Ephemera and apocrypha
2. Sociology

1. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the affine torus.
   (a) Show that every geodesic on T is either (i) dense in T or (ii) the projection of a line of rational slope. Call the latter $\gamma(p/q)$ where $(p,q) = 1$ or $p/q \in \{1/0, 0/1\}$.
   (b) Show that every geodesic lamination on T consists of a single geodesic.
   (c) Show that there is a natural $\mathrm{PSL}(2,\mathbb{Z})$ action on non-dense geodesic laminations and this action preserves the simplicial structure with cells $(p/q, r/s, (p+r)/(q+s))$ where $|ps - qr| = 1$.

2. (a) Find conditions on $\lambda, \mu \in \mathbb{R}$ such that the group G generated by the two elements $\begin{bmatrix} \lambda & -\lambda \\ \lambda & (1-\lambda^2)/2 \end{bmatrix}$ and $\begin{bmatrix} \mu & -\mu \\ -\mu & (1+\mu^2)/2 \end{bmatrix}$ is discrete and has quotient a disjoint union of two punctured tori.
   The group should be Fuchsian of the first kind and hence the hyperbolic metrics on $\mathbb{H}^2$ and $-\mathbb{H}^2$ descend. Let T^* be the quotient of $\mathbb{H}^2$.
   (b) Show that every *closed* geodesic on T^* is either (i) dense in T^* minus some small open neighbourhood of the cusp or (ii) has homology class $p\gamma_0 + q\gamma_\infty$ for $(p,q) = 1$ or $p/q \in \{1/0, 0/1\}$ and with some fixed homology basis $(\gamma_0, \gamma_\infty)$. Even better, there is a bijection between points visible from the origin in $H_1(T^*) \simeq \mathbb{Z}^2$ and non-dense closed geodesics on T^*.
   (c) Show that every geodesic lamination on T^* which does not meet the cusp end ($\iff$ has compact support) consists of a single geodesic.
   (d) Show that there is a natural $\mathrm{PSL}(2,\mathbb{Z})$ action on non-dense geodesic laminations with compact support and that this action preserves the simplicial structure with cells $(p/q, r/s, (p+r)/(q+s))$ where $|ps - qr| = 1$.

3. (We saw the Maskit and Riley slices in the lecture. This problem contains both of them.) Let $G = G(\lambda, \mu, \rho)$ be the group generated by

$$X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}, Y = \begin{bmatrix} \mu & 0 \\ \rho & \mu^{-1} \end{bmatrix};$$

   the generic situation is that G is free and $\Omega(G)/G$ (it is a genus two Schottky group). Suppose $\lambda = 2$, then X represents the transformation $z \mapsto 4z + 2$ which sends the vertical line $\mathrm{Re}\, z = -1$ to the vertical line $\mathrm{Re}\, z = 1$. If $\mu = 2$ then the isometric circles of Y are the circles of radius $1/|\rho|$ centred around $-1/2\rho$ and $2/\rho$.
   (a) Prove that these circles are contained strictly within the vertical strip $-1 < \mathrm{Re}\, z < 1$ iff the two following conditions hold (where $\rho = re^{i\theta}$):

$$r > \frac{1}{2}(\cos\theta + 2) < 1 \text{ and } r > (2\cos\theta + 1).$$

   That is, ρ lies in the mutual exterior of the two cardioids depicted in fig. 1. This gives a rough bound on the family $\mathrm{Fam}(G(\lambda = 2, \mu = 2, \rho), |\rho| \gg 0)$, which is a one-dimensional slice through genus two Schottky space. (Compare this with the Riley slice, which is the slice $\lambda = \mu = 1$ through the boundary of this space.) Which points of the cardioids actually lie on the actual boundary of the deformation space?
   (b) Anyway, carry out an analysis similar to the torus questions for a genus two surface S. Your theory should include such things as:
      i. Maximal geodesic laminations all have three geodesics, and there is a natural partition of the space of laminations given via density in non-zero measure subsets of the surface;
      ii. The geodesics can be viewed in some way as (isotopic to) projections of combinatorially defined curves with respect to the fundamental domain given in (a);

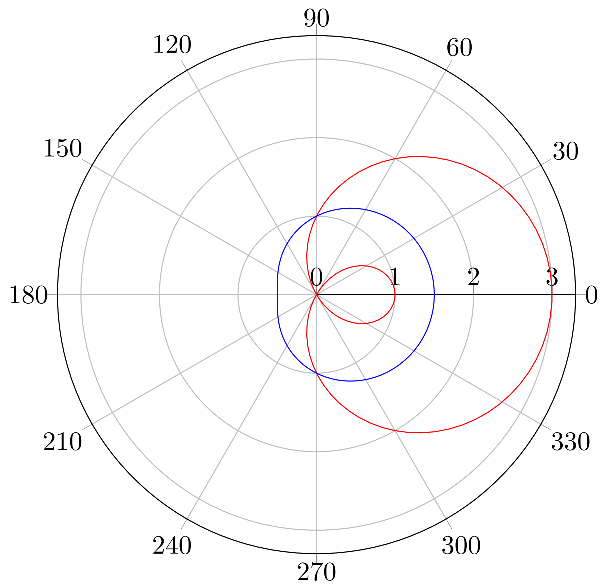

Figure 1: Bounds on the one-dimensional slice through genus two Schottky space.

iii. There is a bijection between the space of non-dense maximal geodesics and some natural subset of $H_1(S)$;

iv. There is an action of some group like $\mathrm{PSL}(3, \mathbb{Z})$ on this subset.

(c) If you take a maximal geodesic lamination with one lamination separating the surface into two tori-minus-discs (each with one of the other laminations) then this lamination can be pinched to a cusp. This reduces (topologically) to question 2 above. In particular, this explains why we ignore geodesics which hit the cusp. Check that your theory contains the theory of question 2.

4. (Open) Some open problems relating Kleinian groups to algebraic curves?

(a) Give a strictly algebraic method for determining the canonical lamination on a Kleinian group.

(b) What algebraic structure can be placed onto an algebraic curve in order to lift a complex structure to the structure of a visual boundary of a 3-manifold? (The answer should be more algebraic than ‘a quadratic differential’, for instance.)

(c) What is the algebraic analogue of the procedure ‘measured lamination $\to$ train track’?