Varieties Over $\mathbb{C}$ And Embeddings Into Projective Space

Lukas Zobernig

The University of Auckland

Introduction

Part 1 (Specific Example)

- Curves
- Weil Divisors on Curves
- Riemann-Roch for Curves
- Ad hoc Example of Projective Embedding

Part 2 (General Theory)

- Cartier Divisors and Line Bundles
- Linear Systems (Linear Series)
- Linear Systems and Rational Maps
- (Very) Ampleness
- Compact Riemann Surfaces

Part 1 (Specific Example)

Curves

Let us look at some simple nontrivial algebraic varieties: Curves.

- Specifically, we consider **projective** curves,
- i.e. projective varieties of dimension 1.
- These can have singular points in general,
- but let us restrict to **smooth** projective curves for simplicity;
- those are smooth everywhere, i.e. no point is a singular point.

Over $\mathbb{C}$, smooth projective curves are simply the compact Riemann surfaces, so we can also keep that picture in mind.

Weil Divisors on Curves

In general, a (Weil) divisor on a variety X is a formal sum of irreducible codimension-1 subvarieties of X.

Definition (Weil Divisors on Curves)

Let C be a curve, then a **Weil divisor** D is a formal sum

$$D = \sum_{P \in C} n_P (P),$$

such that **all but finitely many** $n_P = 0$. Denote by $\deg(D) = \sum n_P$ the **degree** of D.

Definition (Divisor of a Function)

Let $f \in k(C)$ be a function on C, then the divisor $\mathrm{div}(f)$ is defined as

$$\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P(f)(P),$$

where $\mathrm{ord}_P(f)$ is the **order of vanishing** of f at P. Such divisors are called **principal**.

Weil Divisors on Curves

- Divisors form a group under addition: $\{3(P)+2(Q)\}+\{1(Q)\}=3(P)+3(Q)$.
- A divisor is called **effective** if all coefficients are non-negative, write $D \geq 0$. Write $D_1 \geq D_2$ if $D_1 - D_2 \geq 0$.
- We call two divisors D and D' **linearly equivalent**, if there exists $f \in k(C)$ such that $D = D' + \mathrm{div}(f)$ (i.e. if $D - D'$ is principal).
- Divisors up to linear equivalence form the so called **Picard** or **divisor class** group $\mathrm{Pic}(C)$.

Examples

- On $\mathbb{A}^1$ every **prime** divisor (i.e. single point) is the divisor of a single function. Hence $\mathrm{Pic}(\mathbb{A}^1)$ is trivial.
- On $\mathbb{P}^1$ every degree-0 divisor is principal. Hence there is an isomorphism $\mathrm{Pic}(\mathbb{P}^1) \cong \mathbb{Z}$.

Riemann-Roch for Curves

Definition (Riemann-Roch Space)

Let D be a divisor on a curve C and denote by $L(D)$ the finite dimensional k-vector space of "functions with poles no worse than D", i.e.

$$L(D) = \{ f \in k(C)^* \mid \mathrm{div}(f) \geq -D\} \cup \{0\}.$$

Denote its dimension by $\ell(D) = \dim_k L(D)$.

- If $\deg(D) < 0$ then $L(D) = \{0\}$ and $\ell(D) = 0$.
- Linearly equivalent divisors have isomorphic Riemann-Roch spaces.

Theorem (Riemann-Roch for Curves)

Let C be a smooth projective curve and let K_C be a **canonical divisor** *on C. Then the* **genus** *g of C and every divisor D on C satisfy*

$$\ell(D) - \ell(K_C - D) = \deg(D) - g + 1.$$

Riemann-Roch for Curves

- As an immediate corollary we get that if $\deg(D) > 2g - 2$ then $\ell(D) = \deg(D) - g + 1$.
- This means that if a divisor has high enough degree, we can immediately compute the dimension of its associated Riemann-Roch space.
- For example, for genus 1 curves we find the easy relation $\ell(D) = \deg(D)$ if $\deg(D) > 0$.
- We can now use this result and explicit bases for Riemann-Roch spaces to give an ad hoc example of a projective embedding.

Ad hoc Example of Projective Embedding

Definition (Elliptic Curve)

An **elliptic curve** is a pair (E,O), where E is a smooth curve of genus 1 and $O \in E$ a rational point.

- Fix some elliptic curve (E, O). Consider the divisor $n(O)$ for $n \geq 1$ on E.
- By the previous slides we have $\ell(n(O)) = \deg(n(O)) = n$ for all $n \geq 1$.
- The space $L(n(O))$ contains at least the constant functions, so $L(2(O))$ has a basis $\{1, x\}$ for some function $x \in k(E)$.
- Then $L(3(O))$ has a basis $\{1, x, y\}$ for some other function $y \in k(E)$. Hence, x has a pole of exact order 2 at O and similarly y has a pole of exact order 3 at O.
- Continuing, we have bases $L(4(O)) = \{1, x, y, x^2\}$, $L(5(O)) = \{1, x, y, x^2, xy\}$.
- Finally, in $L(6(O))$ we have the **seven** functions $1, x, y, x^2, xy, x^3, y^2$. Hence, there is a linear relation- (which we can normalise to) $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.

Ad hoc Example of Projective Embedding

- The spaces $L(2(O))$ with basis $\{1, x\}$ and $L(3(O))$ with basis $\{1, x, y\}$ give morphisms to $\mathbb{P}^1$ and $\mathbb{P}^2$, respectively.
- We have the $2:1$ ramified double cover

$$\begin{aligned} \phi : E &\to \mathbb{P}^1, \\ (x, y) &\mapsto [x : 1], \\ O &\mapsto [1 : 0]. \end{aligned}$$

  This is not an embedding, since $L(2(O))$ does not contain enough functions.
- We have the closed immersion (which is bijective onto its image)

$$\begin{aligned} \phi : E &\to \mathbb{P}^2, \\ (x, y) &\mapsto [x : y : 1], \\ O &\mapsto [0 : 1 : 0]. \end{aligned}$$

  This is an embedding, and $L(3(O))$ contains enough functions to define it.

Part 2 (General Theory)

Cartier Divisors and Line Bundles

Next, we will change from the language of divisors to the language of **line bundles**.

- This is easier to understand if we work with **Cartier divisors**.
- For **nice enough** varieties Weil and Cartier divisors can be interchanged freely.
- We use that a codimension-1 subvariety of a normal variety is locally defined as the zeroes and poles of a single function.

Definition (Cartier Divisor)

A **Cartier divisor** on a variety X is an equivalence class of collections of pairs $(U_i, f_i)_{i \in I}$ such that:

- The U_i are open sets covering X,
- The f_i are nonzero rational functions in $k(U_i) = k(X)$, and
- $f_i f_j^{-1} \in \mathcal{O}(U_i \cap U_j)^*$, i.e. $f_i f_j^{-1}$ has no poles or zeroes on the overlap $U_i \cap U_j$.

For a function $f \in k(X)$ we have its associated Cartier divisor $\mathrm{div}(f) = \{(X, f)\}$.

Cartier Divisors and Line Bundles

Recall that a **line bundle** $\mathcal{L}$ on a variety X is a **vector bundle** whose fibers are 1-dimension vectors spaces. To a Cartier divisor $(U_i, f_i)_{i\in I}$ we can associate a **line bundle** in the following way:

- Consider the trivial line bundles $U_i \times \mathbb{A}^1 \to U_i$,
- and glue them via the isomorphism

$$\begin{aligned}(U_i \cap U_j) \times \mathbb{A}^1 &\to (U_i \cap U_j) \times \mathbb{A}^1 \\ (x, \lambda) &\mapsto (x, \lambda(f_i f_j^{-1})(x)).\end{aligned}$$

For a divisor D, denote by $\mathcal{O}(D)$ the associated line bundle.

- We have $\mathcal{O}(D + D') = \mathcal{O}(D) \otimes \mathcal{O}(D')$, and
- $\mathcal{O}(-D) = \mathcal{O}(D)^\vee$ (the dual line bundle).

Examples

Hyperplane on $\mathbb{P}^n$

- Denote by $\mathcal{O}(1)$ on $\mathbb{P}^n$ the line bundle associated to a hyperplane.
- The global sections $H^0(\mathbb{P}^n, \mathcal{O}(1))$ are generated by linear forms,
- a possible basis is $\{x_0, \ldots, x_n\}$ (the usual coordinate functions on $\mathbb{P}^n$).

Higher Powers

- Similary, denote by $\mathcal{O}(d)$ the line bundle obtained by tensoring $\mathcal{O}(1)$ with itself d times.
- Then the global sections $H^0(\mathbb{P}^n, \mathcal{O}(d))$ are the homogeneous polynomials of degree d,
- a possible basis is $\{X_1^{i_1} \cdots X_n^{i_n}\}_{i_1 + \cdots + i_n = d}$.

Linear Systems (Linear Series)

Recall that to a divisor D we have associated the Riemann-Roch (vector) space

$$L(D) = \{ f \in k(C)^* \mid \operatorname{div}(f) \geq -D \} \cup \{0\}.$$

The set of effective divisors linearly equivalent to D is then parametrised by the projective space

$$\mathbb{P}(L(D)) \cong \mathbb{P}^{\ell(D)-1}$$

via

$$\begin{aligned} \mathbb{P}(L(D)) &\to \{D' \mid D' \geq 0, D' \sim D\} \\ f \mod k^* &\mapsto D + \operatorname{div}(f). \end{aligned}$$

Definition (Linear System)

A **linear system** (or sometimes **linear series**) on a variety X is a set of effective divisors all linearly equivalent to a fixed divisor D and parametrised by a linear subvariety of $\mathbb{P}(L(D))$. We call the set of all effective divisors linearly equivalent to D a **complete linear system**, and denote it by $|D|$.

More on Linear Systems

Definition (Base Points)

- The set of **base points** of a linear system L is the intersection of the supports of all divisors in L.
- A linear system is called **base point free** if this intersection is empty.
- Similarly a divisor D is base point free if $|D|$ is base point free.

Linear Systems and Line Bundles

There is a useful connection between (complete) linear systems and global sections of line bundles.

- Let D be a divisor, and consider its Riemann-Roch space $L(D)$.
- Then the space of sections $H^0(X, \mathcal{O}(D))$ is in bijection with the functions in $L(D)$.
- Hence, for a line bundle E on X, we find a linear system by choosing a subspace of $H^0(X, E)$.

Linear Systems and Rational Maps

- Let L be a linear system of dimension n, say parametrised by $\mathbb{P}(V) \subset \mathbb{P}(L(D))$.
- Select a basis $f_0, \ldots, f_n$ of $V \subset L(D)$.

Definition

The **rational map associated to** L is the map

$$\begin{aligned} \phi_L : X &\to \mathbb{P}^n, \\ x &\mapsto [f_0(x) : \cdots : f_n(x)]. \end{aligned}$$

This clearly gives a "good rational map" outside the base points of L (recall that a projective point cannot have all coordinates zero).

(Very) Ampleness

The important question to ask is now: When does a linear system L give an embedding, i.e. when is ϕ_L a morphism mapping X isomorphically onto its image $\phi_L(X)$?

Definition ((Very) Ampleness)

- A linear system L on a projective variety X is **very ample** if $\phi_L : X \to \mathbb{P}^n$ is an embedding.
- A divisor D or a line bundle $\mathcal{O}(D)$ is called very ample, if the complete linear system $|D|$ is very ample.
- A divisor D or a line bundle $\mathcal{O}(D)$ is called **ample**, if some positive multiple or power is very ample.

(Very) Ampleness

Theorem (Very Ampleness for General Varieties)

A linear system L on a variety X is very ample if and only if it satisfies the following two conditions:

- *(Separation of points.) For any pair of points $x, y \in X$ there is a divisor $D \in L$ such that $x \in D$ and $y \notin D$.*
- *(Separation of tangents.) For every nonzero tangent $t \in T_x(X)$ there is a divisor $D \in L$ such that $x \in D$ and $t \notin T_x(D)$.*

Theorem (Very Ampleness for Curves)

Let D be a divisor on a curve C.

- *The divisor D is base point free if and only if for all $P \in C$ we have $\ell(D - (P)) = \ell(D) - 1$.*
- *The divisor D is very ample if and only if for all $P, Q \in C$ we have $\ell(D - (P) - (Q)) = \ell(D) - 2$.*

Examples

Recall our example from earlier, the line bundle $\mathcal{O}(1)$ associated to a hyperplane on $\mathbb{P}^n$.

- Let $X = [x_0 : \cdots : x_n] \in \mathbb{P}^n$. Clearly $\mathcal{O}(1)$ is very ample, as $\phi_{\mathcal{O}(1)}(X) = [x_0 : \cdots : x_n]$. This is just the identity embedding of $\mathbb{P}^n \to \mathbb{P}^n$.
- Let $n = 1$, then $\mathcal{O}(2)$ on $\mathbb{P}^1$ gives an embedding $\mathbb{P}^1 \to \mathbb{P}^2$ via

$$\begin{aligned} \phi_{\mathcal{O}(2)} : \mathbb{P}^1 &\to \mathbb{P}^2, \\ [x : y] &\mapsto [x^2 : xy : y^2]. \end{aligned}$$

- In general we see that $\mathcal{O}(d)$ on $\mathbb{P}^n$ is very ample for all $d \geq 1$.
- On the other hand, $\mathcal{O}(d)$ for $d < 1$ has no global sections and hence is not even ample.

Examples

Recall our example from Part 1; we computed the dimensions $\ell(n(O))$ for the divisor $n(O)$ (where O was a rational point on an elliptic curve E).

- We found $\ell(n(O)) = \deg(n(O)) = n$ for all $n \geq 1$.
- From our example we now understand that (O) and $2(O)$ are ample,
- and that $3(O)$ is very ample.
- One could also check this via the theorem on very ampleness for curves we saw earlier:
  - For all points $P, Q \in E$ we have the divisors (P) and $(P) + (Q)$ of degree 1 and 2, respectively.
  - Plugging it into the required relations shows exactly that $3(O)$ is very ample.

Compact Riemann Surfaces

For compact Riemann surfaces (i.e. smooth projective curves over $\mathbb{C}$) this all boils down to the theory of **theta functions**.

- By integrating homology of complex torus C of genus g we find **period lattice** $\Lambda \subset \mathbb{C}^g$ with $\Lambda = \mathbb{Z}^g + \tau\,\mathbb{Z}^g$, and $\mathrm{Pic}(C) \cong \mathbb{C}^g/\Lambda$ as varieties.
- The function $\theta(z,\tau) = \sum_{m\in\mathbb{Z}^g} \exp(\pi i m^T \tau m + 2\pi i m^T z)$ has as divisor a translate of a so call **theta divisor** Θ.
- The divisor Θ is ample and 3Θ is very ample, i.e. we find projective embedding $\mathbb{C}^g/\Lambda \to \mathbb{P}(L(3\Theta))$.
- In genus 1 the Weierstrass $\wp$-function and its derivative $\wp'$ form a basis, and we find the usual model $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$, and the embedding $\mathbb{C}/(\mathbb{Z} + \tau\,\mathbb{Z}) \to \mathbb{P}^2$ via $z \mapsto [\wp(z) : \wp'(z) : 1]$.
- In genus 2 the space is spanned by 16 theta functions and we find an embedding $\mathbb{C}^2/(\mathbb{Z}^2 + \tau\,\mathbb{Z}^2) \to \mathbb{P}^{15}$.

Questions?