The Farey polynomials

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The surfaces we are interested in are spheres with four marked points, together with a curve on the sphere dividing it up into two halves, each half containing two marked points, with the marked points joined up in pairs (one point in each pair in each half), such that the joined points are of the same type (either both punctures, or both cone points with the same angle). We only consider these curves modulo Dehn twists along the curve γ_∞.

Algebraically these surfaces correspond to quotients $\Omega(\Gamma_z^{a,b})/\Gamma_z^{a,b}$ where

$$\Gamma_z^{a,b} := \left\langle X := \begin{bmatrix} e^{2\pi i/a} & 1 \\ 0 & e^{-2\pi i/a} \end{bmatrix}, Y := \begin{bmatrix} e^{2\pi i/b} & 0 \\ z & e^{-2\pi i/b} \end{bmatrix} \right\rangle.$$

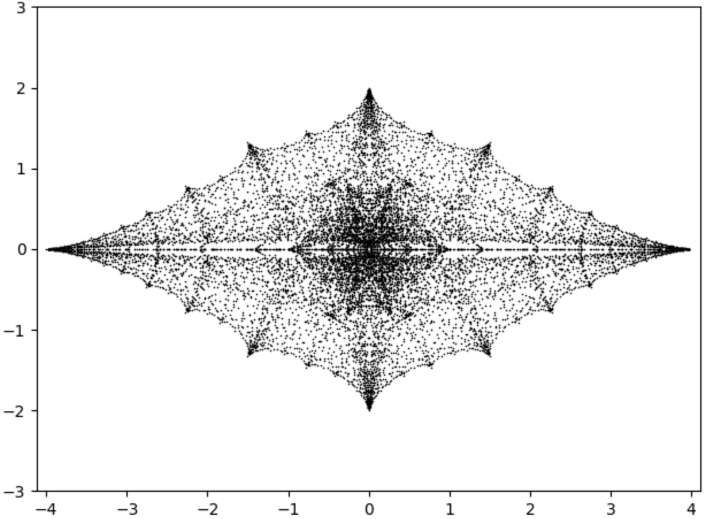
where $\rho \in \mathbb{C}$, and $a, b \in \mathbb{N} \cup \{\infty\}$ (by convention, $\exp(2\pi i/\infty) = 1$).

Here, $\Omega(\Gamma_z^{a,b})$ is the maximal subset of $\hat{\mathbb{C}}$ on which $\Gamma_z^{a,b}$ acts freely discontinuously.

In this talk we will write Γ_z for $\Gamma_z^{a,b}$ to save space.

Every point in the Riley slice corresponds to a 4-marked sphere; we are interested in studying the boundary of the slice, which corresponds to degenerate deformations of the sphere. More precisely, given any of the spheres we take the curve on the sphere and shrink it to a point.

1. List all of the curves which we can shrink down to a point;
2. Study the path along which this shrinking occurs in the Riley slice.


3
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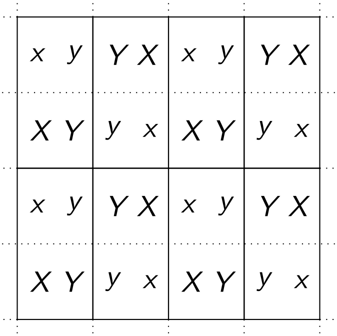
4

The curve enumeration theory comes from two main sources:

1. the paper Birman and Series (1987), and
2. section 2 of Keen and Series (1994) (esp. Proposition 2.2).

It is essentially a version of Dehn's algorithm (for which see chapter 6 of Stillwell (1983)): you can find explicit representatives in $\pi_1(S) = \langle g_1, g_2, ..., g_n \rangle$ for a curve γ on S by drawing a fundamental polygon for $\pi_1(S)$ and writing down the group elements corresponding to the sides crossed by the lift of γ in order.

Here is the relevant polygon tiling:



Consider the marked tiling of $\mathbb{R}^2$ from the previous slide, and let $p/q \in [0,1] \cap \mathbb{Q}$ be in lowest terms. Let $L_{p/q}$ be the line through $(0,0)$ of slope p/q.

Now the **Farey word of slope** p/q, $W_{p/q}$, is defined as follows:

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5. If you are sitting at $(2q, 2p)$ then terminate. Otherwise go back to step 2.

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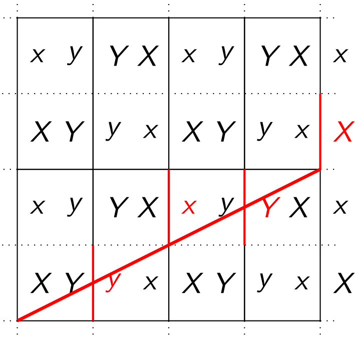
Warning. Our convention at step 4 is *opposite* to that of Keen and Series (1994). This doesn't affect the theory.



This shows that $W_{1/2} = yxYX$.

Lemma

Let $W_{p/q}$ be a Farey word; then the word consisting of the first $2q-1$ letters of $W_{p/q}$ is conjugate to X or Y according to whether the qth letter of $W_{p/q}$ is $X^{\pm 1}$ or $Y^{\pm 1}$ (i.e. according to whether q is even or odd respectively)

Proof.

This identity comes from considering the rotational symmetry of the line of slope p/q about the point (q,p); it is clear from the symmetry of the picture that the first $p-1$ letters of $W_{p/q}$ are obtained from the $(p+1)$th to $(2p-1)$th letters by reversing the order and swapping the case (imagine rotating the line by 180 degrees onto itself and observe the motion of the labelling). $\square$

Theorem (Keen–Series)

- *If* $p/q \in \mathbb{Q}$ *then* $W_{p/q} = W_{p/q+2n}$ *for all n; so all the Farey words are found in* $[0, 2)$*.*
- *There is a bijection between* $[0, 2) \cap \mathbb{Q}$ *and the set of simple closed curves on* S_4 *not homotopic to a puncture modulo* γ_∞*.* $\gamma_{p/q}$ *is represented in* $\Gamma_\rho^{p,q}$ *by the word* $W_{p/q}$*.* $\square$

Now by the standard theory of Kleinian groups,[1] if $\operatorname{tr} W_{p/q} \in \mathbb{R}_{\leq -2}$ then the length ℓ of the curve $\gamma_{p/q}$ that $W_{p/q}$ represents has the property

$$\cosh(\ell/2) = |\operatorname{tr} \gamma/2|;$$

since $\cosh(0) = 1$ we see that in order to shrink ℓ to zero (thus pinching $\gamma_{p/q}$ to a point) we need to send $\operatorname{tr} W_{p/q}$ from $-\infty$ down to -2 along $\mathbb{R}$.

In reality everything is more nuanced and checking that all this actually makes sense is the point of the Keen–Series theory.

[1] see e.g. Maclauchlan and Reid (2003) section 12.1

We now study the combinatorial properties of the Farey words and their traces, independent of the geometric motivation. For convenience, set $\alpha = 2\pi i/a$ and $\beta = 2\pi i/b$ and replace ρ with z so each Farey word $W_{p/q}$ is a word in

$$\left\langle X = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{bmatrix}, Y = \begin{bmatrix} \beta & 0 \\ z & \beta^{-1} \end{bmatrix} \right\rangle;$$

we define the **Farey polynomial of slope** p/q to be

$$\Phi_{p/q} := \operatorname{tr} W_{p/q}$$

which is a monic polynomial in z, of degree q, with coefficients rational functions of α and β.

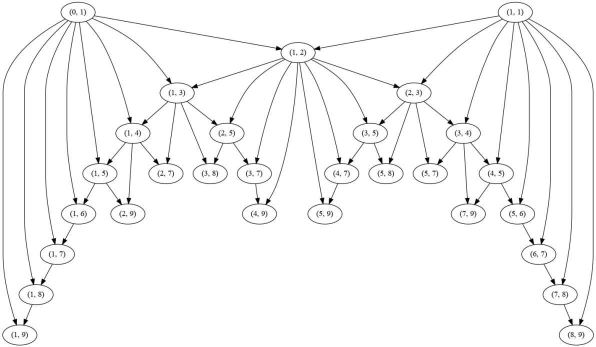
Table: Farey polynomials of slope p/q for small q.

p	q	
0	1	$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}-z$
1	1	$\alpha\beta+\frac{1}{\alpha\beta}+z$
1	2	$2+\left(\alpha\beta-\frac{\alpha}{\beta}-\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}\right)z+z^2$
1	3	$\frac{1}{\alpha\beta}+\alpha\beta+\left(3-\frac{1}{\alpha^2}-\alpha^2-\frac{1}{\beta^2}-\beta^2+\frac{\alpha^2}{\beta^2}+\frac{\beta^2}{\alpha^2}\right)z+\left(\alpha\beta-2\frac{\alpha}{\beta}-2\frac{\beta}{\alpha}+\frac{1}{\alpha\beta}\right)z^2+z^3$
2	3	$\frac{1}{\alpha\beta}+\alpha\beta+\left(-3+\frac{1}{\alpha^2}+\alpha^2+\frac{1}{\beta^2}+\beta^2-\alpha^2\beta^2-\frac{1}{\alpha^2\beta^2}\right)z+\left(-2\alpha\beta+\frac{\alpha}{\beta}+\frac{\beta}{\alpha}-\frac{1}{\alpha\beta}\right)z^2+z^3$

The Farey polynomials are difficult to compute: $\Phi_{p/q}$ is the trace of a product of $2q$ matrices. Thus computing the Farey polynomials is computationally difficult. The goal of this talk is to give an iterative process for computing Farey polynomials which does not depend on computing matrix products.

We say that p/q and r/s are **Farey neighbours** if $ps - qr = \pm 1$ (so if $p/q < r/s$ then $ps - qr = 1$); if p/q and r/s are such then write $p/q \oplus r/s$ for the mediant $(p+r)/(q+s)$. It will be convenient also to have the notation $p/q \ominus r/s$ for the fraction $(p-r)/(q-s)$; we shall only use this when it is known that $(p-r)/(q-s)$ and r/s are Farey neighbours (it is easy to check that if p/q and r/s are neighbours then so are $(p-r)/(q-s)$ and r/s) and that $q - s \neq 0$.

Define a digraph $\mathcal{F}$ with vertices $\mathbb{Q} \cap [0,1]$ such that there is an edge $p/q \to r/s$ if $r/s = p/q \oplus p'/q'$ for some p'/q' a Farey neighbour of p/q.



One might guess, e.g. by analogy with the Maskit slice Mumford, Series, and Wright (2002, p. 277), that $W_{p/q}W_{r/s} = W_{p/q\oplus r/s}$. Let us check:

- $W_{1/2} = yxYX$, $W_{1/1} = YX$, $W_{1/2}W_{1/1} = yxYXYX$, and $W_{1/2\oplus 1/1} = yxyXYX$.
- $W_{1/3} = yXYxYX$, $W_{2/5} = yXYxyXyxYX$, $W_{1/3}W_{2/5} = yXYxYXyXYxyXyxYX$, and $W_{1/3\oplus 2/5} = W_{3/8} = yXYxYXyxYxyXyxYX$.

So our guess is almost correct; the corrected statement is:

Lemma

Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then $W_{p/q \oplus r/s}$ is the word $W_{p/q}W_{r/s}$ with the sign of the $(q+s)$th exponent swapped. $\square$

Our next goal is to translate this lemma on products of Farey words to a result on products of Farey *polynomials*. We do this with trace identities. In particular, we apply the symmetries of Farey words and the standard result $\operatorname{tr} AB = \operatorname{tr} A \operatorname{tr} B - \operatorname{tr} Ab$ [2] to get two such results.

[2] see section 3.4 of Maclauchlan and Reid (2003)

Lemma

Let p/q and r/s be Farey neighbours with $p/q < r/s$. Then the following trace identities hold:

$$\operatorname{tr} W_{p/q}W_{r/s} + \operatorname{tr} W_{p/q \oplus r/s} = \begin{cases} 2 + \alpha^2 + \frac{1}{\alpha^2} & \text{if } q+s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q+s \text{ is odd.} \end{cases}$$

and

$$\operatorname{tr} W_{p/q}W_{r/s}^{-1} + \operatorname{tr} W_{p/q \ominus r/s} = \begin{cases} 2 + \beta^2 + \frac{1}{\beta^2} & \text{if } q-s \text{ is even,} \\ \alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta} & \text{if } q-s \text{ is odd.} \end{cases}$$

□

Our main theorem is then:

Theorem

Let p/q and r/s be Farey neighbours. If $q+s$ is even, then

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 4 + \frac{1}{\alpha^2} + \alpha^2 + \frac{1}{\beta^2} + \beta^2.$$

Otherwise if $q+s$ is odd, then

$$\Phi_{p/q}\Phi_{r/s} + \Phi_{p/q\oplus r/s} + \Phi_{p/q\ominus r/s} = 2\left(\alpha\beta + \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right).$$

Proof.

Suppose $q+s$ is even; then $q-s$ is also even, so

$$\begin{aligned}
&\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} \\
&\quad = \operatorname{tr} W_{p/q} \operatorname{tr} W_{r/s} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\quad = \operatorname{tr} W_{p/q} W_{r/s} + \operatorname{tr} W_{p/q} W_{r/s}^{-1} + \operatorname{tr} W_{p/q\oplus r/s} + \operatorname{tr} W_{p/q\ominus r/s} \\
&\quad = 2 + \alpha^2 + \frac{1}{\alpha^2} + 2 + \beta^2 + \frac{1}{\beta^2}
\end{aligned}$$

where in the final step we used the two trace identities. The odd case is very similar. □

Remark. When $\alpha = \beta = 1$ (i.e. in the parabolic case), the two cases actually unify and independent of the parity of $q+s$ we have

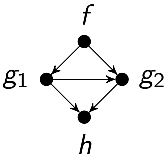
$$\Phi_{p/q}\Phi_{r/s} + \Phi_{(p+r)/(q+s)} + \Phi_{(p-r)/(q-s)} = 8.$$

Our formulae, combined with the starting triple

$$\Phi_{0/1}(z) = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} - z$$
$$\Phi_{1/1}(z) = \alpha\beta + \frac{1}{\alpha\beta} + z$$
$$\Phi_{1/2}(z) = 2 + \left(\alpha\beta - \frac{\alpha}{\beta} - \frac{\beta}{\alpha} + \frac{1}{\alpha\beta}\right) z + z^2$$

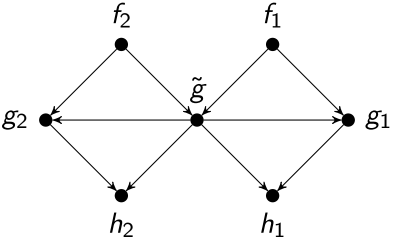
allow us to compute every Farey polynomial recursively down the Farey graph.

The formulae give invariants of certain substructures in the Farey graph: for every directed subgraph of the form



the sum $f + g_1 g_2 + h$ is dependent only on the parity of the denominator of the fraction corresponding to h.

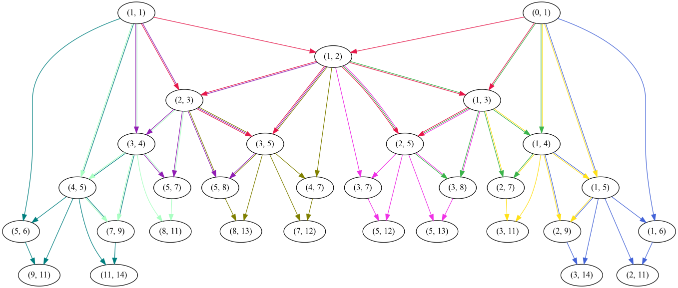
In $\mathcal{F}$, we may take parity pairs of these diamonds to form a 'butterfly' subgraph with the following combinatorial structure:



Labelling the Farey polynomials of the structure according to the scheme above and summing the two identities, we get

$$f_1 + f_2 + g_1\tilde{g} + g_2\tilde{g} + h_1 + h_2 = (\operatorname{tr} X + \operatorname{tr} Y_\rho)^2.$$

Consider again the Farey graph, now coloured according to the butterfly substructures, and delete all except the edges which form the two 'horizontal' edges of the butterfly substructures. The resulting tree is familiar.



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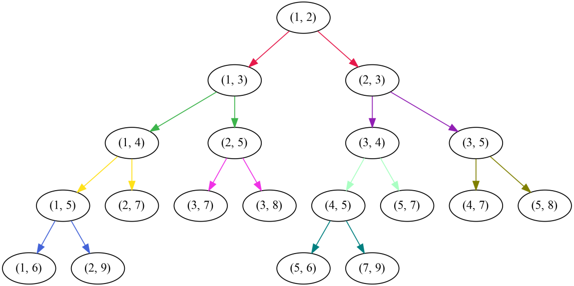


Recall that the Stern-Brocot tree is usually extended to a tree with vertices the entirety of $\hat{\mathbb{Q}}$. By slight modifications to the theory of Chesebro, Emlen, Ke, Lafontaine, McKinnie, and Rigby (2020), we may replace the recursion relation over $\mathbb{Q} \cap [0, 1]$ with a structure that allows us to compute the Farey polynomials over the entire tree: we obtain a family of recursion relations (one for each end of the tree) over $\mathbb{Z}$ which can then be solved for closed-form formulae using standard techniques.

This only works in the parabolic case, since in the (semi-)elliptic case we have a system of two recursion relations which we need to use in turn.

Let $a_q(z)$ be the Farey polynomial $\Phi_{1/q}(z)$; the fractions $(1/q)_{q\in\mathbb{N}}$ form one of the geodesics which we can iterate down locally, and using the standard theory of $\mathbb{Z}$-recursions (solve the homogeneous system generally, find a single non-homogeneous solution, and add them) we get the following:

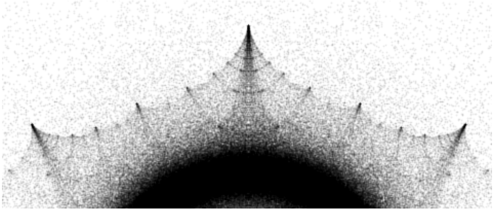
$$a_q(z) = \frac{8}{4-z} + \frac{2^{-q}z}{z-4}\left((z-2-\sqrt{z^2-4z})^q + (z-2+\sqrt{z^2-4z})^q\right)$$

We can solve every local recursion relation similarly, the only problem is knowing the three initial values at the "centre" of the geodesic.

- In the parabolic case, is there a closed form formula for $\Phi_{p/q}(z)$ globally (i.e. on the whole graph)?
- Is there a way to find even local closed forms in the (semi)-elliptic case?
- Is there an algebraic method for determining which roots of $\Phi_{p/q} + 2$ correspond to cusp points?
  Currently to do this we need to look at the asymptotic behaviour of the curves $\Phi_{p/q}^{-1}(\mathbb{R})$, which is difficult to do computationally.

We are actually interested in the real locus $\Phi_{p/q}^{-1}(\mathbb{R})$. This locus is precisely the zero set of the polynomial
$\phi_{p/q}(x,y) = \operatorname{Im}\Phi_{p/q}(x+yi)$ in the two real indeterminates x, y.
In the parabolic case, $\phi_{0/1}(x,y) = -y$, $\phi_{1/1}(x,y) = y$, and
$\phi_{1/2}(x,y) = 2xy$. Note that y divides each of these. We can apply Im to both sides of our recurrence relation to get
$\phi_{p/q}\phi_{r/s} + \phi_{p/q\oplus r/s} + \phi_{p/q\ominus r/s} = 0$. Thus if some polynomial divides $\phi_{p/q}, \phi_{r/s}, \phi_{p/q\ominus r/s}$ then by induction it divides everything below that triple in the Farey graph. In particular, y divides every polynomial $\phi_{p/q}$ $(p/q \in [0,1] \cap \mathbb{Q})$.
Experimentation shows that these polynomials actually seem to factor further in many case. In fact, it looks like the component corresponding to the pleating ray is always of even degree.

- The real locus $\Phi_{p/q}^{-1}(\mathbb{R})$ consists of q different smooth curves. Only one of these corresponds to a pleating ray (actually a pleating ray and its complex conjugate). What do the other components represent geometrically? What if we just restrict to the irreducible factor of $\phi_{p/q}$ corresponding to the pleating ray?
- It looks like $\cup_{p/q\in\mathbb{Q}}\Phi_{p/q}^{-1}(-2)$ is dense in the Riley slice exterior. Is this true?
- Let Φ_α and Φ_β $(\alpha \neq \beta)$ be Farey polynomials; is it ever the case that $\Phi_\alpha^{-1}(\xi) \cap \Phi_\beta^{-1}(\xi) \neq \emptyset$ for some $\xi \in \mathbb{R}_{\geq -2}$? What about $\xi \in \mathbb{R}$?



The approximations to the Riley slice exterior have some rather intricate structure. Here we show a view of such an approximation, zoomed in around the $1/2$-cusp.

Define the **extended pleating ray of slope** p/q to be the preimage of $\mathbb{R}$ defined by the same branch of the inverse of the Farey polynomial. We use $\mathcal{EP}_{p/q}$ to denote this curve.
One sees immediately that there appears to be clustering around extensions of the pleating rays into the exterior; it is known that all discrete hyperbolic 2-bridge knot complement groups lie on these extensions Aimi, Lee, Sakai, and Sakuma (2020) and Akiyoshi, Ohshika, Parker, Sakuma, and Yoshida (2020). Our interest, though, lies in the following property: the extended pleating rays appear to be paired together across a central neighbour to form a smooth curve.

- What is the relationship between the two 'paired' fractions? (It is not that they are Farey neighbours.)
- Suppose p/q and r/s are 'paired'. Conjecture: There exists a complex analytic curve $\mathcal{EP}_{p/q \leftrightsquigarrow r/s}$ such that $\mathcal{EP}_{p/q \leftrightsquigarrow r/s} \cap (\mathbb{C} \setminus \mathcal{R}) = \mathcal{P}_{p/q} \cup \mathcal{P}_{r/s}$.
- The curve $\mathcal{EP}_{p/q \leftrightsquigarrow r/s}$ meets the extension of $\mathcal{P}_{(p+r)/2(q+s)}$ at a single point.

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