailed discussion (in our case, a nice expository paper we wrote) and (2) a clear warning that it is not necessary to understand the concrete details of the paper.
- Then seperately give a short introduction to the bare minimum concrete mathematics needed to actually do some arithmetic with the main theorem.

Background

History of the Farey polynomials ¶

Warning! This section contains references to hyperbolic geometry and low-dimensional topology. If you are interested in the combinatorics, none of this is needed to understand the rest of the page!

The original definition appears in [KS94] with corrections in [KS98]; in these papers, they appear as the traces of certain words in the parabolic Riley groups (Kleinian groups freely generated by two parabolic elements) and represent the lengths of non-boundary-parallel geodesics on four-times punctured spheres. To be more precise, they represent geodesics on the boundary at infinity of certain hyperbolic 3-manifolds; see [EMS22b] (which we wrote for the average non-expert mathematical audience). For further background on the hyperbolic geometry and the geometric interpretations of trace polynomials like these, we particularly recommend [MSW02]. Some further work on the Farey words was done by David Wright after the publication of that book [W05].

We recently extended the Keen and Series theory to allow groups generated by elliptics (the parabolic theory is just a limiting case of this), in our paper [EMS22c]. In order to understand this case, we defined a family of elliptic Farey polynomials which include the parabolic polynomials as a limiting case. We found that computing with these polynomials was a difficult business since they are defined in terms of a long word in a matrix group (so all calculations involve lots of matrix products and are slow); this motivated our study in [EMS22a] of the polynomials from a purely combinatorial point of view. In the process we found many nice number-theoretic properties of the polynomials.

Number theoretic background

We will make use of a little bit of classical number theory, from the subfield of *Farey sequences*; for full details see Chapter III of [HW08], or Section 3 of [EMS22a], or Chapter 7 of the highly recommended [MSW02]. Define the following **Farey addition** operation on $\hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$:

$$\frac{p}{q} \oplus \frac{r}{s} := \frac{p+q}{r+s}.$$

Of course, this is not well-defined. In order to make it well-defined, we restrict its domain: we only allow ourselves to write $(p/q) \oplus (r/s)$ when $|ps - qr| = \pm 1$. We call pairs of rational numbers which satisfy this condition **Farey neighbours**. Two remarks:

1. This condition implies that both summands are written in least terms (i.e. $(p, q) = (r, s) = 1$).
2. Secretly, this is the determinant of a matrix in $\mathrm{PSL}(2, \mathbb{Z})$ and in the background is an ideal triangulation of the hyperbolic plane being generated by the action of this Fuchsian group on the upper half-plane (this is the **Farey triangulation** of $\mathbb{H}^2$).

The Farey neighbour relation together with the Farey addition has beautiful number-theoretic properties (related for example to continued fractions) and geometric properties (for which we also should mention the harder monograph [ASWY07]).

For our purposes it is useful to also define an inverse operation

$$\frac{p}{q} \ominus \frac{r}{s} := \frac{p-q}{r-s}.$$

which is again only defined on Farey neighbours.

Mathematical content

- **The definition of the main objects studied.** This is given very concretely (that this is possible is the main result of the paper), rather than the classical definition (also concrete but requiring a lot of geometry to even state)
- **Examples of the objects.** Important for two reasons: (1) reproducibility, for those working through the paper; (2) to illustrate the structures which are visible in the objects, in order to convince the reader that they are interesting.
- **Conjectures.** It is easier to discuss conjectures and interesting unproved observations in an informal venue like this, and they are good for advertising purposes.

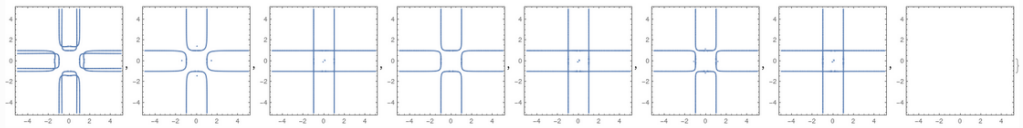
Mathematical content

Elzenaar's GitHub (the relevant package is `farey.py`). We remark that these recurrence relations in the parabolic case were known to Keen and Series (see the Remark following Corollary 4.3 of [KS94]).

```
p/q, \Phi_{p/q}(z)
0/1, 2 - z
1/1, 2 + z
1/2, 2 + z^2
2/3, 2 - z - 2 z^2 - z^3
3/5, 2 + z + 2 z^2 + 3 z^3 + 2 z^4 + z^5
5/8, 2 + 4 z^4 + 8 z^5 + 8 z^6 + 4 z^7 + z^8
8/13, 2 - z - 2 z^2 - 5 z^3 - 12 z^4 - 22 z^5 - 32 z^6 - 44 z^7 - 54 z^8 - 53 z^9 - 38 z^{10} - 19 z^{11} - 6 z^{12} - z^{13}
13/21, 2 + z + 2 z^2 + 7 z^3 + 14 z^4 + 31 z^5 + 64 z^6 + 124 z^7 + 214 z^8 + 339 z^9 + 498 z^{10} + 699 z^{11} + 936 z^{12} + 1148 z^{13} + 1216 z^{14}
21/34, 2 + z^2 + 8 z^4 + 24 z^5 + 68 z^6 + 192 z^7 + 516 z^8 +1256 z^9 + 2834 z^{10} + 5912 z^{11} + 11460 z^{12} + 20816 z^{13} +  35598 z^{14} + 57248
34/55, 2 -z - 4 z^2 - 10 z^3 - 34 z^4 - 103 z^5 - 286 z^6 - 791 z^7 - 2078 z^8 - 5221 z^9 - 12680 z^{10} - 29824 z^{11} -  67872 z^{12} - 149896 z^{13}
```

Observe that there are already some interesting properties visible, for instance unimodularity (up to sign).

Examples are given in a way that is easy to copy-and-paste.



Here is the Mathematica code used to generate these plots. It needs to be run in the same directory as the file fareys_general.txt containing the list of Farey polynomials (see Software implementations below for the relevant Mathematica code).

```
fareyList = << "fareys_general.txt";

fareyList =
  Sort[fareyList,
  Function[{a, b},
    a[[1]][[1]]/a[[1]][[2]] < b[[1]][[1]]/b[[1]][[2]]]];

fareyPlots =
  Map[ContourPlot[
      Coefficient[#[[2]], z] == 0, {a, -5, 5}, {b, -5, 5}] &,
    fareyList];
```

Some computer code is given, but most computational detail is left to the bibliographic sections so as not to overwhelm the interesting mathematics. Pictures are given—but not enough of them.

Bibliographic information

Three kinds of bibliographic information.

- Authorship information.
- Information directly relating to the paper and its mathematical content.
- Information about computational matters.

Note. Unlike many of the pages in this repository, there is no nontrivial software directly related to the paper: one can very easily compute examples by hand. Computers are only necessary in order to compute 1000s of polynomials at once in order to produce pictures or for other applications, and all the computer scripts of relevance can be reduced to 2–3 lines. This can be seen in the relative sizes of the bibliography and the computational information. It is important to make decisions like this consciously—think about the emphases you want to give to the reader!

Final remarks

- **How to measure effectiveness?** Hard to do this in any way for this page yet, since the main paper which it is supplementing is not yet available. Of course, measuring effectiveness is hard in general anyway! It depends very much on the reasons you put the page up.
- Think about ongoing maintenance, especially if the page includes a software component. Perhaps it is better to host software somewhere like Bitbucket or Github, but this is a question for other people.

Advertisement

Our expository paper “Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds” is written entirely for a non-expert mathematical audience, and is available at arXiv:2204.11422. It contains historical information, a lot of the background for understanding the mathematics here more deeply, and announces our major results (including the classification of the torsion arithmetic groups).