# RIGID SYSTEMS OF POLES AND HINGES

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ABSTRACT. We describe relationships between the computational geometry of rigid frameworks (a subfield of real algebraic geometry) and the theory of deformation spaces of $d$-dimensional Kleinian groups.

## 1. BAR-AND-JOINT FRAMEWORKS

**Definition 1.1** ([3, Definition 4.1])**.** Let $G$ be a (simple undirected loopless) graph with finite vertex set $V$ and edge set $E$. A *bar-and-joint framework* in $\mathbb{R}^d$ for $G$ consists of *joints*, which are points $q_i \in \mathbb{R}^d$ for $i \in V$, and *bars*, which are (Euclidean) line segments $[q_i, q_j]$ for $\{i, j\} \in E$. If the canonical map $V \to \mathbb{R}^d$ is non-injective then the framework is *degenerate*.

The problem of whether or not a bar-and-joint framework is rigid is an important one for applications. There are two major kinds of rigidity (Figure 1):

- A framework $(q_i)$ is *locally rigid* if there are no differentiable families of frameworks $t \mapsto (q_i(t))_{i \in V}$ with $q_i(0) = q_i$ that preserve lengths of edges (all defined in the obvious sense), except for those induced by Euclidean isometries of $\mathbb{R}^d$.
- A framework $(q_i)$ is *rigid* if every framework for $G$ with the same edge lengths is congruent to $(q_i)$ via a Euclidean isometry.

Observe that if $(q_i)$ is rigid in either sense then every subdivision of $q_i$ (defined by adding vertices interior to edges) is also rigid.

Fix a framework $(q_i)$ on the graph $G$. Let $\widetilde{\mathbb{M}}_d$ be the space of conformal automorphisms of $\mathbb{S}^d$, and let $\mathbb{M}_d$ be the orientation-preserving half of

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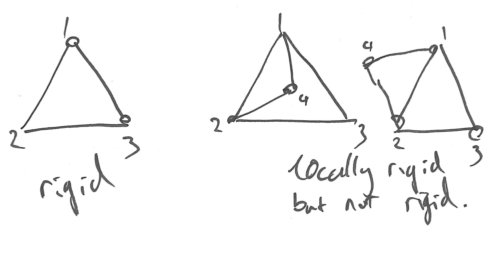



FIGURE 1

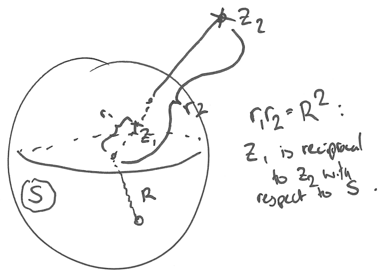



FIGURE 2

$\widetilde{\mathbb{M}}_d$. It is well-known that $\widetilde{\mathbb{M}}_d$ is generated by reflections in spheres, Figure 2 [1, II, 18.10.4 and 20.6.3]. A discrete subgroup of $\widetilde{\mathbb{M}}_d$ is called a ($d$-dimensional) *Kleinian group* [7, §I.2].

Following [5, Construction 1.2], we define a map which assigns to each bar-and-joint framework a $d$-dimensional Kleinian group.

**Construction 1.2.** We define a $d$-dimensional Kleinian group $\tilde{\Gamma}$ in the following way :

(1) If two edges intersect, add a vertex/joint at the intersection.
(2) For each vertex $q_i$, let $q_{i'}$ be the vertex closest to $q_i$ in the Euclidean metric. If $q_i$ is not joined by an edge to $q_{i'}$ then subdivide all edges of $G$ incident with $q_i$ into two, adding new joints midway between $q_i$ and its neighbouring vertices. Continue doing this until the graph stabilises.
(3) Around each vertex $q_i$ draw a sphere with radius $\frac{1}{2}\min_{j\in V\setminus\{i\}} |q_i - q_j|$.
(4) For each edge, consider the two spheres centred at its endpoints. If these spheres are not tangent, subdivide the edge at the midpoint of the subsegment of the edge not covered by the spheres, and draw a new sphere centred at this new joint which is tangent to the two existing spheres.
(5) Now let $\tilde{\Gamma}$ be the group generated by the reflections in all the spheres.

The construction admits many nontrivial deformations, and is far from optimal. Is there a method of picking circles at vertices of a bar-and-joint framework which better reflects the geometry in the situation where the bars are not of uniform length—what is the optimal way of picking interpolating circles?

## 2. Two dimensions

We recall from undergraduate complex analysis that $\mathbb{M} = \mathbb{M}_2$ is identified with $\mathsf{PSL}(2, \mathbb{C})$ and is the group of fractional linear transformations, i.e. acts on $\mathbb{S}^2 \simeq \hat{\mathbb{C}}$ like

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z := \frac{az+b}{cz+d}$$

In addition one easily sees that every orientation-reversing element $f \in \widetilde{\mathbb{M}}$ is of the form

$$f(z) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tilde{\cdot} z := \frac{a\overline{z}+b}{c\overline{z}+d}.$$

It can be explicitly decomposed into a product of Euclidean motions (i.e. products of reflections in Euclidean planes) and a single sphere reflection in some sphere $I^-(f)$ [8, §I.C.1]. This sphere has the unique property that the restriction of $f$ to this sphere is an Euclidean isometry. It is called the (repelling) *isometric circle* of $f$.

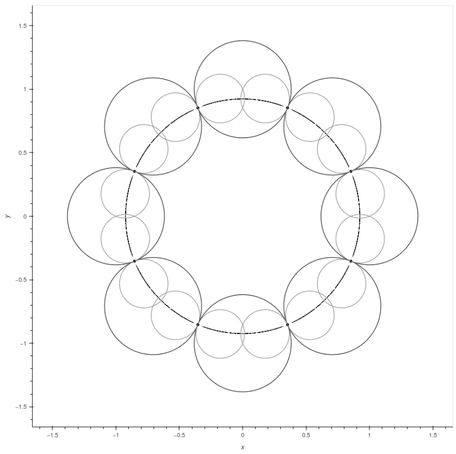
Now the group $\tilde{\Gamma}$ produced by Construction 1.2 is identified with a discrete subgroup of $\mathsf{PSL}(2, \mathbb{C})$. The orientation-preserving half $\Gamma$ of $\tilde{\Gamma}$ is generated by all products $\phi\phi'$ where $\phi$ and $\phi'$ are reflections in two tangent circles. If the framework is a topological circle, then $\Gamma$ lies on the boundary of Schottky space (it is Schottky-type); this can be seen by noting that the quotient $\Omega(\Gamma)/\Gamma$ is a handlebody-with-rank-one-cusps using the Poincaré polyhedron theorem.

**Example 2.1** ([5, Example 1.3])**.** In Figure 3, we show (in black) the defining circles $C_i$ for four different necklace groups; the points chosen in each case are respectively:

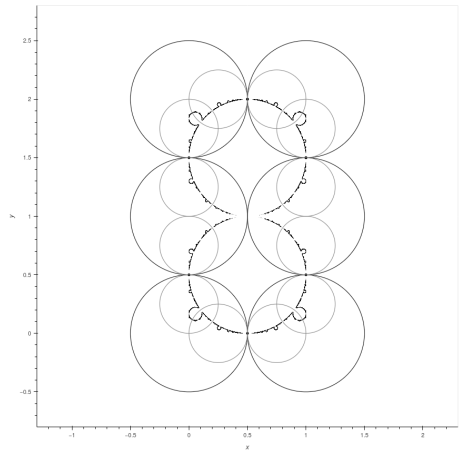
(A) The 8th roots of unity;
(B) $\{0, 1, 1+1i, 1+2i, 2i, 1i\}$;
(C) $\{0, 1, 2, 1.5+3i, 4i, 5i, 6i-1, 4i-3, 2i-2\}$; and
(D) $\{0, 1+1i, 1+\sqrt{2}+1i, 2+\sqrt{2}, 1+\sqrt{2}-1i, 1-1i, 0, -1+1i, -1-\sqrt{2}+1i, -2-\sqrt{2}, -1-\sqrt{2}-1i, -1-1i\}$ (compare [8, §VIII.F.5]).

The light gray circles are the isometric circles of the chosen generators for the orientation-preserving half. At each point of tangency of the black reflection circles there is also a tangent pair of isometric circles, and the product of the reflections in the two black circles maps the interior of one grey circle onto the exterior of the other grey circle. By taking sufficiently small vertex circles and sufficiently small interpolating circles in the construction, one can obtain groups whose limit sets are arbitrarily close to the starting polygon.

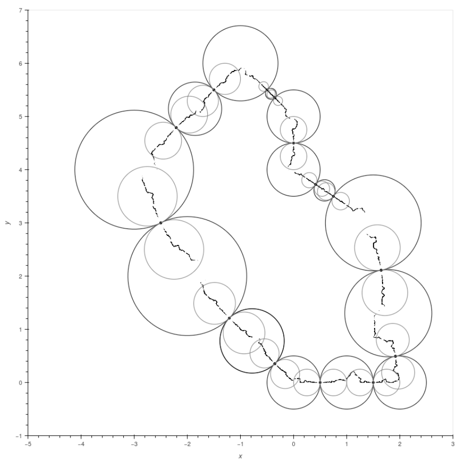
**Example 2.2.** We take a regular hexagon and compute a group on the boundary of genus 6 Schottky space (subset of $X(F_6)$), Figure 4. Hexagons tile the plane, and if we generalise the notion of a bar-and-joint framework to discrete embeddings of graphs in $\mathbb{R}^d$ then we obtain groups with less generators and a simpler character variety, $X((\mathbb{Z} \oplus \mathbb{Z}) * \mathbb{Z})$ [5, Example 2.1]. The point is that we can pack space with circles by taking translates of the discs at the hexagon vertex; the group generated by the



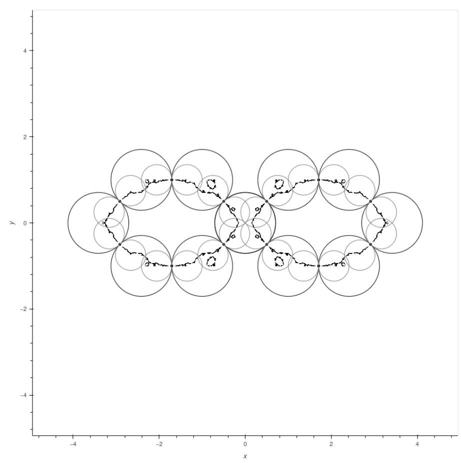
(A) Roots of unity



(B) Rectangle



(C) A complicated polygon



(D) Figure 8

FIGURE 3. Four necklace groups.

reflections in all these circles is not finitely generated (the corresponding orbifold is not geometrically finite), but if we extend it by adding the translation group arising from the rank 2 lattice of hexagon centres then this extended group is finitely generated. See Figure 5 for a fundamental domain that shows how this additional rank two subgroup glues up the genus 6 surface: the two 6-punctured spheres are glued onto each other inside-out and then there are additional foldings of this glued surface to produce a 2-punctured torus.

Recall that the deformation spaces of Schottky groups and Schottky-type groups are very complex. A subgroup of $\widetilde{\mathbb{M}}$ is discrete if and only if its orientation-preserving half is discrete. Therefore one would expect the deformation spaces of groups arising from these configuration spaces to be quite complex. But this is not the case. Smooth motions of the bar-and-joint framework give smooth deformations of the generators:

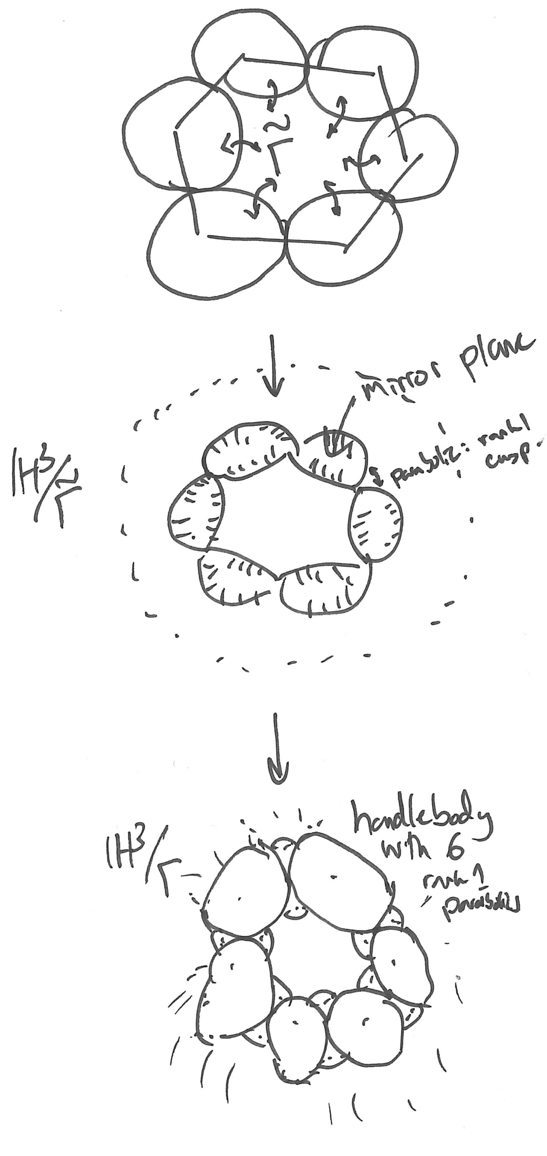



FIGURE 4

keep the same circle radii (this is OK since the edge lengths are fixed), hence a smooth parameterisation of the group in its character variety.

**Theorem 2.3.** *The subset of quasiconformal deformation space parameterised by a (general 2D) bar-and-joint framework is a real semi-algebraic set.*

*Proof.* The groups fail to be discrete when (new) parabolics or elliptics form (not quite true, but the boundary is dense with such points, and if these points lie in a semialgebraic set then the whole boundary must since it is closed). This occurs whenever non-intersecting circles

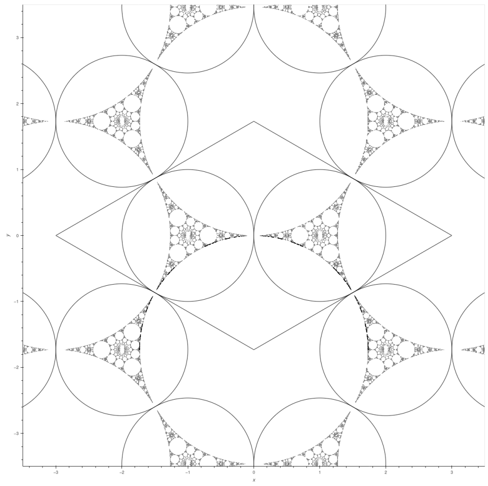


FIGURE 5

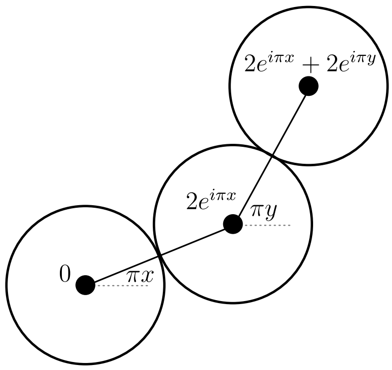



FIGURE 6. Parametrising the phase space of the double pendulum.

intersect (fixed points of the product of the two circle reflections are the intersection pts). ☐

**Example 2.4** ([5, Example 3.9]). Consider two rigid rods of length 2, connected to form a double pendulum. Fixing one of the ends at 0, the configuration space is parameterised by two angles $\pi x$ and $\pi y$ and we may draw circles of radius 1 centred at the three vertices (Figure 6). A generating set of the orientation-preserving half (before normalising to determinant 1) is

$$M_1 = \begin{bmatrix} -e^{i\pi x} & 2e^{2i\pi x} \\ -2 & 3e^{i\pi x} \end{bmatrix}, \ M_2 = \begin{bmatrix} -4e^{i\pi x} - e^{i\pi y} & 8e^{2i\pi x} + 8e^{i\pi(x+y)} + 2e^{2i\pi y} \\ -2 & 4e^{i\pi x} + 3e^{i\pi y} \end{bmatrix}, \text{ and}$$

$$M_3 = \begin{bmatrix} -7e^{i\pi(x+y)} - 4e^{2i\pi y} - 4e^{2i\pi x} & 2e^{i\pi(2x+y)} + 2e^{i\pi(x+2y)} \\ -2e^{i\pi x} - 2e^{i\pi y} & e^{i\pi(x+y)} \end{bmatrix}.$$

If we parameterise this by the complex plane $t = x + iy$ (with $0 \leq x \leq 2$ and $0 \leq y \leq 2$) we may visualise the entire phase space of the double

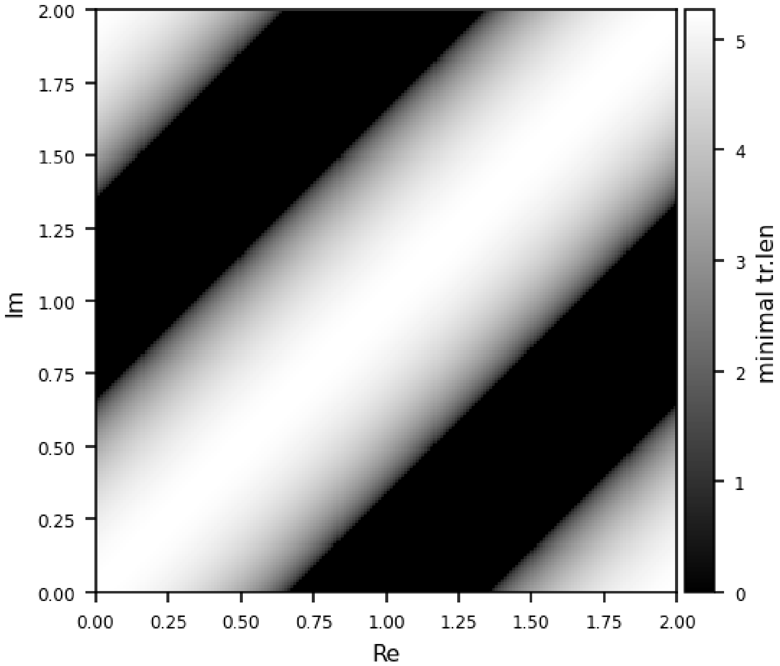



FIGURE 7. A linear slice through $X(F_3)$ which arises from the configuration space of the double pendulum.

pendulum, Figure 7. The groups $\langle M_1, M_2, M_3\rangle$ generically lie in the boundary of genus 3 Schottky space, and the lines in Figure 7 which appear straight are actually straight (not fractal), corresponding to the condition that the three circles become mutually tangent in a triangle.

If a framework is rigid, the image of any map of this form is a single point. Consider the possible deformations of the group in (larger) deformation space; they correspond to growing and shrinking the reflecting circles while keeping them tangent. In other words, it corresponds to changing the distance between the mirrors in $\mathbb{H}^3/\tilde{\Gamma}$.

This suggests an interpretation of the deformation spaces we have constructed: they correspond to spaces of $\mathbb{H}^3$-orbifolds with mirror planes, with constraints on the 'mirror diameter' of the 3-fold (minimal distance between mirror planes).

Here is another application. The deformation space of a geometrically finite Kleinian group is the quotient of a product of Teichmüller spaces (this is a version of Marden–Tukia rigidity, see [10, Theorems 5.26 and 5.27]), i.e. it is connected. However, parameterisations are not usually connected: e.g. the map $\mathcal{R} \to \mathrm{QH}(\Gamma)$ where $\Gamma$ is a Riley group is a double-cover (the covering map comes from a Dehn twist around the compression disc) [6]. Our maps go from configuration spaces to these parameter spaces. There exist algorithms for determining whether two bar-and-joint frameworks with the same edge lengths are deformable into each other (e.g. the two 'locally rigid but not rigid' frameworks in

Figure 1 are not deformable into each other in $\mathbb{R}^2$, only $\mathbb{R}^3$) [4, Chapter 5]. These can be modified to provide constructive connectedness proofs for our parameterisations of subsets of quasiconformal deformation spaces.

## 3. Higher dimensions

We also get quantifiable maps into quasiconformal deformation spaces of higher-dimensional Kleinian groups. Here is one fun example [8, §VIII.F.4]:

**Example 3.1.** Take a string of beads. Lift it into $\mathbb{S}^3$, cut it open at a point of tangency, and reglue it. The limit set of the result is a topological circle embedded in $\mathbb{S}^3$ as a wild knot.

We already saw that we get the best results when our linkages have high levels of symmetry in their lengths, e.g. are polygons with all equal lengths. In higher dimensions there are many nice symmetric objects, e.g. the Leech lattice $\Lambda_{24}$ [2]. From these objects we can define highly structured subsets of deformation spaces similar to the lattice-invariant groups of [5, §2]. Euclidean groups generated by reflections in planes arising from the Leech lattice are well-studied, see e.g. [9] and references therein, but the groups generated by the reflections in the sphere packing which it determines do not seem to be so popular.

**Question 3.2.** Define a group $G$ which is generated by the reflections in the Leech sphere packing (Figure 8). There are a couple of natural geometrically finite extensions analogous to the construction we carried out before:

- Adjoin the Leech lattice itself (i.e. a $\mathbb{Z}^{\oplus 24}$). This is the tightest extension, in some sense: it is the Euclidean extension with the shortest translation lengths, so the smallest/least complex conformal 24-fold at infinity.
- Adjoin the translations in $\operatorname{Aut}\Lambda_{24}$. This is a direct generalisation of Example 2.2.
- Adjoin the entire automorphism group $\operatorname{Aut}\Lambda_{24}$.

These groups are interesting since the Leech lattice is the tightest sphere packing in $\mathbb{R}^{24}$, so (just like the group generated by the triangular packing in $\mathbb{R}^2$ to) the corresponding group has conformal quotient $\Omega/G$ of minimal Euclidean volume. What are the hyperbolic 25-orbifolds uniformised by the extensions of $G$? Does they have similar extremal properties, e.g. minimal hyperbolic convex core volume? Unlike in Example 2.2 there are conformal surfaces which have varying 'types' (parameterised by the so-called 'deep holes' of the Leech lattice)—how do these show up in the hyperbolic structure?

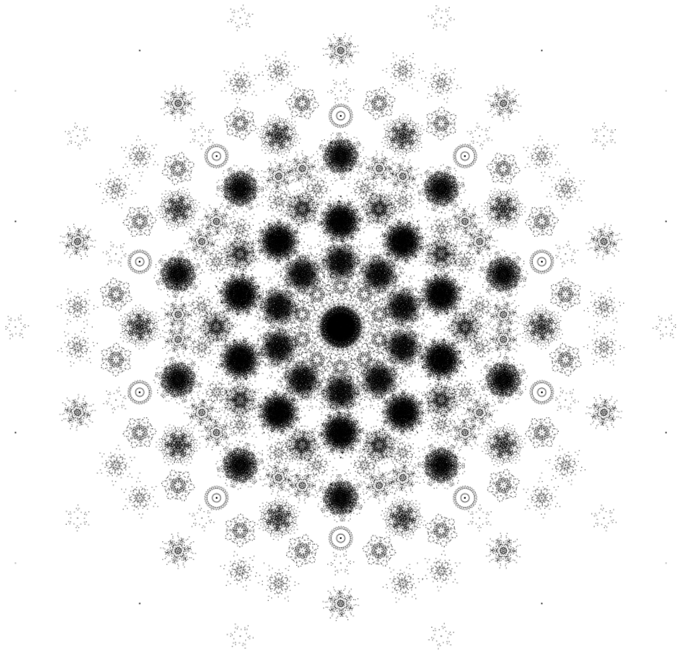


FIGURE 8. Projection to 2D of the 196,560 vertices of the Leech polytope (convex hull of shortest vectors in $\Lambda_{24}$). Gro-Tsen, answer to Examples of unexpected mathematical images, https://mathoverflow.net/a/338245/150082

## 4. Notes

(1) If we arbitrarily subdivide the framework, the subset of quasiconformal deformation space which is parameterised gets closer and closer to being parameterised by the entire deformation space of the rigid framwork but you never get a bijection (both kernel and cokernel are nontrivial). Is it possible to quantify this in some way (suspect not).
(2) Our construction above is 'greedy'. Investigate minimal numbers of circles needed to construct groups from configurations. These give smaller but more "canonical"/"natural" subsets of deformation space.
(3) Sharpness of these sets—'extremal' configurations of the framework must always give cusp points, if the cusp hasn't already been hit earlier in the deformation. Can we always choose a subdivision of edges of the original framework so as to be able to 'hit' arbitrary cusps sharply?
(4) Maybe a better way of putting (1)–(3): determine the optimal subdivision algorithm for a fixed starting configuration to get

a maximally large image in some quasiconformal deformation space.

(5) Combinatorial description based on $G$ of the cusps which can/-can't occur. In the case of Schottky-type groups (those arising from generalised polygons), we get knotted cusp groups

(6) Carefully write down the relationship between configuration rigidity and group rigidity—the above algorithm does not give rigid groups, only single-point images of the defined map. Can this be fixed?

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