The Riley slice

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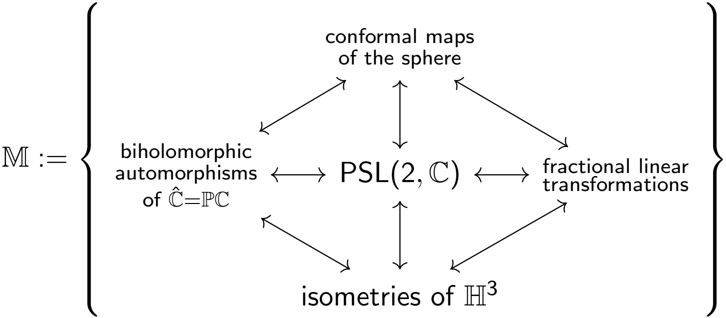
In this talk, $\mathbb{H}^3$ will denote hyperbolic 3-space with sphere at infinity identified with $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We can extend isometries of $\mathbb{H}^3$ to a conformal action on the boundary. Conversely, conformal maps on $\hat{\mathbb{C}}$ extend uniquely to isometries on the interior. Thus all of the following are 'the same':

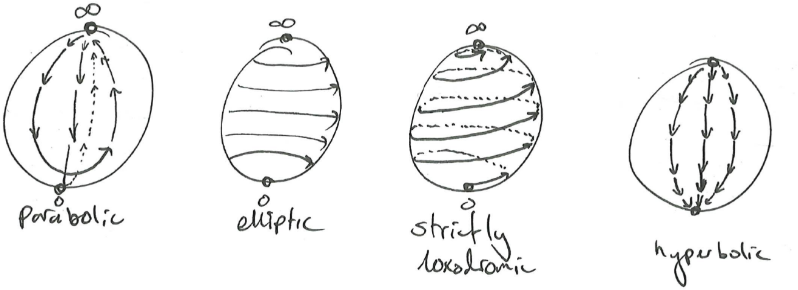


We are interested in subgroups $G \leq \mathbb{M}$ which have 'geometric' actions. That is, we wish for $\mathbb{H}^3/G$ to be an orbifold. The most useful assumption on G which guarantees this is discreteness. Discrete subgroups of $\mathbb{M}$ are called **Kleinian**.

Discreteness is not enough to ensure the action on $\hat{\mathbb{C}}$ gives a nice geometric quotient, since orbits of points often accumulate on $\hat{\mathbb{C}}$ making the quotient non-Hausdorff. If we delete these points, everything works nicely:

- The **limit set** of a Kleinian group G is the set $\Lambda(G) \subseteq \hat{\mathbb{C}}$ of accumulation points of the orbits of G on $\overline{\mathbb{H}^3}$.
- The **regular set** is $\Omega(G) := \hat{\mathbb{C}} \setminus \Lambda(G)$.
- **Theorem:** $\mathcal{S}(G) := \Omega(G)/G$ is a (marked, possibly disconnected) Riemann surface.

We can classify the elements of a Kleinian group according to their orbits on $\hat{\mathbb{C}}$:



The orbit types are distingushed by the tr^2 of matrix representatives for the element in $\mathrm{PSL}(2, \mathbb{C})$.

The study of the surface $\mathcal{S}(G)$ is a very important tool in the study of Kleinian groups. The easiest model for this is the theory of Schottky groups.

Every subgroup of $\mathbb{M}$ generated by two non-loxodromics with distinct fixed point sets is equivalent (up to conjugacy) to one of the groups

$$\Gamma_\rho^{p,q} := \left\langle X := \begin{bmatrix} e^{2\pi i/p} & 1 \\ 0 & e^{-2\pi i/p} \end{bmatrix}, Y := \begin{bmatrix} e^{2\pi i/q} & 0 \\ \rho & e^{-2\pi i/q} \end{bmatrix} \right\rangle.$$

where $\rho \in \mathbb{C}$ and $p, q \in \mathbb{N} \cup \{\infty\}$ (by convention, $\exp(2\pi i/\infty) = 1$).

For ρ sufficiently large ($|\rho| > 4$), this group is a deformed Schottky group with one pair of circles for each of X and Y_ρ. Thus, for $|\rho| \gg 0$, $\Gamma_\rho^{p,q}$ discrete with the only relations being those directly induced by having finite-order generators, and $\mathcal{S}(\Gamma_\rho)$ is supported on a 4-marked sphere with marked points identified in pairs (each pair being of same order):

If we fix the orders of the two generators to be p, q (allowing ∞ for parabolics) then we get a family of groups $\Gamma_\rho^{p,q}$, indexed by one complex parameter ρ. Within one of these families, for $|\rho| >> 0$ every surface $\mathcal{S}(\Gamma_\rho)$ is of the same topological type.
Define the (p,q)-**Riley slice** to be the set $\mathcal{R}^{p,q}$ of all $\rho \in \mathbb{C}$ such that $\mathcal{S}(\Gamma_\rho^{p,q})$ is of this type.

Theorem

- *Each group in $\mathcal{R}^{p,q}$ is discrete, and the only relations are those imposed by finite-order generators.*
- *$\mathcal{R}^{p,q}$ is an open set homeomorphic to an open annulus.*

The boundary $\partial\mathcal{R}^{p,q}$ consists of two types of groups:

- Groups $\Gamma_\rho^{p,q}$ with $\Omega(\Gamma_\rho^{p,q}) = \emptyset$; and
- Groups $\Gamma_\rho^{p,q}$ with $\mathcal{S}(\Gamma_\rho^{p,q})$ consisting of a disjoint union of two triply marked spheres, each sphere having marked points corresponding to each of the two generators plus one additional puncture.

This is explained with the theory of Keen and Series.

We explain briefly the theory of Keen and Series (1994), extended to the 2-marked case.
The first step is a result of Lyubich and Suvorov (1988):

Theorem

$\mathcal{R}^{p,q}$ is biholomorphic to the quotient of Teich(S_4) *by the subgroup generated by a single Dehn twist along γ_∞.*

Picture:

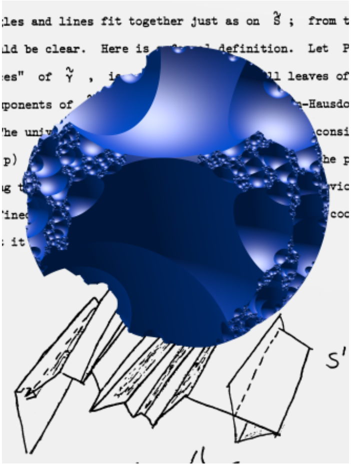
The previous result shows that the space $\mathcal{R}^{p,q}$ is the space of 'ways to glue two identical two-punctured discs along their boundary'. We can see easily that every closed simple curve on S_4 is represented, up to (1) homotopy and (2) factors of γ_∞, by one constructed by taking the projection of a line of rational slope in $\mathbb{R}^2 \setminus \mathbb{Z}^2$ through a quotient. We therefore label these curves, which correspond to the gluing boundaries of the discs, by rational numbers: the curve of slope r/s is denoted $\gamma_{r/s}$.

One now thinks about moving to the boundary by pinching one of these curves $\gamma_{r/s}$. This corresponds to deforming the group element representing it, $W_{r/s}(\rho)$, so as to send it to a parabolic. That is, we consider curves such that $\operatorname{tr}^2 W_{r/s}(\rho) \to 4$ as ρ moves to the boundary along the curve; and this corresponds geometrically to the deformation sending a 4-marked sphere to a pair of 2-marked spheres with an additional puncture each.

The main idea of Keen and Series is to use this, together with the work of Thurston on measured laminations, to study the structure of the slice.

The two discs embedded in $\mathcal{S}(\Gamma_\rho)$ correspond to two non-conjugate Fuchsian subgroups, each generated by a pair of parabolic or elliptic elements A, B such that the word $W_\infty(\rho)$ of Γ_ρ representing γ_∞ is given by walking about each marked point on the disc in turn (i.e. $W_\infty(\rho)$ is in the free homotopy class of AB). This system of two groups with shared subgroup $\langle W_\infty(\rho)\rangle$ accomplishes the gluing algebraically.

These embedded subgroups manifest geometrically in $\Lambda(\Gamma_\rho)$. Thurston (1979) studied a combinatorialisation construction on hyperbolic 3-manifolds: for a Kleinian group G, $\mathrm{h.conv}(\Lambda(G))/G$ is a deformation retract of $(\mathbb{H}^3 \cup \Omega(G))/G$. Taking boundaries gives us the construction which we need here: $\partial\,\mathrm{h.conv}(\Lambda(G))/G$ is a pleated surface.



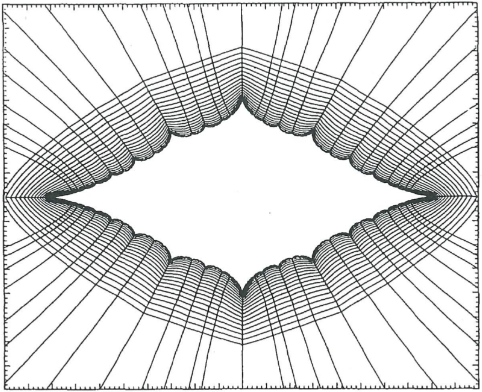
(Image due to Jeffrey Brock and David Dumas, `https://www.dumas.io/convex/`.)

We want the projection onto S_4 to have a single pleat, exactly along the curve $\gamma_{r/s}$. This corresponds to the pleats at the intersections of domes erected over the limit sets of the two Fuchsian groups being the only ones which are 'visible' in the quotient.

A picture showing the desired pleated surface structure on S_4:

Definition

The **rational pleating ray of slope** r/s is the set $\mathrm{pl}(r/s)$ of $\rho \in \mathcal{R}^{p,q}$ such that the pleating locus of the hyperbolic convex hull boundary projects exactly to the curve $\gamma_{r/s}$ on the quotient.



(Plot is due to David Wright, and reproduced from Keen and Series (1994).)

Theorem

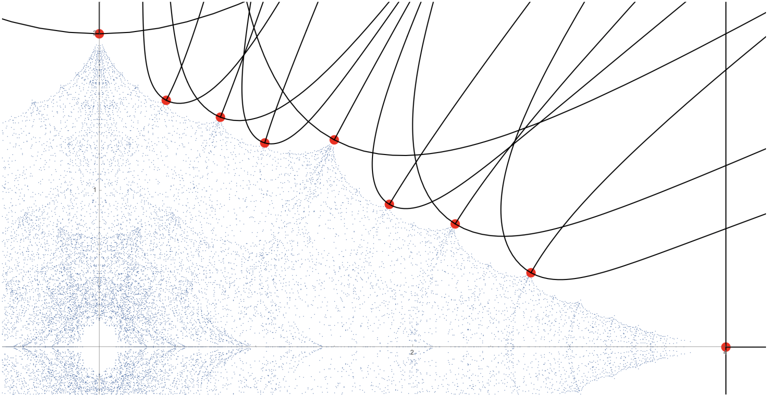
For each $r/s \in \mathbb{Q}$, there is a polynomial $\Phi^{p,q}_{r/s}$ (which we call a **Farey polynomial***) with the property that*

$$\mathrm{pl}(r/s) \approx (\Phi^{p,q}_{r/s})^{-1}((-\infty, -2))$$

where we use $\approx$ to mean 'up to choosing the right connected component of the inverse image".

Theorem (Open neighbourhood theorem: E., Martin, Schillewaert (2021))

Let H be the half-plane $\{z \in \mathbb{C} : \operatorname{Re} z < -2\}$*. Then the connected component of* $(\Phi_{r/s}^{\infty,\infty})^{-1}(H)$ *containing the pleating ray is an open subset of the Riley slice.*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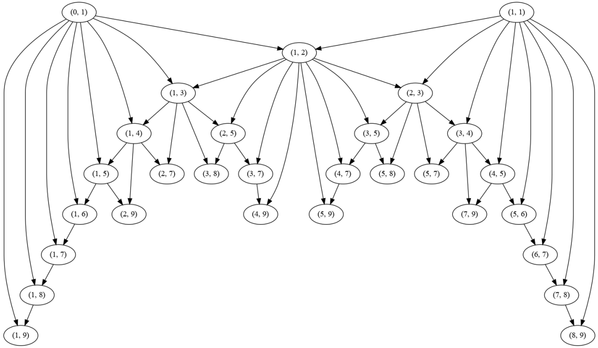
The interest here is that we can find a neighbourhood 'pushing outwards' from a cusp near the (fractal) slice boundary. The proof is a slight modification of the Keen-Series argument: the main difficulty is showing that the analogues of the peripheral discs converge to peripheral discs in such a way as that the limit disc is also peripheral (and so is glued in the limit surface to form a 4-times punctured sphere).

Since McMullen (1991) proved that cusps are dense in the boundary, these neighbourhoods fill the Riley slice and so give a concrete method for deciding whether a given point lies in the Riley slice (the Keen-Series theory as originally developed does not allow this since the rational pleating rays form a subset of measure 0 even though they are dense).

The Farey polynomials are difficult to compute: $\Phi_{p/q}^{r,s}$ is the trace of a product of $2q$ matrices. Thus computing the Farey polynomials is computationally difficult. We have found an iterative process for computing Farey polynomials which does not depend on computing matrix products.

Let p/q and r/s be rational numbers in least terms; the **mediant** is the operation defined by $p/q \oplus r/s := (p+r)/(q+s)$. If $ps - qr = \pm 1$, then the mediant is the fraction in $(p/q, r/s)$ with minimal denominator and we say that p/q and r/s are **Farey neighbours**.

Define a digraph $\mathcal{F}$ with vertices $\mathbb{Q} \cap [0,1]$ such that there is an edge $p/q \to r/s$ if $r/s = p/q \oplus p'/q'$ for some p'/q' a Farey neighbour of p/q.



Theorem

Let $p, q \in \mathbb{N} \cup \{\infty\}$; define $\alpha = 2\pi/p$ and $\beta = 2\pi/q$. Then, if r_1/s_1 and r_2/s_2 are Farey neighbours,

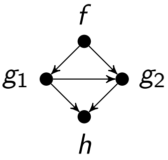
$$\Phi_{r_1/s_1}\Phi_{r_2/s_2} + \Phi_{r_1/s_1 \oplus r_2/s_2} + \Phi_{r_1/s_1 \ominus r_2/s_2} = \begin{cases} 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2 & (s_1 + s_2 \textit{ even}) \\ 2\left(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) & (s_1 + s_2 \textit{ odd}). \end{cases}$$

This, combined with the starting triple

$$\Phi_{0/1}(z) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z$$
$$\Phi_{1/1}(z) = \alpha\beta + \frac{1}{\alpha\beta} + z$$
$$\Phi_{1/2}(z) = 2 + \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z + z^2$$

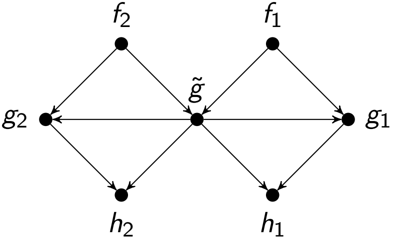
allow us to compute every Farey polynomial recursively down the Farey graph.

The formulae give invariants of certain substructures in the Farey graph: for every directed subgraph of the form



the sum $f + g_1 g_2 + h$ is dependent only on the parity of the denominator of the fraction corresponding to h.

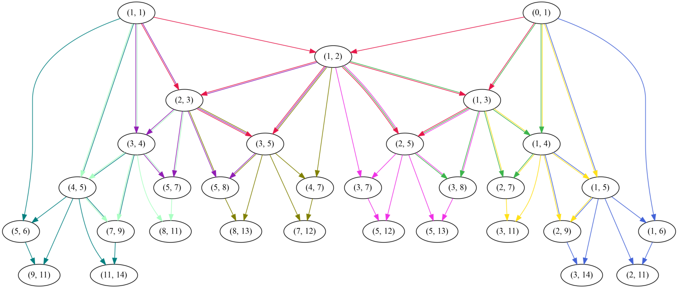
In $\mathcal{F}$, we may take parity pairs of these diamonds to form a 'butterfly' subgraph with the following combinatorial structure:



Labelling the Farey polynomials of the structure according to the scheme above and summing the two identities, we get

$$f_1 + f_2 + g_1\tilde{g} + g_2\tilde{g} + h_1 + h_2 = (\operatorname{tr} X + \operatorname{tr} Y_\rho)^2.$$

Consider again the Farey graph, now coloured according to the butterfly substructures, and delete all except the edges which form the two 'horizontal' edges of the butterfly substructures. The resulting tree is familiar.



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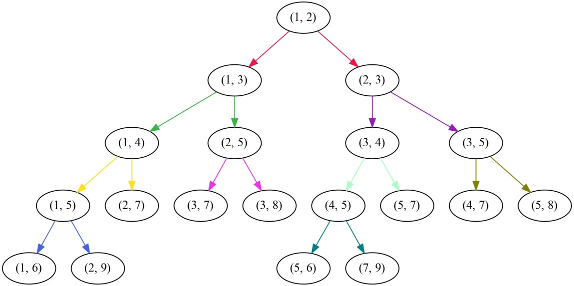

Recall that the Stern-Brocot tree is usually extended to a tree with vertices the entirety of $\hat{\mathbb{Q}}$. By slight modifications to the theory of Chesebro, Emlen, Ke, Lafontaine, McKinnie, and Rigby (2020), we may replace the recursion relation over $\mathbb{Q} \cap [0, 1]$ with a structure that allows us to compute the Farey polynomials over the entire tree: we obtain a family of recursion relations (one for each end of the tree) over $\mathbb{Z}$ which can then be solved for closed-form formulae using standard techniques.

Let $a_q(z)$ be the Farey polynomial $\Phi_{1/q}(z)$; the fractions $(1/q)_{q\in\mathbb{N}}$ form one of the geodesics which we can iterate down locally, and using the standard theory of $\mathbb{Z}$-recursions (solve the homogeneous system generally, find a single non-homogeneous solution, and add them) we get the following:

$$a_q(z) = \frac{8}{4-z} + \frac{2^{-q}z}{z-4}\left((z-2-\sqrt{z^2-4z})^q + (z-2+\sqrt{z^2-4z})^q\right)$$

We can solve every local recursion relation similarly, the only problem is knowing the three initial values at the "centre" of the geodesic.

- Is there a closed form formula for $\Phi_{p/q}(z)$ in general?
  One way of looking at this is as a local-global problem (we can compute closed form solutions along many geodesics on the tree which overlap in compatible ways; the issue is that, to find the initial values for the local closed form solutions 'far down' the tree requires knowing the solutions higher up beforehand).
- Is there an algebraic method for determining which roots of $\Phi_{p/q} + 2$ correspond to cusp points?
  Currently to do this we need to look at the asymptotic behaviour of the curves $\Phi_{p/q}^{-1}(\mathbb{R})$, which is difficult to do computationally.