# What is...a Kleinian group?


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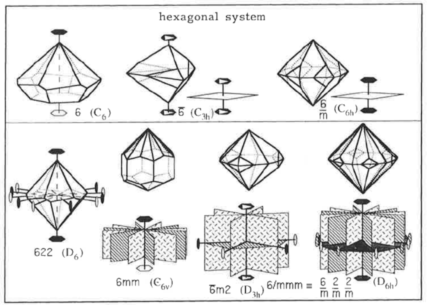


# Tiling

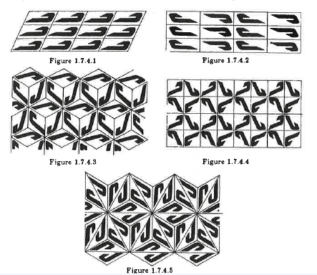


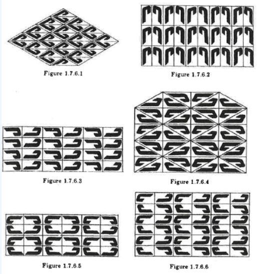
Jones, *The grammar of ornament*.

# Tiling

A tiling consists of a polyhedron $T$ in some geometric space $X$, together with a discrete subgroup $G \leq \mathrm{Isom}^+(X)$ such that

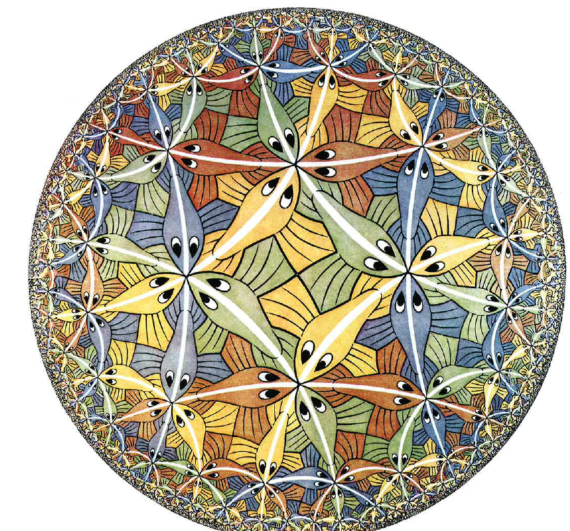
1. (The tiles cover $X$): $G.T = X$.
2. (The tiles overlap only on their edges): if $g.\operatorname{int} T$ and $h.\operatorname{int} T$ intersect nontrivially, then $g.T = h.T$.




Berger, *Geometry I*, p. 13 and p. 19.

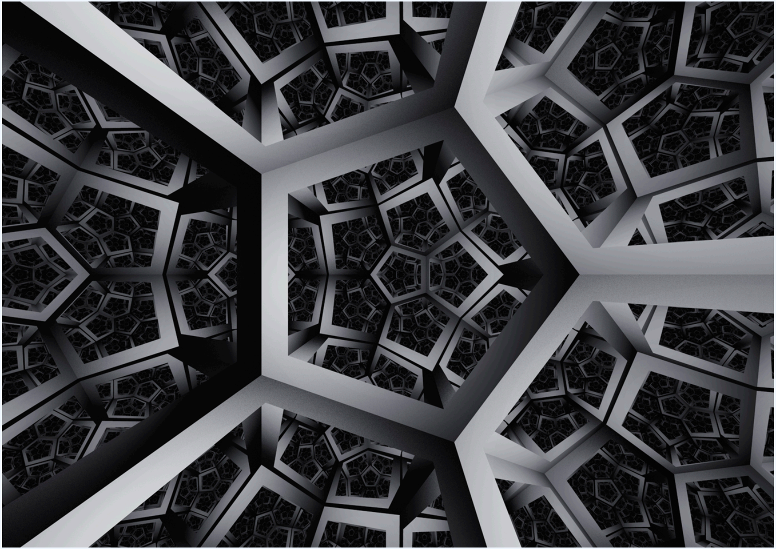
# Tiling the hyperbolic plane



244. *Circle Limit III*, woodcut, 1959

Ernst, *The magic mirror of M. C. Escher*, p. 113.

# Tiling hyperbolic space



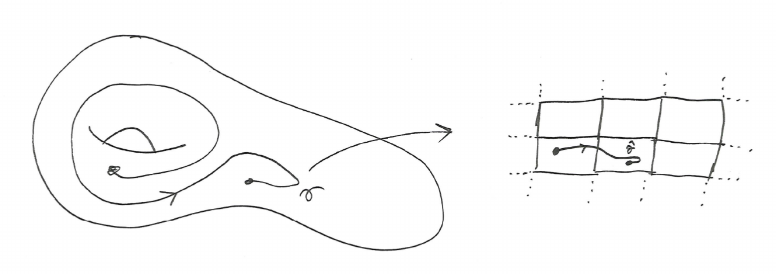
The Geometry Center, *Not knot*.

If $G \leq \mathrm{Isom}^+(X)$ is sufficiently nice[1], then the quotient space $G/X$ is a manifold locally modelled on $X$.
If $G$ comes from a tiling with tile $T$, then $X/G$ is induced by gluing the sides of $T$.

[1]acts freely and discretely

# Developing surfaces

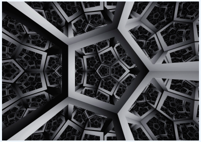
Let $S$ be a surface. Then there exists a way of cutting the surface open and flattening it onto its universal cover $X$, which is one of $S^2$, $\mathbb{H}^2$, or $\mathbb{R}^2$. The symmetry group of this covering gives an embedding $\pi_1(S)\ \ \mathrm{Isom}^+(X)$; the image is called the **holonomy group** of $S$, and $S \simeq_{\text{isometric}} X / \mathrm{Hol}(S)$.



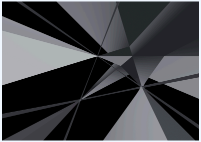
**Theorem**

*Thurston-Perelman, 1982–2002 Let $M$ be a 3-manifold. Then there exists a way of cutting the manifold into pieces such that each piece is of the form $X/G$, where $X$ is a Thurston geometry and $G$ is a discrete subgroup of $\mathrm{Isom}^+(X)$.*

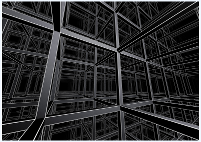
A manifold of the form $X/G$ for a geometry $X$ is called geometric; $G$ is then the holonomy group of $X$.



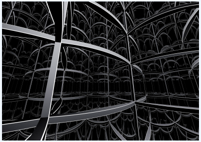
$\mathbb{H}^3$



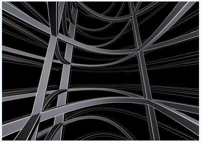
$\mathbb{S}^3$



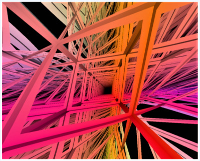
$\mathbb{E}^3$



$\mathbb{H}^2 \times \mathbb{E}^1$

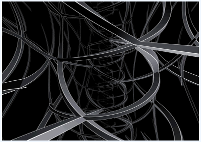


$\mathbb{S}^2 \times \mathbb{E}^1$



$\widetilde{\mathrm{PSL}}(2, \mathbb{R})$

Nil



Sol

All due to Pierre Berger (`http://www.espaces-imaginaires.fr/works/ExpoEspacesImaginaires2.html`)

except $\widetilde{\mathrm{PSL}}(2, \mathbb{R})$ due to Tiago Novelloa, Vinícius da Silvab, Luiz Velhoa, Mikhail Belolipetsky

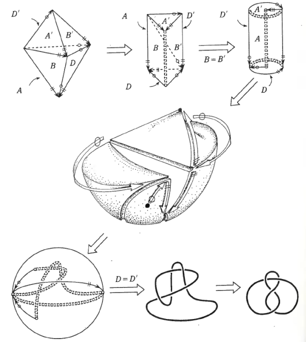
(`https://arxiv.org/abs/2005.12772`, p.27)

### Definition

A **Kleinian group** is one of the following equivalent things:

1. a holonomy group of some hyperbolic 3-manifold.
2. a discrete subgroup of the isometry group of $\mathbb{H}^3$.

# The figure 8 knot




**Example (William Thurston, 1972ish)**

The complement $M := \mathbb{B}^3 \setminus k$ ($k$ the figure 8 knot) is obtained by guing the faces of a hyperbolic tetrahedron with vertices at infinity. Hence $M$ is a hyperbolic 3-manifold

**Theorem (Robert Riley, 1972)**

*There exists a faithful, discrete representation* $\mathrm{Hol}(M) \to \mathrm{PSL}(2, \mathbb{C})$ *with image*

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\exp(2\pi i/3) & 1 \end{bmatrix} \right\rangle.$$

How do we see this matrix action geometrically?

Every matrix in $\mathrm{PSL}(2,\mathbb{C})$ acts naturally on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} . z = \frac{az+b}{cz+d}.$$

Every fractional linear transformation is a product of reflections in circles (a Möbius transformation), so is a conformal map. Every conformal map arises in this way. Hence

$$\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Conf}\,\hat{\mathbb{C}}$$

(they are even isomorphic as topological groups).

1. Given any point $x \in \mathbb{H}^3$ and any point $z \in \partial\mathbb{H}^3 = \hat{\mathbb{C}}$, there is a unique geodesic ray through $x$ towards $z$. Conversely, any geodesic ray through $x$ hits the boundary at exactly one point.
2. Since parallel lines diverge (i.e. hit different points at infinity, unlike in $\mathbb{R}^n$), every isometry of $\mathbb{H}^3$ gives a different conformal map on the boundary (given by moving geodesics and looking at where the ends go) and vice versa.
3. This induces a natural isomorphism (again as topological groups), $\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{Conf}\, \hat{\mathbb{C}}$.

# Summing up the global identifications

There is a natural correspondence:

$$
\begin{tikzcd}
 & \text{biholomorphic automorphisms of } \hat{\mathbb{C}}=\mathbb{P}\mathbb{C} \arrow[d, leftrightarrow] & \\
\text{orientation-preserving Möbius transformations} \arrow[ur, leftrightarrow] \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathrm{PSL}(2,\mathbb{C}) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \text{fractional linear transformations} \arrow[ul, leftrightarrow] \arrow[dl, leftrightarrow] \\
 & \text{isometries of } \mathbb{H}^3 &
\end{tikzcd}
$$

**Definition**

A **Kleinian group** is one of the following equivalent things:

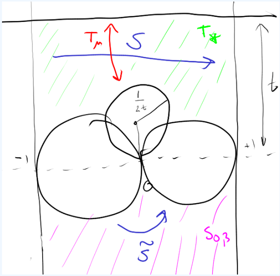
1. a holonomy group of some hyperbolic 3-manifold.
2. a discrete subgroup of the isometry group of $\mathbb{H}^3$.
3. a discrete group of fractional linear transformations.
4. a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.
5. a discrete group of conformal maps of the sphere.

# WARNING!

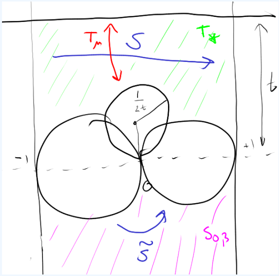
A Kleinian group $G$ acts properly and freely on $\mathbb{H}^3$, and the quotient $\mathbb{H}^3/G$ is a hyperbolic 3-orbifold.
It is **not** true that $G$ acts properly and freely on $\hat{\mathbb{C}}$. In other words, even if there is a tile $T \subseteq \hat{\mathbb{C}}$ with such that $G$ glues $T$ up to a Riemann surface, it is not necessarily true that $T$ tiles the whole $\hat{\mathbb{C}}$.
The maximal subset which $G$ tiles is called its **domain of discontinuity**, $\Omega(G)$. It might be empty.

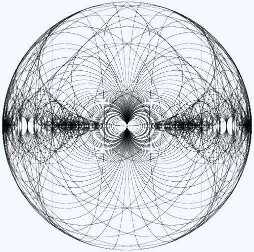
Suppose $\mu = r + ti \in \mathbb{C}$ and define

$$G_\mu := \left\langle S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \tilde{S} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, T_\mu = \begin{bmatrix} -i\mu & i \\ i & 0 \end{bmatrix} \right\rangle$$



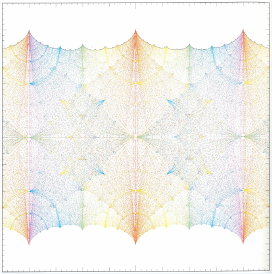
# A punctured torus group



If $\mu \gg 0$ then the group glues the top region up to a punctured torus and the bottom one up to a 3-times punctured sphere. The two regions respectively tile the upper half-plane and the lower half-plane, leaving $\hat{\mathbb{R}}$.

What is the set of all $\mu$ such that $G_\mu$ glues the upper half-plane up to a punctured torus and the lower half-plane up to a 3-times punctured sphere?



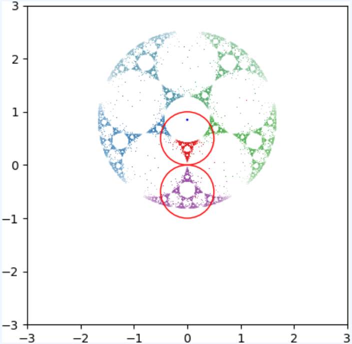
Mumford, Series, and Wright, *Indra's pearls*, p.288.

This is the so-called **Maskit embedding**. It is a Bers slice through the boundary of the quasi-Fuchsian moduli space of punctured torus groups.

# A circle packing group

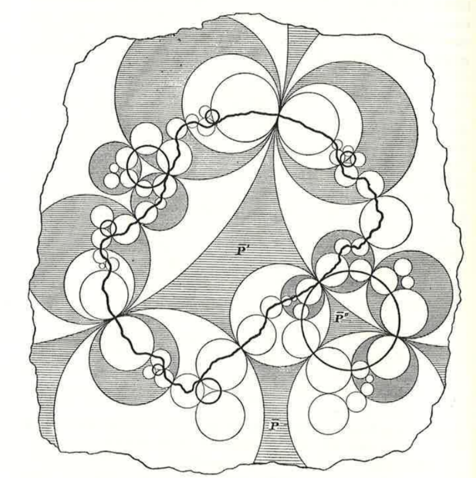
Consider the group

$$\Gamma_{2i}^{5,\infty} = \left\langle \begin{bmatrix} e^{\pi i/5} & 1 \\ 0 & e^{-\pi i/5} \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix} \right\rangle$$

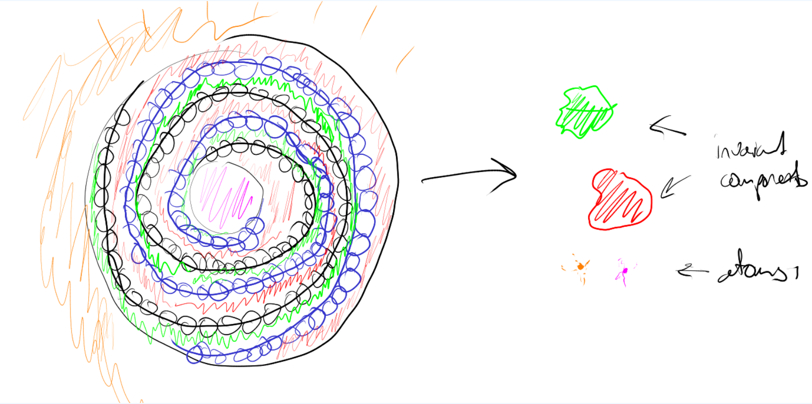


# Bead groups

## Klein and Fricke, 1897:




Fricke and Klein, *Vorlesungen über die Theorie der automorphen Funktionen* 1, fig. 156.



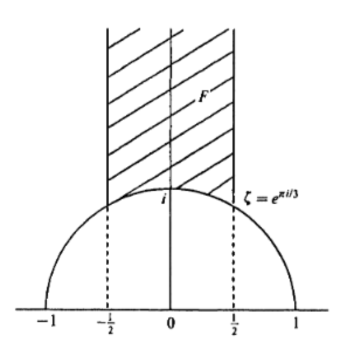

Example due to Accola, see Maskit §VIII.F.7.

# PSL(2, $\mathbb{Z}$)

It is a standard result that

$$\mathrm{PSL}(2,\mathbb{Z}) = \left\langle X = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \right\rangle$$

Any group of finite index in PSL(2, $\mathbb{Z}$) is called a **modular group**.




Miyake, *Modular forms*, fig. 4.1.1.

Let $N \in \mathbb{N}$. Define

$$\Gamma(N) = \left\{A \in \mathrm{PSL}(2, \mathbb{Z}) : A \equiv I_2 \pmod{N}\right\}.$$

A group $G \leq \mathrm{PSL}(2, \mathbb{Z})$ is a **congruence subgroup** if $\Gamma(N) \leq G$ for some $N \in \mathbb{N}$.

**Theorem (Taniyama-Shimura modularity theorem)**

*All rational elliptic curves arise from modular forms.*

We can give a more precise formulation on the level of complex analysis (not the full theorem). If $E$ is an elliptic curve with Weierstrass form $Y^2 = 4X^3 - pX - q$ then define $j(E) := 1728p^3/(p^3 - q^2)$. This is an isomorphism invariant of $E$.

**Theorem**

*If $j(E)$ is rational, then $E$ is the holomorphic image of a Riemann surface $\mathbb{H}^2/G$ for some congruence subgroup $G$.*

More generally, we can define Kleinian groups carrying arithmetic data from more complicated global fields than $\mathbb{Q}$. These will come from embeddings of quaternion algebras.

## Definition

A **quaternion algebra** $A$ over a field $k$ is a four-dimensional $k$-algebra with additive basis $\{1, i, j, k\}$, such that 1 is a multiplicative identity, $i^2 = a1$, $j^2 = b1$, and $ij = -ji = k$ for some $a, b \in k^*$. We write $(a, b|k)$ for this algebra. A quaternion algebra admits a natural conjugation $\bar{\cdot}$, $(1, i, j, k) \mapsto (1, -i, -j, -k)$.

For example, $(-1, -1|\mathbb{R})$ is the usual Hamiltonian quaternion algebra; and $(1, 1|k) = \mathrm{Mat}_{2\times 2}(k)$.

Let $k$ be a number field with at least one complex embedding $\sigma$, let $R$ be its ring of integers, and let $A/k$ be a quaternion algebra. An **order** in $A$ is a finitely generated $R$-submodule $\mathcal{O} \subseteq A$ such that $\mathcal{O} \otimes_R k \simeq A$, which is also a subring (with 1).

### Theorem

*Let $\rho : A \to \mathrm{Mat}_{2\times 2}(\mathbb{C})$ be an embedding such that $\rho \restriction_{Z(A)} = \sigma$, and let $\mathcal{O}$ be an order of $A$. Write $\mathcal{O}^1 = \{x \in \mathcal{O} : x\overline{x} = 1\}$. Then $P\rho(\mathcal{O}^1)$ is a Kleinian group.*

A Kleinian group $G$ is called **arithmetic** if there exists such an algebra and order such that $G \cap P\rho(\mathcal{O}^1)$ is of finite index in both $G$ and $P\rho(\mathcal{O}^1)$ (i.e. $G$ and $P\rho(\mathcal{O}^1)$ are **commensurable**).

Let $d$ be a positive square-free number, so $\mathbb{Q}(\sqrt{-d})$ is a quadratic imaginary number field. Let $O_d$ be the ring of integers of $d$. The groups $\mathrm{PSL}(2, O_d)$ are called Bianchi groups.

**Theorem**

*If $G$ is an arithmetic Kleinian group with $\mathrm{Vol}(\mathbb{H}^3/G) = \infty$, then $G$ is commensurable with a Bianchi group.*

# The figure 8 knot (again)

Recall that the holonomy group of the figure 8 knot complement is the Kleinian group

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\exp(2\pi i/3) & 1 \end{bmatrix} \right\rangle;$$

this is arithmetic (in fact, is index 12 in $\mathrm{PSL}(2, O_3)$).

**Theorem**

*The figure 8 knot is the only knot with arithmetic holonomy group. There are infinitely many links with arithmetic holonomy groups.*

**Theorem**

*Jørgensen and Thurston, c.1979*

1. *Volume is a finite-to-one function of complete hyperbolic 3-manifolds of finite volume.*
2. *The set of volumes of complete hyperbolic 3-manifolds of finite volume is a well-ordered closed subset of $\mathbb{R}_{>0}$. In particular, there is a manifold of minimal volume.*

# The Matveev-Fomenko-Weeks manifold

The following is is the unique smallest-volume closed orientable hyperbolic 3-manifold [Gabai/Meyerhoff/Milley, 2009], Vol ≈ 0.942707...:

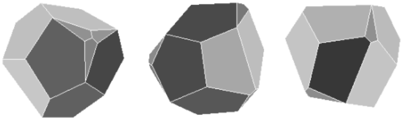


FIG. 12. Three views of the fundamental domain of the Weeks manifold.

TABLE I. The Minkowski coordinates for the 26 vertices of the fundamental polyhedron for the Weeks manifold, which is the comoving spatial section of this paper's model.

| | $X_1$ | $X_2$ | $X_3$ | $X_4$ |
|---|---|---|---|---|
| G | 0.00000000 | 0.00000000 | 0.81159257 | 1.28789848 |
| O | 0.00000000 | 0.10664201 | 0.78401962 | 1.27517029 |
| A | 0.47654246 | 0.27513192 | 0.61537675 | 1.29671849 |
| B | 0.62025991 | −0.16851487 | 0.16978597 | 1.20081093 |
| D | 0.43218382 | 0.62623778 | 0.21703147 | 1.27517029 |
| T | 0.68871325 | 0.03742749 | −0.38772738 | 1.27517029 |
| C | 0.54663073 | 0.54283761 | −0.29664965 | 1.29671849 |
| X | 0.65134973 | −0.00044773 | −0.48417540 | 1.28789848 |
| W | 0.57995255 | 0.06697606 | −0.53406790 | 1.27517029 |
| R | −0.09235469 | −0.05332100 | 0.78401962 | 1.27517029 |
| H | −0.47654246 | 0.27513192 | 0.61537675 | 1.29671849 |
| L | −0.16419180 | 0.62141827 | 0.16978597 | 1.20081093 |
| I | −0.75842974 | 0.06116328 | 0.21703147 | 1.27517029 |
| Q | −0.37676978 | 0.57772942 | −0.38772738 | 1.27517029 |
| K | −0.74342652 | 0.20197730 | −0.29664965 | 1.29671849 |
| Y | −0.32528712 | 0.56430928 | −0.48417540 | 1.28789848 |
| U | −0.34797925 | 0.46876561 | −0.53406790 | 1.27517029 |
| S | 0.09235469 | −0.05332100 | 0.78401962 | 1.27517029 |
| E | 0.00000000 | −0.55026383 | 0.61537675 | 1.29671849 |
| J | −0.45606811 | −0.45290341 | 0.16978597 | 1.20081093 |
| F | 0.32624592 | −0.68740106 | 0.21703147 | 1.27517029 |
| P | −0.31194347 | −0.61515691 | −0.38772738 | 1.27517029 |
| M | 0.19679579 | −0.74481490 | −0.29664965 | 1.29671849 |
| Z | −0.32606261 | −0.56386155 | −0.48417540 | 1.28789848 |
| V | −0.23197331 | −0.53574168 | −0.53406790 | 1.27517029 |
| N | 0.00000000 | 0.00000000 | −0.66479086 | 1.20081093 |

TABLE II. The faces of the fundamental polyhedron and the generators that relate them: $F_k = \gamma_k(F_{k+9})$, $k = 0\text{–}8$. $a$ and $b$ are defined in Table III.

| $k$ | $F_k$ | $F_{k+9}$ | $\gamma_k$ |
|---|---|---|---|
| 0 | ABTCD | HLDAO | $a^2b^{-1}$ |
| 1 | SEFBA | FMPJE | $ab$ |
| 2 | RHIJE | IKQLH | $b^{-1}ab^{-1}$ |
| 3 | FMXTB | UKZVN | $a$ |
| 4 | WNUYC | IJPZK | $b$ |
| 5 | CYQLD | MXWNV | $ba^{-1}$ |
| 6 | CTXW | HOGR | $b^2a^{-1}b$ |
| 7 | KQYU | ERGS | $ba^{-1}b^2a^{-1}b$ |
| 8 | MPZV | ASGO | $b^{-1}a^{-1}b^{-1}a^{-1}$ |

position), the others being at points $P_n$, $n=1\text{–}30$, chosen pseudorandomly. With each generator applied to these points we get $\gamma_k(P_n)$, $k=0\text{–}17$, $n=0\text{–}30$, as potential [16] indirect images; the figure represents all images with redshift $Z \leq 3.1$, as seen from $P_0$. The apparent anisotropy in the figure with some voids and concentrations is suggestive but is not meaningful in the present context, because we only have a partial covering of the space of images $H^3$, namely the FP itself and the 18 cells $\gamma_k$(FP). Now, a degree of anisotropy is expected in such cosmologies, as explained in QGAII and in agreement with Arp's observations [17]. Kokubun and I [18] are working on a more detailed computer simulation of the QGAII model, to estimate this anisotropy in the case of quasar-galaxy

Lehoucq, Luminet, and Uzan, "Topological lens effects in Universes with Non-Euclidean Compact Spatial Sections".

# Bedtime reading


- George K. Francis, *A topological picturebook*. Springer, 1987.
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- Albert Marden, *Hyperbolic manifolds*. Cambridge, 2015.
- Jessica Purcell, *Hyperbolic knot theory*. AMS, 2021.
- Title picture: D. Schwarzenbach, *Crystallography*, p. 51. Wiley, 1996.