$V/\mathbb{C}$

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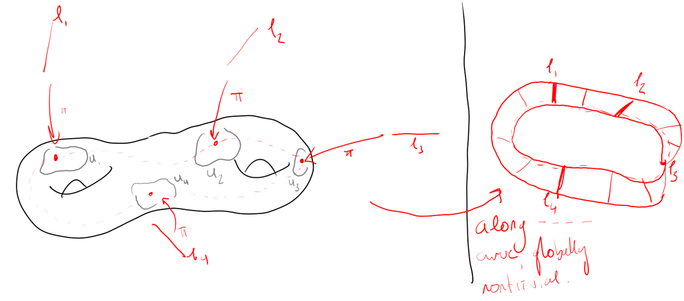
(MPI-MIS)

May 24, 2022

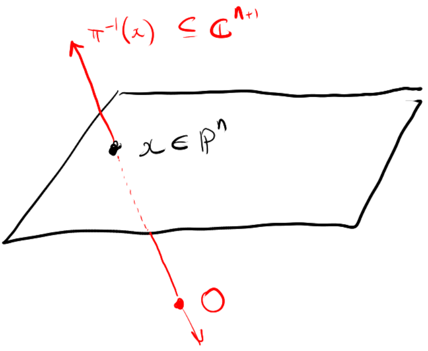
Warnings

- This talk will necessarily be sketchy.
- We always assume (usually without explicit statement) that the objects are ‘nice enough to make the theorems true’.
- Usually this means all varieties are complete and smooth over $\mathbb{C}$.



Line bundle over X: A space $\mathcal{E}$ with a map $\pi : \mathcal{E} \to X$ such that

- $\pi^{-1}(x)$ is a 1-dimensional vector space for all $x \in X$, and
- around each $x \in X$ there is a neighbourhood U such that $\pi^{-1}(U)$ is isomorphic (as a vector space) to $U \times \mathbb{R}$ in a way compatible with the two projections π and $U \times \mathbb{R} \to U$.



The tautological line bundle $\pi : \mathbb{P}^n \times \mathbb{C}^{n+1} \to \mathbb{P}^n$ with $\pi^{-1}(x)$ the line in $\mathbb{C}^{n+1}$ defining x. The local trivialisation cover can be chosen to be the usual open cover of $\mathbb{P}^n$ by $n + 1$ affine spaces.

Invertible sheaves

- Let $\mathcal{E}$ be a line bundle over X. Let (U_i) be an open cover giving a local trivialisation.
- For each U_i set $\mathcal{E}(U_i)$ to be the set of continuous sections of $\mathcal{E}$ over U_i, i.e. the set of maps $\phi : U_i \to \mathcal{E}$ such that $\pi\phi = \mathrm{id}$.
- If X is actually a variety then we replace 'continuous' with 'polynomial' and take lines to be 1D over $\mathbb{C}$, so $\mathcal{E}$ becomes a subsheaf of $\mathcal{O}_X$.
- The sheaves associated to line bundles are called **invertible sheaves**.

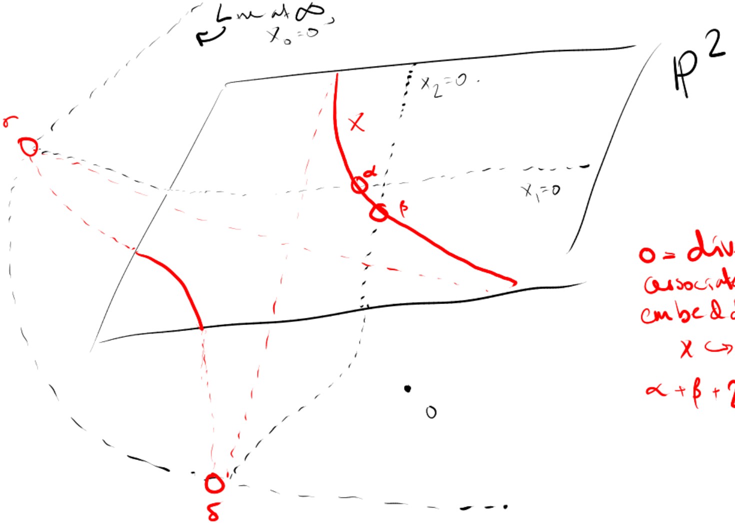
Weil divisors and line bundles

- Suppose $\mathcal{L}$ is a line bundle whose associated sheaf has global sections generated by $s_0, \ldots, s_n$.
- Then there is a natural Weil divisor associated to $\mathcal{L}$, namely the union of the zeros of the s_i.
- Conversely, if $D = \sum a_i D_i$ is a Weil divisor then we can associate an invertible sheaf $\mathcal{L}(D)$, namely the 'sheaf whose sections are functions who locally have poles at of order at worst $-a_i$.

The dual bundle: the Serre twisting sheaf

- The dual vector bundle to the tautological bundle is the bundle $\mathcal{O}(1)$ whose fibre over x is the set of hyperplanes in $\mathbb{C}^{n+1}$ orthogonal to the line defining x (i.e. the 1-dimensional vector space of linear functionals on $\mathbb{C}^{n+1}$ which kill the line).
- Let $f : \mathbb{P}^n \to \mathbb{C}$ be a global section of $\mathcal{O}(1)$; then f is a polynomial map $\mathbb{P}^n \to \mathrm{Hom}(\mathbb{C}^{n+1}, \mathbb{C}) \simeq \mathbb{C}$; this map has to behave linearly when passing between the affine open subsets and so must be of degree 1.
- Thus the global sections of $\mathcal{O}(1)$ are the linear forms on $\mathbb{P}^n$.
- Hence there is a global set of section generators for $\mathcal{O}(1)$, the coordinate forms $x_0, \dots, x_n$.

- Let X be a smooth variety over $\mathbb{C}$. Let $\varphi : X \to \mathbb{P}^n$ be a morphism.
- The pullback $\varphi^*\mathcal{O}(1)$ entirely determines X, since points in $\mathbb{P}^n$ are determined by the set of hyperplanes through them (= the fibres of $\varphi^*\mathcal{O}(1)$).
- The invertible sheaf $\varphi^*\mathcal{O}(1)$ has an associated Weil divisor: the intersection of $\varphi(X)$ with the standard coordinate hyperplanes of $\mathbb{P}^n$ (= the individual zero sets of the global generators of the pullback sheaf).
- **Theorem.** Conversely, if $\mathcal{L}$ is an invertible sheaf on X and if $s_0, \ldots, s_n$ generate $\mathcal{L}(X)$, then there exists a unique morphism $\varphi : x \to \mathbb{P}^n$ such that $\mathcal{L} \simeq \varphi^*\mathcal{O}(1)$ and $s_i = \varphi^*(x_i)$ for each i.

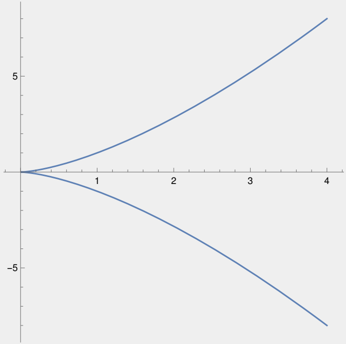


o = divisor associated to embedding $X \hookrightarrow \mathbb{P}^2$,

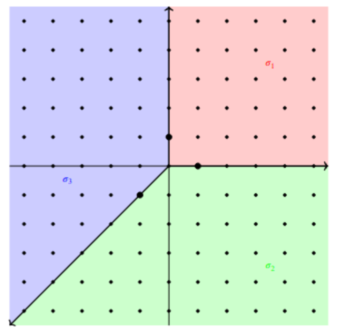
$\alpha+\beta+2\gamma+2\delta$.

- The choice of embedding $\varphi : X \to \mathbb{P}^n$ has an associated degree $\deg \varphi$.
- $\mathbb{Z}_{>0}$-linear combinations of divisors which give embeddings also give embeddings.
- **Question.** What is $\deg(\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k)$?
- **Answer.** It is a homogeneous polynomial in the variables $\lambda_1, \ldots, \lambda_k$, called the **volume polynomial** of $\varphi_1, \ldots, \varphi_k$.

- **Toric variety:** a variety $V/\mathbb{C}$ which contains an open subvariety isomorphic to $(\mathbb{C}^*)^n$ such that the action of $(\mathbb{C}^*)^n$ on itself by multiplication extends algebraically to the entirety of V.
- **Examples.** $\mathbb{A}^n$; $\mathbb{P}^n$ (the torus is the projection of $(\mathbb{C}^*)^{n+1}$); the cusped cubic $\mathrm{Z}(Y^2 - X^3)$.



The toric variety structure is determined locally (on torus-invariant open sets) by the semigroup of characters $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$ which extend regularly to the whole open set. The gluing of the open sets to form the variety is reflected in the gluing of the character semigroups into a polyhedral fan.



- Let $X = X_\Sigma$ for some fan Σ over M. The divisor class group of X_Σ is generated by the characters χ^m.
- Let D be a torus-invariant divisor of X_Σ. Then D is the closure of an orbit corresponding to one of the rays of Σ. The group of torus-invariant divisors is $\langle D_\tau : \tau \in \Sigma(1)\rangle$.
- **Lemma.** $\operatorname{Div}\chi^m = \langle m, u_\tau\rangle D_\tau$ where u_τ is a minimal ray generator for τ.
- **Theorem.** If $D = \sum a_\tau D_\tau$ then $\chi^m \in \mathcal{L}(D)(X_\Sigma)$ iff $\langle m, u_\tau\rangle \geq -a_\tau$ for all τ.
- **Corollary.** The set of torus-invariant Weil divisors of X_Σ is in bijection with the set of polyhedra.

Theorem

Let $D_1, \ldots, D_k$ be torus-invariant (ample) divisors on a toric X_Σ of (complex) dimension n with corresponding embeddings φ_i and polyhedra P_i. Then

$$\mathrm{Vol}(\lambda_1\varphi_1 + \cdots + \lambda_k\varphi_k) = (\lambda_1 D_1 + \cdots + \lambda_k D_k)^{\cdot n} = k!\,\mathrm{Vol}(\lambda_1 P_1 + \cdots + \lambda_k P_k).$$

Matching coefficients:

Corollary

$V(P_1, \ldots, P_n) = \frac{1}{k!}(D_1 \cdot \ldots \cdot D_n)$.

Suppose X is a complex manifold. There is an induced vector bundle isomorphism $I : TX \to TX$ with $I^2 = -1$ given locally by multiplication by i in each dimension separately; this is called a **almost complex structure**. A Riemannian metric g on X is **Hermitian** if $g_x(I(\cdot), I(\cdot)) = g_x(\cdot, \cdot)$ on T_xX. The **fundamental form** associated to the triplet (X, g, I) is the differential 2-form given locally by

$$\omega \coloneqq g(\cdot, I(\cdot)).$$

The manifold X is called a **Kähler manifold** if the fundamental form ω is closed; ω is called a **Kähler form** and the **Lefschetz operator** is

$$\begin{aligned} L : \bigwedge^k X &\to \bigwedge^{k+2} X \\ \alpha &\mapsto \alpha \wedge \omega \end{aligned}$$

Write $\Omega^{1,0}$ for the space of 1-forms f such that $f(I(v)) = if(v)$ for tangent vectors v (i.e. the form is a combination of differential forms that look like dz), and write $\Omega^{0,1}$ for the space of 1-forms f such that $f(I(v)) = -if(v)$ (a combination of differential forms that look like $d\overline{z}$).
Then the space of forms of **type** (p, q) is

$$\Omega^{p,q} = \bigwedge^{p} \Omega^{1,0} \wedge \bigwedge^{q} \Omega^{0,1}.$$

Theorem

Let X be a compact Kähler manifold. There is a vector space decomposition

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{R}) \otimes \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the set of cohomology classes representable by a closed form of type (p, q).

General situation

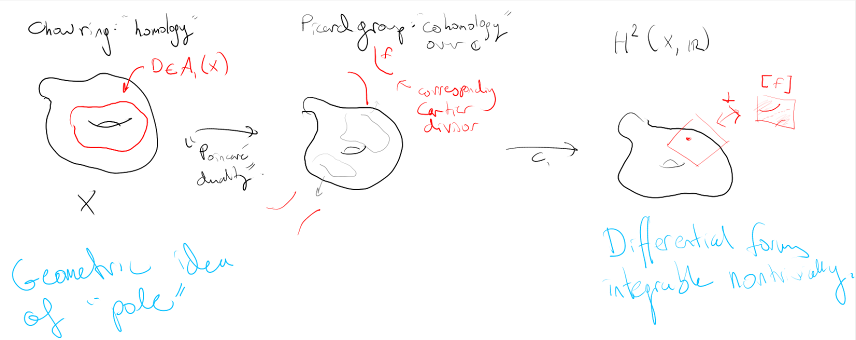
Theorem (Hodge index theorem)

Let X be a compact Kähler surface (e.g. a projective surface over $\mathbb{C}$, with Kähler structure induced by pulling back the Serre twisting sheaf). Then the standard middle cohomology pairing

$$\begin{aligned} H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) &\to H^4(X,\mathbb{R}) \simeq \mathbb{R} \\ (\alpha,\beta) &\mapsto \langle \alpha,\beta\rangle := \int_X \alpha \wedge \beta \end{aligned}$$

satisfies $\langle \alpha,\alpha\rangle < 0$ if α is taken in $H^{1,1}(X) \cap H^2(X,\mathbb{R})$, the orthogonal complement of the space of cohomology classes satisfying $\langle \beta,\beta\rangle > 0$.

- Then $D_1.D_2 = \langle c_1(D_1), c_1(D_2)\rangle$ is the usual intersection product.
- This generalises to give information about the middle cohomology product of higher-dimensional Kähler


Chow ring: "homology"
D ∈ A1(X)
X
"Poincaré duality"
Picard group: "cohomology over C"
f
corresponding Cartier divisor
c1
H2(X, R)
[f]
Geometric idea of "pole"
Differential forms integrable nontrivially.

Corollary

If D_1, D_2 are divisors on X, and $D_1, D_1 > 0$, then $(D_1 \cdot D_2)^2 \geq (D_1 \cdot D_1)(D_2 \cdot D_2)$.

Now let P_1 and P_2 be full-dimensional lattice polyhedra in $\mathbb{R}^n$ and let D_1, D_2 be corresponding divisors giving toric embeddings.

$$V(P_1, P_2)^2 = \frac{1}{4}(D_1 \cdot D_2)^2 \geq \frac{1}{4}(D_1 \cdot D_1)(D_2 \cdot D_2) = V(P_1, P_1)V(P_2, P_2).$$

A similar argument gives the full Alexandrov-Fenchel inequality for k polyhedra.

In general if the relevant divisors are ample then the volume polynomial

$$\mathrm{Vol}(\sum \lambda_i \varphi_i)$$

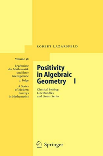
is strictly Lorentzian.[1] In fact, one can prove a 'Hodge index theorem' over arbitrary loopless matroids and deduce that arbitrary Lorentzian polynomials satisfy the corresponding symmetry results.

[1] A Lorentzian polynomial is a limit of strictly Lorentzian polynomials, and so if you replace ampleness with a the relevant weakened positivity criterion, **nefness** — roughly, a divisor is nef if it is a limit of ample divisors — you get Lorentzian polynomials.

Reading Group:
Positivity in Intersection Theory

We will take a very geometric approach
to a very geometric subject:
divisors with positive intersection numbers.

We will be interested particularly in concrete examples and classical geometric problems (e.g. volume calculation).



We will be reading *Positivity in Algebraic Geometry* by Robert Lazarsfeld.

Rough background required:

basic algebraic geometry (as found in Hartshorne chapters I and II) and differential topology (e.g. as found in Bott/Tu). We will take time to deal with detailed background as needed.

Tentative start date & time:

Wed. 27 Jan, 2021 @ 2pm, location tbd

Please email aelz176@aucklanduni.ac.nz to express interest.

~Alex Elzenaar (masters student)

Further reading

- Robert Lazarsfeld, *Positivity in Algebraic Geometry* (Springer)
- William Fulton, *Introduction to Toric Varieties* (Princeton)
- Claire Voisin, *Hodge Theory and Complex Algebraic Geometry* (Cambridge)
- Daniel Huybrechts, *Complex Geometry* (Springer)
- Yuriĭ D. Burago and Viktor A. Zalgaller, *Geometric Inequalities* (Springer)
- Matthew Baker, "Hodge Theory in Combinatorics" (survey article, `https://arxiv.org/abs/1705.07960`)