HECKOID GROUPS

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ABSTRACT. This is based on [EMS24].

1. Algebraic groups

A familiar concept:

Definition 1.1. A *Lie group* is a smooth manifold G equipped with C^∞ maps $\mu : G \times G \to G$, $e : \{1\} \to G$, and $\iota : G \to G$ satisfying the usual group axioms.

Important groups are the classical matrix groups: $\mathsf{SL}(2, \mathbb{C})$, $\mathsf{PSL}(4, \mathbb{R})$, $U(67)$, $\mathsf{Sp}(238, \mathbb{C})$ etc etc. But we can define these groups over other rings/fields easily, where we can no longer apply Lie theory. How do we make sense e.g. of $\mathsf{PSL}(3048, \mathbb{Z})$?

"Definition" 1.2. A *scheme* is a topological space with an atlas of local charts; each chart is the intersection of the zerosets of some polynomials in a several variables with coefficients from a commutative ring R.

The actual definition is a little more complicated, because we want to allow different rings on each chart, we don't want to prioritise any particular coordinate system (so everything should be defined only up to rational changes of coordinates with controlled singularities), and we also have to deal with the fact that R does not have nice geometric topology (the most classical topology is the Zariski topology[1], which has massive open sets—one manifestation of this is that we don't have classical partitions of unity which are the most basic tool in differential geometry, so we need to be careful when patching things together). A classic treatment of the theory (but quite outdated in some aspects) is Mumford's Red Book [Mum99]. A more modern treatment is [Vak24].

Example 1.3. $\mathsf{SL}(2, \mathbb{Z})$ is the set of points $(a, b, c, d) \in \mathbb{Z}^4$ satisyfing the equation $ad - bc - 1 = 0$; since it only consists of one piece, it is an *affine scheme*.

The example $\mathsf{SL}(2, \mathbb{Z})$ is a very simple scheme: it is still reduced and irreducible, so we can think of it as 'variety over $\mathbb{Z}$'. This eventually

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[1]Definition: the closed sets of $\operatorname{Spec} R$ are the sets of the form $\mathbf{Z}(\mathfrak{a})$ for $\mathfrak{a} \trianglelefteq R$, where $\mathbf{Z}(\mathfrak{a})$ is the set of prime ideals $\mathfrak{p} \supseteq \mathfrak{a}$.

runs into psychological problems, but this line of thinking leads us to note that we can also define $\mathsf{SL}(2,\mathbb{Z}) = \mathsf{SL}(2,\mathbb{C}) \cap \mathbb{Z}^4$. The point is if something is defined in terms of equations over a ring R, and we have a map $R \to S$ for some S, then we can ask for solutions of the equations in S^n and not just R^n. Actually, knowing the relations between these ring-valued points is enough to recover all the algebraic data again: by Yoneda's Lemma, we can view schemes as (contravariant) functors $X : \mathsf{Ring} \to \mathsf{Set}$, where a ring R is sent to the set $X(R)$ of solutions of a bunch of polynomial systems in the ring R [Mum99, §II.6].

"Definition" 1.4. An *algebraic group* over a ring R is a scheme X/R together with morphisms $\mu : X \times_{\mathrm{Spec}(R)} X \to X$, $e : \mathrm{Spec}(R) \to X$, and $\iota : X \to X$ which satisfy the usual group axioms when restricted to $X(R)$. An *arithmetic group* is $X(\mathbb{Z})$ where X is an algebraic group (all schemes are defined over $\mathbb{Z}$ since $\mathbb{Z}$ is initial in Ring). A *thin group* is an infinite-index subgroup of the integer points of its Zariski closure.

Remark. If R is a field, then $\mathrm{Spec}(R)$ is just a singleton corresponding to an abstract point with coordinates in R. If R is not a field, then things are weird; in particular $\mathrm{Spec}\,\mathbb{Z}$ is not a singleton, it is a 1D scheme with closed points corresponding to the primes $p \in \mathbb{Z}$ and a single open point (with closure $\mathrm{Spec}\,\mathbb{Z}$) corresponding to the zero ideal. Note that X as a set is not a group, only the $X(R)$ points for fixed R is a group. For the full formal definition see [Stacks, Tag 022S]. Of course the classical reference is SGA III.

Remark. We are not quite giving the full definition of an arithmetic group either. See [Kon+18; Wit01].

Intuition: an algebraic group is a Lie group intersect with $\mathbb{Z}^n$ (this is very wrong, though); i.e. it is a 'lattice'. A thin group is a subgroup of an arithmetic group which is infinite covolume but still has the 'same rank' as a set of discrete points—see Examples 1–3 of [Kon+18].

2. Arithmetic Kleinian groups

Actually working with the definition of arithmetic groups is hard since it is so abstract. We have a good concrete characterisation if we are living in $\mathsf{PSL}(2,\mathbb{C})$.

Theorem 2.1 ([MR03, Thm 8.2.2 and Thm 10.3.7])**.** *A Kleinian group $\Gamma \leq \mathsf{PSL}(2,\mathbb{C})$ is arithmetic (resp. thin) iff it is finite (resp. infinite) covolume,*

$$k\Gamma^{(2)} = \mathbb{Q}(\{\mathrm{tr}^2\, g : g \in \Gamma\})$$

has one field embedding into $\mathbb{C}$, and for all field embeddings $\rho : k\Gamma^{(2)} \to \mathbb{R}$ the algebra

$$A_0\Gamma^{(2)} = \{\textstyle\sum \rho(a_i)\gamma_i : a_i \in k\Gamma^{(2)}, \gamma_i \in \Gamma\}$$

is isomorphic to the Hamiltonian quaternions. □

Theorem 2.2 (Maclachlan and Martin, 1999 [MM99])**.** *There are only finitely many conjugacy classes of arithmetic or thin Kleinian groups generated by two parabolic or elliptic elements.* □

3. Parabolic case

A 2001 conjecture of Agol proved in 2020 by Aimi/Lee/Sakui/Saguma [Aim+20] and Akiyoshi/Ohshika/Parker/Sakuma/Yoshida [Aki+21] classified all Kleinian groups generated by two parabolics, $\langle X, Y\rangle$ which are not Fuchsian or elementary:

Theorem 3.1. *If X and Y are parabolic, and $G = \langle X, Y\rangle$ is Kleinian and non-Fuchsian, then G falls into one of the following mutually exclusive categories:*

(1) *Split as a free product $\langle X\rangle * \langle Y\rangle$:*
   (a) *Groups in $\overline{\mathcal{R}}$; i.e. torsion-free groups with comanifold interior homeomorphic to a 3-ball with two drilled ideal arcs. We categorise these further as cusp groups $\langle X, Y : W^\infty = 1\rangle$, non-cusp groups on the boundary of the Riley slice, (which are geometrically infinite), and Riley groups.*

(2) *Don't split:*
   (a) *Heckoid groups: $\langle X, Y : W^n = 1\rangle$ for $n > 1$*
   (b) *2-bridge link complements: $\langle X, Y : W = 1\rangle$*
   (c) *Manifolds representing quotients of the manifolds in the previous classes by involutions preserving unknotting tunnels and similar.*

These groups are all geometrically finite except for the non-cusp groups (this follows by the density theorem, since all components of geometrically finite groups accumulate at $\overline{\mathcal{R}}$.)

Note that the study of all these groups was initiated by Riley in the 1970s (pre-Thurston) see e.g. [Ril72; Ril75a; Ril75b; Ril92].

Theorem 3.2. *Out of the Kleinian groups generated by 2 parabolics:*

- *(Gehring-Maclachlan-Martin, 1998 [GMM98]): exactly 4 are arithmetic (the fig 8 knot and 3 links of 4,5,6 crossings).*
- *(Elzenaar-Martin-Schillewaert, 2024 [EMS24]): exactly 3 are thin, 2 Hopf link Heckoid groups and 1 quotient.*

Proof. Enumerate all quadratic integers in $\mathbb{C} \setminus \mathcal{R}$ and check whether the corresponding group lies in one of the known discrete families. □

Remark. The general case of groups on 2 elliptics: the classification of possible types has been carried out by Chesebro-Martin-Schillewaert (to appear). There are approx. 150 thin groups (Elzenaar-Martin-Schillewaert, to appear)—essentially the proof proceeds by (i) bounding the degree of the algebraic integers which can occur, and then (ii) enumerating all of the finitely many of these in some large disc and applying [Elz+24; EMS; EMS23].

Conjecture 3.3. *A classification of all Kleinian groups $\langle X, Y\rangle$ which are non-Fuchsian and which are quasiconformally rigid can be obtained by replacing, in Theorem 3.1 case (2), '2-bridge link complement' with 'geometrically finite hyperbolic 3-manifold with no conformal boundary and one or two rank 2 cusps that admits an unknotting tunnel'. Note that such manifolds do not necessarily embed into $\mathbb{S}^3$. For further discussion see [Elz25].*

If one restricts the Conjecture to groups uniformising 3-orbifolds embeddable in $\mathbb{S}^3$, then this corresponds to restricting to tunnel number one links. From [AR96; Kuh96; MS91; MSY96] we obtain a classification of all tunnel number one links in $\mathbb{S}^3$ including a study of involutions. From [CM09] we obtain a correspondence between these links and representations of genus two Schottky groups obtained by gluing 2-handles. One can then attempt to modify [Aim+20; Aki+21] (perhaps starting with [LS12; LS13; ORS08] which generalise nicely to the setting of [CM09]) to obtain a proof of the special case. In general it is not clear how to proceed as the various results that go into the two-bridge theorem strongly depend on embedding the knot into $\mathbb{S}^3$. The ambient manifold is, instead, obtained by a 'genus two Dehn filling' of $\mathbb{S}^3$ (i.e. a manifold with Heegaard genus 2).

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