CONVEX GEOMETRY and MIXED VOLUMES

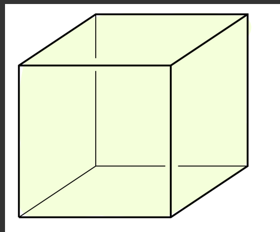
1. POLYHEDRA

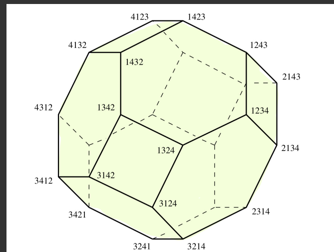
IF $Y = \{y_1, \dots, y_k \in \mathbb{R}^n\}$ THEN cone $(Y) := \{\lambda_1 y_1 + \dots + \lambda_k y_k \mid \lambda_i \in \mathbb{R}, \lambda_i \geq 0\}$

$\{\lambda_1 x_1 + \dots + \lambda_k x_k \mid 0 \leq \lambda_i \leq 1 \quad \sum \lambda_i = 1\}$

DEF A POLYTOPE IS THE CONVEX HULL OF FINITELY MANY POINTS $\{x_1 \dots x_k\}$

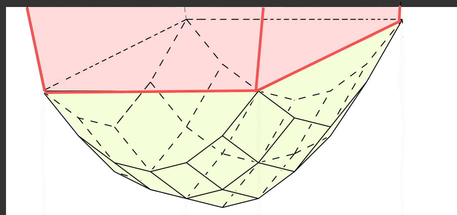
EQUIVALENTLY IS THE BOUNDED INTERSECTION OF HALF SPACES





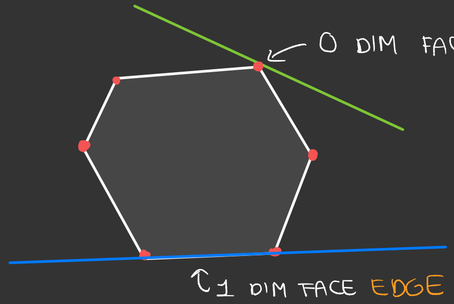
POLYHEDRON IS THE CONVEX HULL OF FINITELY MANY POINTS + cone(Y)

EQUIVALENTLY IS THE ~~BOUNDED~~ INTERSECTION OF HALF SPACES



DEF LET P BE A POLYHEDRON IN $\mathbb{R}^n$. A LINEAR INEQUALITY $\underline{c}^T \underline{x} \leq c_0$ WITH $\underline{c} \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$ IS VALID FOR P IF IT IS SATISFIED BY ALL POINTS OF P.

A FACE IS THE INTERSECTION OF $\underline{c}^T \underline{x} = c_0$ AND P.

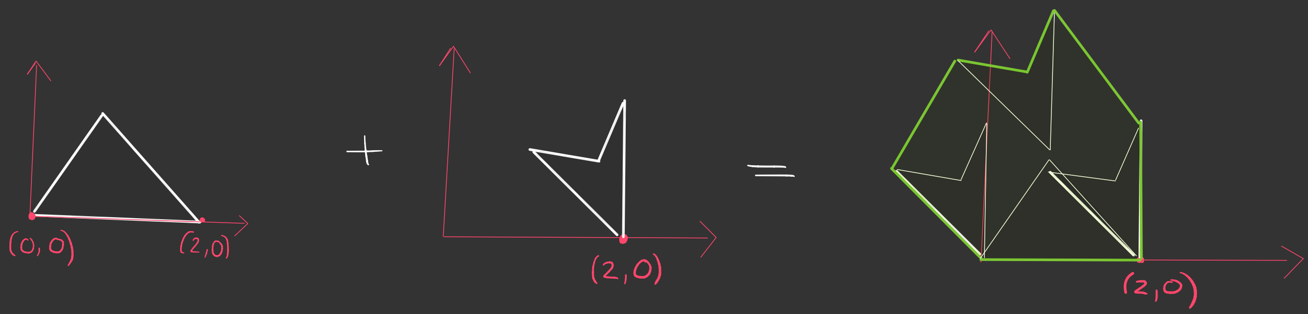


REMARK IF YOU WANT TO CREATE, VISUALIZE AND PLAY WITH POLYTOPES, CHECK OUT polymake

2. MIXED VOLUMES and CROSS-SECTIONAL MEASURES

DEF GIVEN TWO SUBSETS A, B IN $\mathbb{R}^n$, WE DEFINE THE MINKOWSKY SUM

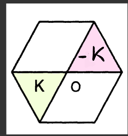
$$A + B := \{ x + y \in \mathbb{R}^n \mid x \in A,\ y \in B \}$$



PROPERTIES

- IF A, B COMPACT AND CONVEX, $A+B$ IS COMPACT AND CONVEX
- IF P, Q ARE POLYHEDRA THEN $P+Q$ IS STILL A POLYHEDRON AND ITS VERTICES ARE CONTAINED IN $\{ V_P + V_Q \mid V_P \text{ VERTEX OF } P,\ V_Q \text{ VERTEX OF } Q \}$
- $(A+B)+C = A+(B+C)$

REMARK • IN GENERAL $(-P)$ IS NOT THE NEGATIVE OF P W.R.T THE MINKOWSY SUM



NOTICE NOW THAT THE FOLLOWING MAKES SENSE

$$\lambda_1 A_1 + \dots + \lambda_k A_k \qquad \lambda_i \in \mathbb{R}, \lambda_i \geq 0$$

THEOREM LET $A_1, \dots, A_k$ CONVEX and COMPACT IN $\mathbb{R}^n$ AND $\lambda_i \in \mathbb{R}, \lambda_i \geq 0$ THEN $\mathrm{Vol}_n(\lambda_1 A_1 + \dots + \lambda_k A_k)$ IS A HOMOGENEOUS POLYNOMIAL OF DEGREE n in $\lambda_1, \dots, \lambda_k$

$$\mathrm{vol}_n(\lambda_1 A_1 + \dots + \lambda_k A_k) = \sum_{i_1, \dots, i_n = 1}^{k} V(A_{i_1}, \dots, A_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}$$

$V(A_1, A_2)\lambda_1\lambda_2 + V(A_2, A_1)\lambda_2\lambda_1$

DEF ARRANGING THE COEFFICIENTS S.T $V(A_{i_1}\dots A_{i_n}) = V(A_{\sigma(i_1)},\dots,A_{\sigma(i_n)})$ FOR EVERY PERMUTATION σ OF $\{1,\dots,k\}$, WE CALL $V(A_{i_1},\dots,A_{i_n})$ THE MIXED VOLUME OF $A_{i_1},\dots,A_{i_n}$.

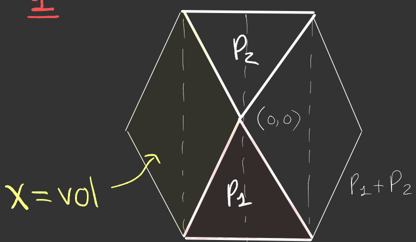
PROPERTIES

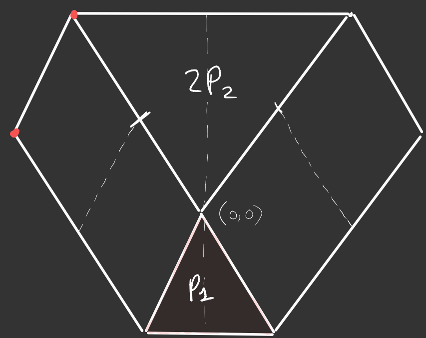
- FOR A CONVEX, COMPACT IN $\mathbb{R}^n$. $V(\underbrace{A,\dots,A}_{n \text{ COPIES}})$ IS THE VOLUME $\mathrm{vol}_n(A)$.
- MULTILINEARITY $V(aA+bB, C_2,\dots,C_n) = aV(A,C_2,\dots,C_n) + bV(B,C_2,\dots,C_n)$

  $a,b \in \mathbb{R}_{\geq 0}$

$n=2$

EXAMPLE 1





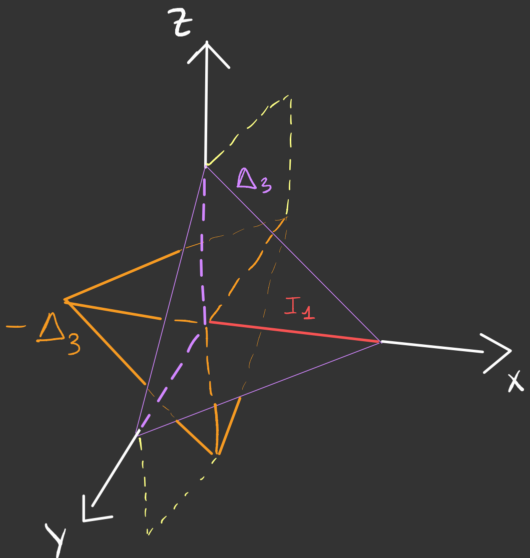
$V(P_1,P_2) = X$

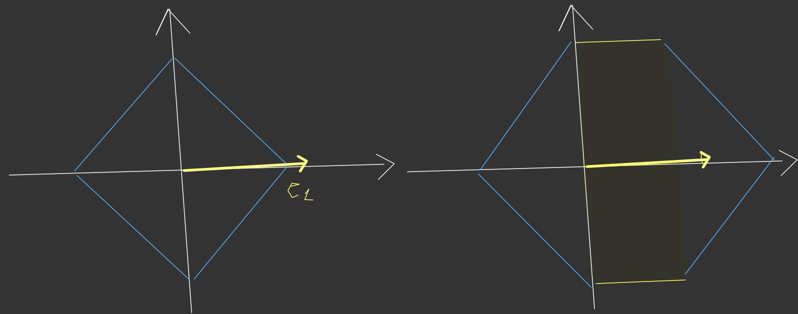
$$\mathrm{vol}_2(\lambda_1 P_1 + \lambda_2 P_2) = \lambda_1^2 \mathrm{vol}_2\left(P_1 + \frac{\lambda_2}{\lambda_1} P_2\right) = \lambda_1^2 \mathrm{vol}_2(P_1) + \lambda_2^2 \mathrm{vol}_2(P_2) + 2\lambda_1\lambda_2 \cdot X$$

$V(P_1,P_2)\lambda_1\lambda_2 + V(P_2,P_1)\lambda_2\lambda_1$

2. CONSIDER $P_1 = \Delta_3$, $P_2 = -\Delta_3$, $P_3 = [0, e_1]$. WE WANT TO FIND $V(P_1, P_2, P_3)$

$$\mathrm{vol}_3(\lambda_1 P_1 + \lambda_2 P_2 + \lambda_3 P_3) - \lambda_3 \cdot \mathrm{vol}_2\big(\pi(\lambda_1 P_1 + \lambda_2 P_2)\big) = \mathrm{vol}_3(\lambda_1 P_1 + \lambda_2 P_2)$$





BUT THEN

$$3!\, V(P_1, P_2, P_3) = 2!\, V(\pi_1(P_1), \pi_1(P_2))$$

WHERE π_1 PROJECTION OVER $e_1^{\perp}$

SOME GEOMETRIC INTUITION

DEF WE DEFINE THE m-th CROSS-SECTIONAL MEASURE OF A COMPACT, CONVEX K IN $\mathbb{R}^n$

$$V_m(K) := V(\underbrace{K, \dots, K}_{m \text{ TIMES}}, B^n \dots, B^n)$$

B^n → n-dim UNIT BALL

REMARK $V_n(K) = \mathrm{vol}_n(K)$

$$V_{n-1}(K) = n \cdot \underbrace{S(K)}_{\text{SURFACE AREA OF } K}$$

THEOREM THE VALUE OF $V_m(K)$ COINCIDES (UP TO A FACTOR DEPENDING ONLY ON m AND n) WITH THE MEAN INTEGRAL VALUE OF m-DIMENSIONAL VOLUMES OF THE PROJECTION OF K ON ALL POSSIBLE m-DIMENSIONAL SUBSPACES $\nu = \mathbb{R}^m \subset \mathbb{R}^n$.

$$V_m(K) = \frac{\mathrm{vol}_n(B^n)}{\mathrm{vol}_m(B^m)} \cdot \int_{Gr(m,n)} \mathrm{vol}_m(\pi_\nu(K)) \cdot d\omega(\nu)$$

ω → HAAR MEASURE

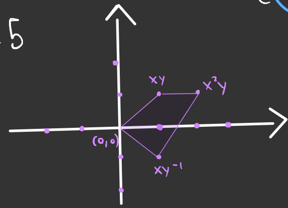
3. BKK THEOREM

THIS IS A VERY NICE RESULT EVIDENCING THE IMPORTANCE OF MIXED VOLUMES

DEF GIVEN A LAURENT POLYNOMIAL $f \in \mathbb{C}[[t_1, \dots, t_n]]$ WHERE $f = \sum_I c_I \, t^{a_I}$ $a_I \in \mathbb{Z}^n$.

THE NEWTON POLYTOPE IS THE CONVEX HULL OF THE EXPONENTS $\{a_I\}$

example $2xy^{-1} + 3xy + 4x^2y + 5$



THEOREM (BERNSTEIN, KUSHMIRENKO & KHOVANSKI)

LET $f_1, \dots, f_n \in \mathbb{C}[[t_1, \dots, t_n]]$ LAURENT POLYNOMIALS S.T

$$\alpha := \# V(f_1, \dots, f_n) = \{ x \in (\mathbb{C}^*)^n \mid f_1(x) = \dots = f_n(x) = 0 \} < +\infty$$

LET $P_i := N(f_i)$ THE NEWTON POLYTOPE OF f_i (= CONVEX HULL OF EXPONENT VECTORS OF MONOMIALS IN f_i)

THEN

$$\alpha \leq n! \, V(P_1, \dots, P_n)$$

AND THE EQUALITY HOLDS FOR GENERIC COEFFICIENTS.

4. ALEXANDROV-FENCHEL'S INEQUALITY

THEOREM (ALEXANDROV-FENCHEL'S INEQUALITY) LET $K_1, K_2, P_1, \dots, P_{n-2}$ BE CONVEX BODIES IN $\mathbb{R}^n$. THEN

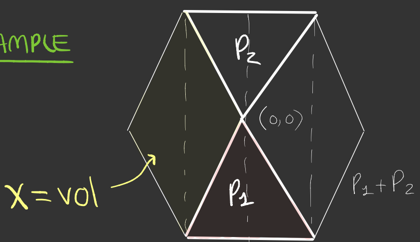
$$V(K_1, K_2, P_1, \dots, P_{n-2})^2 \geq V(K_1, K_1, P_1, \dots, P_{n-2}) \cdot V(K_2, K_2, P_1, \dots, P_{n-2})$$

FROM NOW ON $\mathcal{C} = \{P_1, \dots, P_{n-2}\}$. IN OTHER WORDS THIS THEOREM SAYS THAT

$$t \longrightarrow V(K_1 + tK_2, K_2 + tK_1, \mathcal{C})$$

IS A QUADRATIC POLYNOMIAL WITH NON-NEGATIVE DISCRIMINANT.

EXAMPLE



IN THIS CASE $n=2$ AND THE INEQUALITY GIVES

$$V(P_1, P_2)^2 \geq vol(P_1) \cdot vol(P_2)$$

$$X \geq vol(P_1)$$

SOME GEOMETRIC CONSEQUENCES

FROM THE AF INEQUALITY, WE CAN DEDUCE SOME CLASSICAL GEOMETRIC INEQUALITIES.

- ISOPERIMETRIC INEQUALITY IF K CONVEX COMPACT IN $\mathbb{R}^n$

$$\big(\underset{\text{SURFACE AREA}}{S}(K)\big)^n \geq n^n \operatorname{vol}_n(\underbrace{B^n}_{\text{n-dim UNIT BALL}}) \cdot \big(\operatorname{vol}_n(K)\big)^{n-1}$$

<u>PROOF</u> FROM THE AF INEQUALITY WE GET

$$\big(V(\underbrace{K,\dots,K}_{n-1}, S^n)\big)^n \geq \operatorname{vol}_n(B^n) \cdot \big(\operatorname{vol}_n(K)\big)^{n-1}$$

BUT $n \cdot V(K,\dots,K,S^n)$ IS EXACTLY $S(K)$.

THIS INEQUALITY IS STATING THAT THE SPHERE HAS THE SMALLEST SURFACE AREA PER GIVEN VOLUME, SINCE THIS EQUALITY HOLDS WHEN $K = B^n$.