The moulding of hyperbolic clay

Deformation spaces of Kleinian groups

Alexander Elzenaar

Supervisors:
Gaven Martin (Massey)
Jeroen Schillewaert

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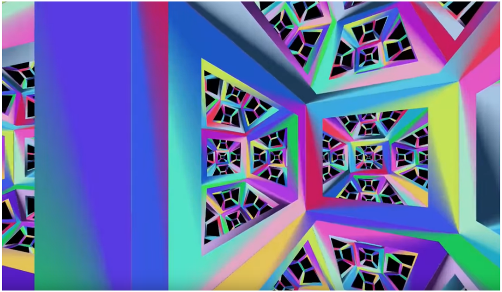
Definition. A Kleinian group is a discrete[1] group of (orientation-preserving) isometries of hyperbolic 3-space $\mathbb{H}^3$.



Screengrab from `https://www.youtube.com/watch?v=jfSTwqmrQDc`

[1] With respect to the topology of the Lie group $\mathrm{Isom}^+(\mathbb{H}^3)$.

Limit points of groups

Let Γ be a group of homeomorphisms of a space X. A point $x \in X$ is a **limit point** of Γ if x is an accumulation point of some orbit of Γ (i.e. if there exists a point x_0 and a sequence of distinct elements $(\gamma_i \in \Gamma)$ such that $\gamma_i x_0 \to x$.)

Lemma

The action of a Kleinian group on $\mathbb{H}^3$ has no limit points.

If a discrete group Γ acts on a locally compact space X without limit points, then X/Γ is Hausdorff; all these conditions are satisfied for the action of a Kleinian group on $\mathbb{H}^3$, and in fact we get more:

Theorem

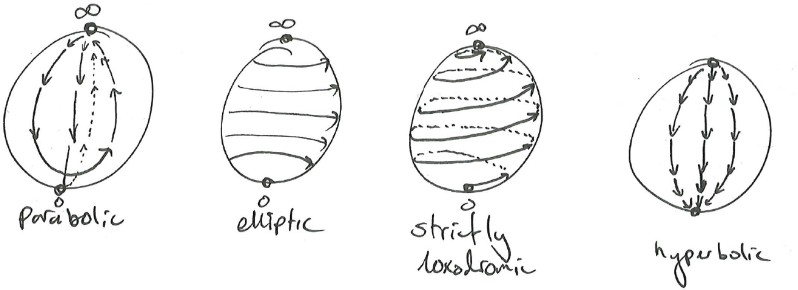
If Γ is a Kleinian group, then $\mathbb{H}^3/\Gamma$ is a hyperbolic orbifold.

The action at infinity

- We can extend the action of an isometry f of $\mathbb{H}^3$ to an action on the sphere at infinity S^2_∞ of hyperbolic space: given any point z at infinity, move an oriented geodesic ending at z by f and look at the new location of the end. (This can be made precise depending on your definition of S^2_∞.)
- It turns out that this action is conformal. We can classify all of the conformal bijective maps on S^2: they are precisely the Möbius transformations (all the transformations generated by reflecting in circles.)

The action at infinity: dynamical properties

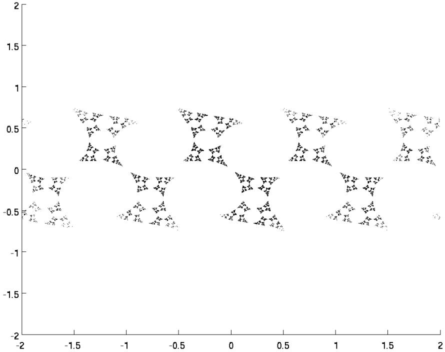
We can classify the elements of a Kleinian group according to their orbits on S^2_∞:



Observe, *we can have limit points at infinity*.

The action at infinity: dynamical properties

By stereographic projection, view $S^2_\infty = \mathbb{C} \cup \{\infty\}$. Here is an example limit set:



The action at infinity: dynamical properties

If Γ is a Kleinian group, we write $\Lambda(\Gamma)$ for its limit set, and $\Omega(\Gamma)$ for the complement $S^2_\infty \setminus \Lambda(\Gamma)$. This second set is the **ordinary set** or **domain of discontinuity**.

Lemma

If Γ is a Kleinian group, then $\Omega(\Gamma)/\Gamma$ is a (possibly singular) Riemann surface.

Putting everything together, it turns out that the Kleinian manifold

$$\frac{\Omega(\Gamma) \cup \mathbb{H}^3}{\Gamma}$$

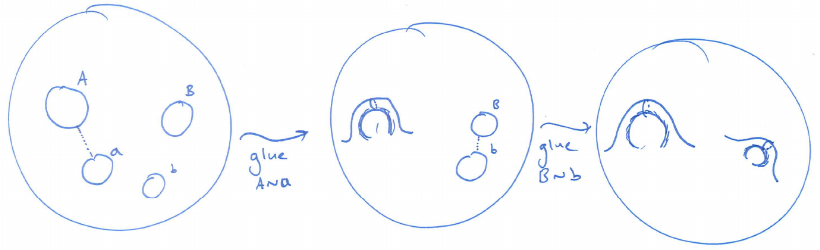
is an orbifold-with-boundary, with interior $\mathbb{H}^3/\Gamma$ and boundary $\Omega(\Gamma)/\Gamma$.

Circle pairings: a 2-torus

Consider four disjoint circles in the plane, all disjoint. Put the circles in pairs:

- There are conformal bijections f and g which act on $\hat{\mathbb{C}}$ such that $f(\operatorname{int} A) = \operatorname{ext} B$ and $g(\operatorname{int} b) = \operatorname{ext} B$. The group $\Gamma = \langle f, g \rangle$ is a Kleinian group, called the **classical Schottky group** obtained from the given data.
- (**Poincaré polyhedron theorem**) The quotient surface $\Omega(\Gamma)/\Gamma$ can be obtained by taking the common exterior of the four circles and gluing a to A and b to B with the identifications given by f and g.

Circle pairings: a 2-torus

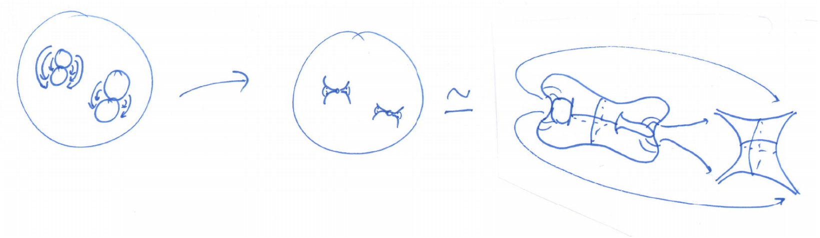


Circle pairings: 4-times punctured spheres

- Consider now the following deformation. We take a and A and push them together, and take b and B and push them together, such that the paired circles become tangent. As long as the two pairs remain mutually disjoint and the paired circles have the same radius, the Poincaré polyhedron theorem still holds.
- However, the Riemann surface now has *cusps* at the point of tangency. Hence we obtain a 4-times punctured sphere as the quotient.
- Observe, we have two real parameters: the relative angle of $a \cup A$ to $b \cup B$, and the distance between them. In fact, all of the groups we obtain are parameterised by one complex number, ρ.[2]

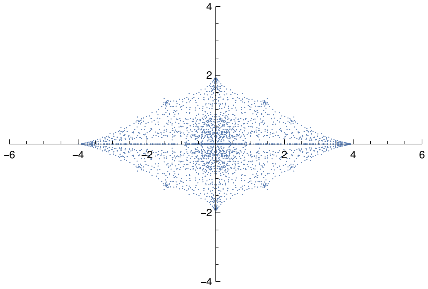
[2] Actually the parameters are the angle and distance between the two hyperbolic lines joining the two intersection points in each pair...

Circle pairings: 4-times punctured spheres



The Riley slice

After suitable normalisation, the parameter values for which we obtain 4-times punctured spheres can be plotted; they form the *exterior* of the following figure:



This exterior is called the **Riley slice**. We denote it by $\mathcal{R}$.

The Riley slice: properties

Theorem (Bers, Lyubich, Maskit, Suvorov, Swarup, and others)

The Riley slice is a connected open set, and in fact is topologically an annulus. If $\omega \in \partial\mathcal{R}$, then G_ω is discrete and either $\Omega(G_\omega) = \emptyset$ or $\Omega(G_\omega)/G_\omega$ is a pair of 3-times punctured spheres. Points in the latter category are called **cusp points**.

It is worth separating out the following, as the theorem of which this is a special case is part of the work for which McMullen won a Fields Medal in 1998:

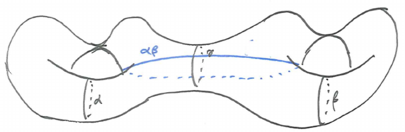
Theorem (Curtis McMullen, 1991)

Cusp points are dense on $\partial\mathcal{R}$.

Deformations

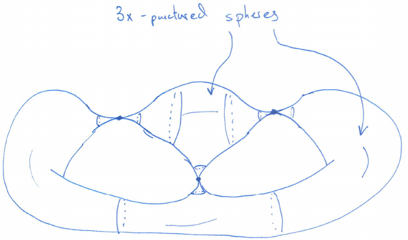
Observe that we obtained the 4-punctured sphere by 'deforming' Schottky groups. In some sense we are justified in saying that the Riley slice is a subset of the boundary of the space of Schottky groups (this can be made precise).

We now study the boundary of the Riley slice. Suppose we take the 4-punctured sphere and begin to shrink the geodesic marked as $\alpha\beta$ in the following picture:



Deformations

After pinching the marked geodesic to zero, we obtain a pair of 3-punctured spheres:



The fundamental group

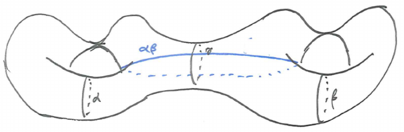
In order to understand what is happening algebraically, we need to study the relationship between $\pi_1(\Omega(\Gamma)/\Gamma)$ and Γ. A figure like the common exteriors of the circles which we used to define the Schottky groups is called a **fundamental domain**. In this case, the fundamental domain W is the common exterior of two pairs of tangent circles. Observe, this is topologically an annulus. Basic algebraic topology and the Poincaré polyhedron theorem gives the following:

$$\Gamma \simeq \frac{\pi_1(\Omega(\Gamma)/\Gamma)}{p_*\pi_1(W)}$$

where $p : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$ is the canonical projection.

The fundamental group

In particular, elements of Γ lift to homotopy classes of closed loops on the 4-punctured sphere; and every homotopy class is obtained except for the one which corresponds to the nontrivial loop in W (this is the geodesic γ):



One can check that the geodesic labelled as $\alpha\beta$ is the lift of the element fg from $\Gamma = \langle f, g\rangle$ to the fundamental group of the 4-punctured sphere.

Curves to the boundary

- It is possible to enumerate all of the curves which, when shrunk to zero, deform the 4-punctured sphere to a pair of 3-punctured spheres. They are in bijective correspondence with rational numbers $0 < r/s < 1$ according to the theory of Dehn's **cutting sequences**.
- We let $\mathscr{W}$ be the set of all words in the generators of Γ which correspond to such curves; each word corresponds (via the isomorphism $\mathrm{PSL}(2,\mathbb{C}) \simeq \mathrm{Isom}^+(\mathbb{H}^3)$) to a matrix; and the trace of the matrix determines the dynamical type of the object.
- Pinching a geodesic to zero corresponds to deforming the relevant lift to a parabolic element; this corresponds to sending the trace to ± 2.

Farey polynomials

Definition

A polynomial (in the single complex parameter ρ) obtained by taking the trace of an element of $\mathscr{W}$ is called a **Farey polynomial**. Each polynomial is associated bijectively with a rational number $0 < r/s < 1$, the **slope**. Write $\Phi_{r/s}$ for the polynomial with slope r/s.

Thus we wish to study 'limits of preimages of Farey polynomials' as the image tends to ± 2.

Pleating rays

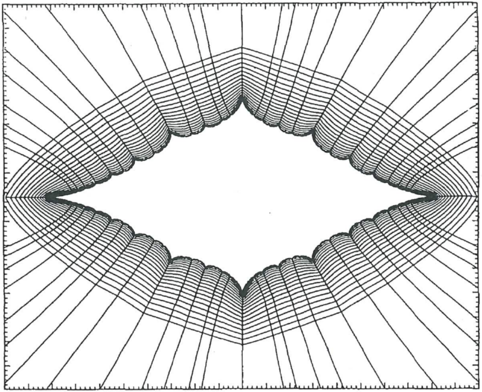
Theorem (Linda Keen and Caroline Series, c.1992)

There is a lamination of the Riley slice obtained in the following way: for each Farey polynomial $\Phi_{r/s}$, there are two connected components of the inverse image

$$\Phi_{r/s}^{-1}\left((-\infty,-2)\cup(2,\infty)\right)$$

*with respective asymptotic slopes $\pi(1+r/s)$ and $\pi(1-r/s)$. These components (called the r/s-**pleating rays**) are complex conjugate nonsingular curves, with unique complex conjugate endpoints on $\partial\mathcal{R}$ corresponding to the "± 2–limit"; these endpoints are cusp groups, hence are dense in the boundary.*

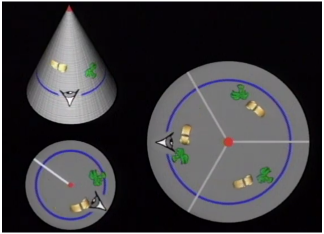
Pleating rays (picture)



(Plot is due to David Wright, and reproduced from L. Keen and C. Series (1994). "The Riley slice of Schottky space". In: *Proceedings of the London Mathematics Society* 3.1 (69), pp. 72–90.)

The 2-elliptic case

Take the space of Riemann surfaces with four paired *ramification* points instead of punctures. This corresponds to the quotient space of a group with two elliptic (rotation) elements, which determine the cone angles.



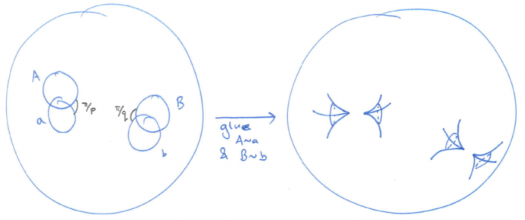
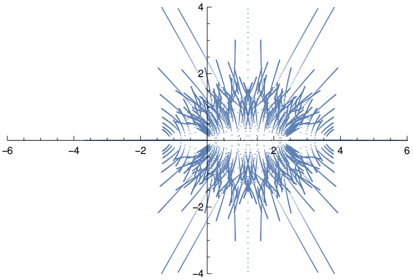
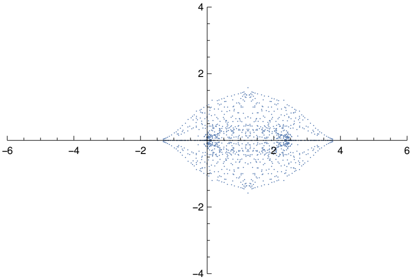


Figure on left from C. Gunn, D. Maxwell, and D. Epstein (1991). *Not Knot*. URL: https://www.youtube.com/watch?v=4aN6vX7qXPQ.

Pleating rays for cone angles $\pi/3$ and $\pi/4$





Big question

The Farey polynomials determine the shape of the Riley slice.
So, what are the combinatorial properties of the Farey polynomials?