Some properties of 2×2 matrices

Deformation spaces of Kleinian groups

Alexander Elzenaar

Supervisors:
Gaven Martin (Massey)
Jeroen Schillewaert

June 8, 2021

Definition

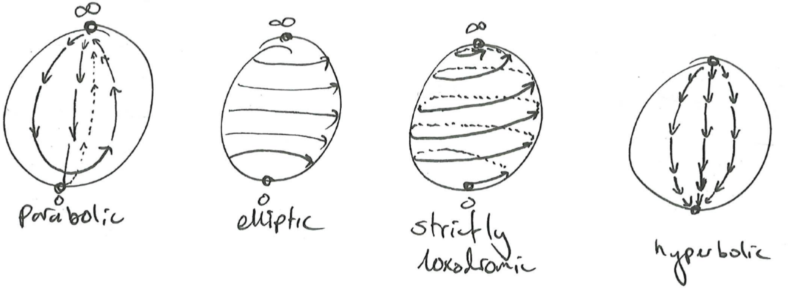
There is a natural correspondence:

$$\begin{tikzcd}
 & \text{conformal (angle-preserving) maps of the sphere} \arrow[dl, leftrightarrow] \arrow[d, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{biholomorphic automorphisms of } \hat{\mathbb{C}}=\mathbb{P}\mathbb{C} \arrow[r, leftrightarrow] & \mathrm{PSL}(2,\mathbb{C}) \arrow[r, leftrightarrow] & \text{fractional linear transformations} \\
 & \text{isometries of } \mathbb{H}^3 \arrow[ul, leftrightarrow] \arrow[u, leftrightarrow] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

A **Kleinian group** is a *discrete* subgroup of $\mathrm{PSL}(2,\mathbb{C})$.
(There are many different definitions of ‘Kleinian group’ depending on the level of abstraction desired.)

Dynamical properties

Elements of a Kleinian group may be categorised according to their fixed points.



Limit sets

Let Γ be a Kleinian group; the **limit set** $\Lambda(\Gamma)$ is the set of accumulation points of orbits of Γ.
That is,

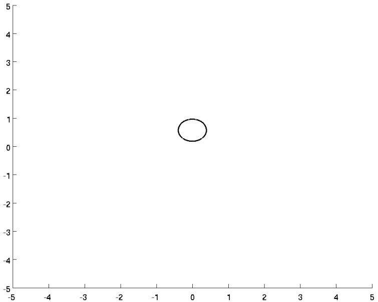
$$x \in \Lambda(\Gamma)$$

$$\Updownarrow$$

there is a sequence (γ_i) and a point $z \in \hat{\mathbb{C}}$ such that

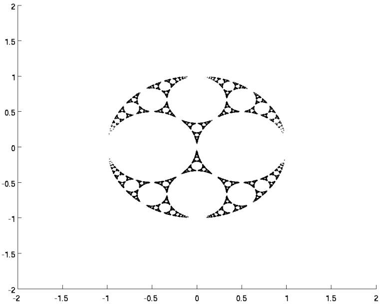
$$x = \lim_{i\to\infty} \gamma_i(z).$$

Limit sets: Fuchsian group



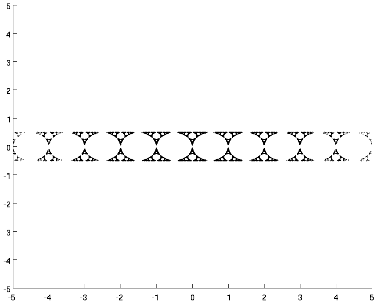
$$\left\langle \begin{bmatrix} \exp(2\pi i/3) & 1 \\ 0 & \exp(-2\pi i/3) \end{bmatrix}, \begin{bmatrix} \exp(2\pi i/7) & 0 \\ 5 & \exp(-2\pi i/7) \end{bmatrix} \right\rangle$$

Limit sets: The Apollonian Gasket



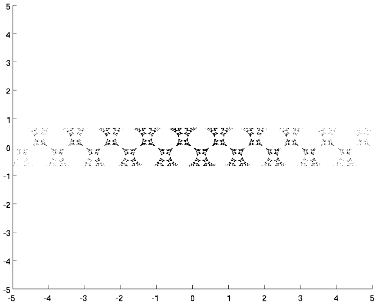
$$\left\langle \begin{bmatrix} 1 & 0 \\ -2i & 1 \end{bmatrix}, \begin{bmatrix} 1-i & 1 \\ 1 & 1+i \end{bmatrix} \right\rangle$$

Limit sets: cusp group



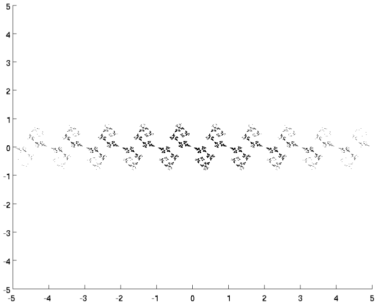
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2i & 1 \end{bmatrix} \right\rangle$$

Limit sets: cusp group II



$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1.5 + 1.3i & 1 \end{bmatrix} \right\rangle$$

Limit sets: 2-parabolic group



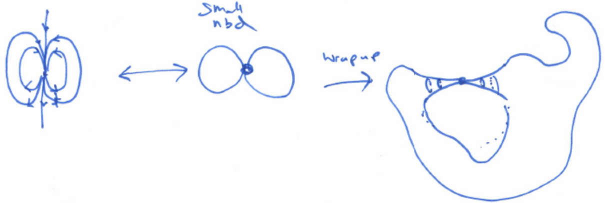
$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+2i & 1 \end{bmatrix} \right\rangle$$

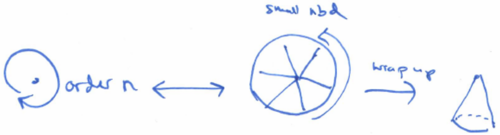
Regular sets

Let G be a Kleinian group. The complement of $\Lambda(G)$ in $\hat{\mathbb{C}}$ is the **regular set** of G, denoted $\Omega(G)$.
The quotient $\Omega(G)/G$ is a marked Riemann surface.

$$\begin{array}{ccc} \hat{\mathbb{C}} & & \Omega(G)/G \\ \text{elliptic fixed points in } \Omega(G) & \to & \text{cone points} \\ \text{parabolic fixed points in } \Lambda(G) & \to & \text{cusp points} \end{array}$$

Marked points





Hyperbolic 3-manifolds

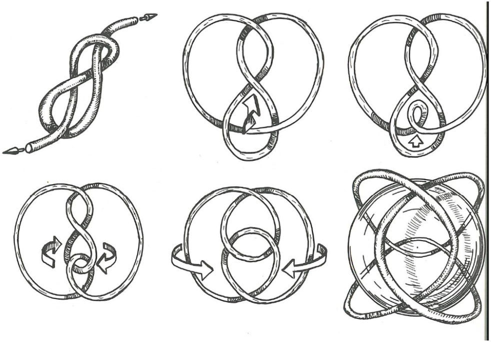
We can view $\hat{\mathbb{C}}$ as the boundary at infinity of $\mathbb{H}^3$. The basis of the quinchotomy in the first slide is essentially that the group of conformal maps on $\hat{\mathbb{C}}$ is naturally identified with the group of isometries of $\mathbb{H}^3$. Thus if G is a Kleinian group, we may construct the quotient orbifold

$$\frac{\Omega(G) \cup \mathbb{H}^3}{G}$$

(sometimes called the **Kleinian manifold** of G), which has a hyperbolic structure.

Every hyperbolic 3-manifold arises in this way: if M is a hyperbolic 3-manifold without boundary, then $M \simeq \mathbb{H}^3/G$ where G is the holonomy group of M.

The figure 8 knot



(Image from G. K. Francis (1987). *A topological picturebook*. Springer, p. 150)

The figure 8 knot complement

Theorem (Riley, 1975)

The complement M of the figure 8 knot admits a hyperbolic structure.

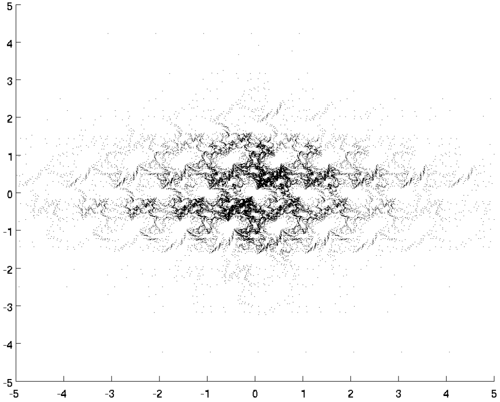
Idea of the proof.

Let Π be the fundamental group of the knot complement M. Then Π is a 2-generated group, say $\Pi = \langle x_1, x_2 : \text{some known relations} \rangle$. Define a representation $\theta : \Pi \to \mathrm{SL}(2, \mathbb{C})$:

$$\theta x_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \theta x_2 := \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}$$

where ω is a primitive 3rd root of unity. Then θ is faithful and so $\Pi \simeq \theta\Pi$; by Mostow rigidity then, $M \simeq \mathbb{H}^3/\theta\Pi$. □

Limit sets: figure 8 complement



Two theorems of Thurston

Generalisations:

Theorem

If $K \subseteq S^3$ is a knot, then $S^3 \setminus K$ admits a hyperbolic structure iff K is not one of two degenerate families of knots.[1]

Theorem

Suppose $L \subseteq S^3$ is an indecomposable link of $m \geq 2$ components. If L satisfies two criteria similar to the previous theorem, then $S^3 \setminus L$ has a hyperbolic structure.

[1] Is neither a torus knot or a satellite knot.

2-bridge knots

A **2-bridge knot** is (roughly speaking) a knot which admits a diagram of the form

that is, a diagram made up of braids connected via noncrossing lines such that there are only 2 'turning points' to the far left.

(Diagram from W. R. Lickorish (1997). *An introduction to knot theory*. Graduate Texts in Mathematics 175. Springer, p. 9.)

Riley's presentation again

Recall the presentation of Riley for the figure 8 group. For $\omega \in \mathbb{C}$ define

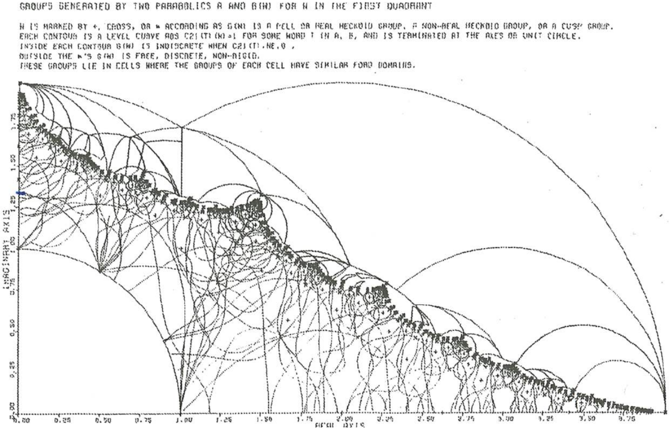
$$G_\omega := \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \omega & 1 \end{bmatrix} \right\rangle.$$

(Up to conjugacy in $\mathrm{SL}(2,\mathbb{C})$, every group of isometries of $\mathbb{H}^3$ generated by two parabolics is of this form.)

Theorem (Adams, 1991)

G_ω is discrete and $\operatorname{Vol} \mathbb{H}^3/G_\omega$ is finite if and only if G_ω is isomorphic to $\pi_1(S^3 \setminus L)$ for some 2-bridge link complement L.

Riley studied the complex numbers ω such that G_ω corresponds to a 2-bridge link group. He produced the following plot:



(Plot is reproduced from H. Akiyoshi, M. Sakuma, M. Wada, and Y. Yamashita (2007). *Punctured torus groups and 2-bridge knot groups I*. Lecture Notes in Mathematics 1909. Springer, p. VIII.)

Definition

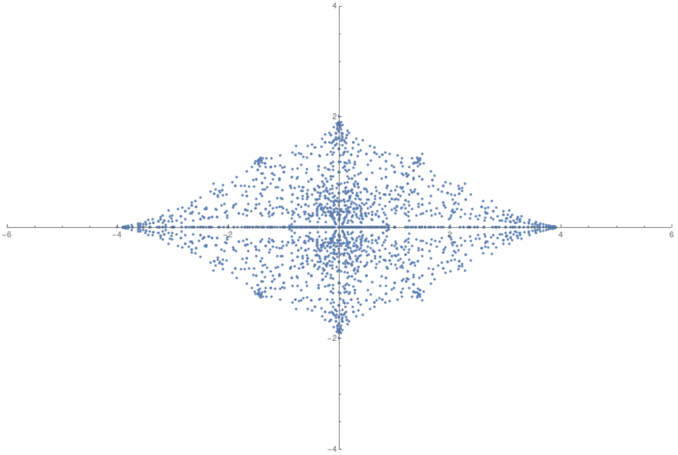
The set of complex numbers obtained by Riley outlines the following domain:

Definition

The **Riley slice** (of Schottky space) is the subspace $\mathcal{R}$ of $\mathbb{C}$ consisting of all ω such that G_ω is discrete, free, and the quotient $\Omega(G_\omega)/G_\omega$ is a 4-times punctured sphere.

A picture of $\mathcal{R}$

The blue dots below approximate the exterior of the Riley slice in $\mathbb{C}$.



In the remainder of this talk, we will discuss the structure of the Riley slice as described by Linda Keen and Caroline Series.
One should also note: the theory of Jørgensen allows this study to be extended to the *exterior* of the Riley slice.

Initial properties

Theorem (Bers, Lyubich, Maskit, Suvorov, Swarup, and others)

The Riley slice is a connected open set, and in fact is topologically an annulus. If $\omega \in \partial\mathcal{R}$, then G_ω is discrete and either $\Omega(G_\omega)\emptyset$ or $\Omega(G_\omega)/G_\omega$ is a pair of 3-times punctured spheres. Points in the latter category are called **cusps**.

Theorem (Special case of McMullen, 1991)

Cusp points are dense on $\partial\mathscr{R}$.

McMullen won a Fields Medal in 1998, partially for the general theorem of this type.

Farey polynomials

Fix $\omega \in \mathcal{R}$, and write G for G_ω. Here, and from now on, we use X and Y for the two generators of G.

To each $m \in \mathbb{Q}$, we may associate a nontrivial simple closed geodesic $\gamma(m)$ on the 4-times punctured sphere. All of these geodesics are homotopically distinct, and all but one homotopy class of simple closed curves arise in this way.

Farey polynomials

Theorem

If $p/q \in [0,2) \cap \mathbb{Q}$, then the lift of the simple closed curve $\gamma(p/q) \subseteq \Omega(G)/G$ to $\Omega(g)$ has a connected component left invariant by a word

$$V_{p/q} = X^{\varepsilon_1} Y^{\varepsilon_2} \ldots X^{\varepsilon_{2q-1}} Y^{\varepsilon_{2q}}$$

where each $\varepsilon_i \in \{\pm 1\}$.
The trace $\operatorname{tr} V_{p/q}$ is a polynomial in the indeterminate ω of degree q, with integral coefficients.

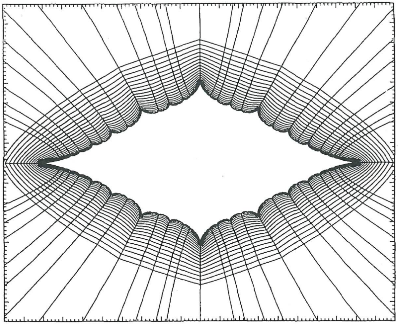
Rational pleating rays

Let $m \in \mathbb{Q}$. The **rational pleating ray** P_m is the union of the connected components of the inverse image of $(2, \infty)$ under the polynomial function $\operatorname{tr} V_m$.

- Two pleating rays P_n and P_m intersect iff $m = n$.
- The union of the pleating rays is dense in $\mathcal{R}$.
- The endpoints on $\partial\mathcal{R}$ of the pleating rays are cusps, and the element V_m in the corresponding group is parabolic.

Picture

P_0 is the ray $(-\infty, -4)$; P_1 is the ray $(4, \infty)$.



(Plot is due to David Wright, and reproduced from L. Keen and C. Series (1994). "The Riley slice of Schottky space". In: *Proceedings of the London Mathematics Society* 3.1 (69), pp. 72–90.)

The theory of Keen and Series shows the following:

The structure of the Riley slice and its boundary are determined by the combinatorial properties of the polynomials $\operatorname{tr} V_{p/q}$.

In particular, the hyperbolic structures on many 3-manifolds are determined by the combinatorics of these polynomials.

Question

Given the success of the study of groups on two parabolic generators, might the methods of Keen and Series work to study the moduli space of Kleinian groups on two *elliptic* generators?

Definitions

Let $p, q \in \mathbb{N}$. Define the group $G_\mu = \langle X, Y_\mu \rangle$, where

$$X = \begin{bmatrix} \zeta & 1 \\ 0 & \overline{\zeta} \end{bmatrix} \qquad Y_\mu = \begin{bmatrix} \xi & 0 \\ \mu & \overline{\xi} \end{bmatrix}$$

$$\text{(where)} \qquad \zeta = \exp\left(\frac{2\pi}{p}\right), \quad \xi = \exp\left(\frac{2\pi}{q}\right),$$

The elliptic Riley slice will be the space of all $\mu \in \mathbb{C}$ such that G_μ is discrete and the quotient $\Omega(G_\mu)/G_\mu$ is a sphere with four cone points.

Some pictures

The following video shows the conjectured elliptic Riley slices for $p = 3$ and q varying in steps of 0.05 from 0.05 to 20.