Ephemera and apocrypha
1. Kleinian groups

Question 1: Isometric circles

For this question fix $f \in \mathbb{M}$ such that $f(\infty) \neq \infty$.

(a) Show that there exists some $r \in \mathbb{R}_{>0}$ such that if C is a circle of radius greater than r centred at $f^{-1}(\infty)$ then $f(C)$ is a circle of radius less than r about $f(\infty)$.

(b) Improve the result of (a) enough that you can apply the intermediate value theorem to conclude the existence of a circle of radius r about $f^{-1}(\infty)$ that is mapped to a circle of the same radius about $f(\infty)$. Show that these are the only two circles of the same Euclidean radius paired by f. These are the **isometric circles** of f.

(c) True or false: f is parabolic if and only if its isometric circles are tangent.

(d) Give the most general theorem which you can that relates intersection properties of the isometric circles of f and the dynamical properties of f.

Question 2: Schottky groups

(a) Let C_i for $i \in \{1, 2\}$ be the circle of radius $\rho_i > 0$ about $x_i \in \mathbb{C}$. Suppose $C_1 \neq C_2$. Write down all transformations $f \in \mathbb{M}$ such that $f(C_1) = C_2$.

(b) A **classical Schottky group** is given by the following data: (i) $2n$ disjoint circles, $C_1, \ldots, C_n, C'_1, \ldots, C'_n$, which bound a common exterior U; and (ii) for each i, a loxodromic transformation g_i which sends C_i to C'_i. Describe the homeomorphism class of the hyperbolic 3-manifold which it uniformises. Describe the conformal structure at infinity.

(c) For $n = 2$ and $n = 3$, compute as many qualitatively different limit sets as possible for classical Schottky groups on $2n$ circles. How do the limit sets vary (qualitatively) as the coefficients vary?

(d) Give an example, for arbitrary $n \in \mathbb{N}$, of a one-parameter family G_t $(t \in (0, 1))$ of classical Schottky groups on $2n$ circles such that as $t \to 1$ the family converges (as a matrix group) to a group whose quotient surface is exactly a union of thrice-punctured spheres and as $t \to 0$ every generator is an involution in $\mathbb{M}$ (i.e. conjugate to $z \mapsto -1/z$).

Question 3: Fuchsian groups

Let $\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the hyperbolic plane.

(a) Show that $A \in \mathrm{PSL}(2, \mathbb{C})$ preserves $\mathbb{H}^2$ iff $A \in \mathrm{PSL}(2, \mathbb{R})$.

(b) Recall that the metric ϱ on $\mathbb{H}^2$ is given by $\cosh \varrho(w, z) = 1 + \frac{|w-z|^2}{2(\operatorname{Im} w)(\operatorname{Im} z)}$. Show that every element of $\mathrm{PSL}(2, \mathbb{R})$ is an isometry of $\mathbb{H}^2$. (Remark: the converse is also true, $\mathrm{PSL}(2, \mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2)$.)

(c) A Kleinian group which preserves $\mathbb{H}^2$ (i.e. a discrete group of isometries of $\mathbb{H}^2$) is called **Fuchsian**. Show that a discrete G is Fuchsian iff $\Lambda(G) \subseteq \mathbb{R}$.

Question 4: The (∞, ∞, ∞)-triangle group

Let C_1, C_2, C_3, C_4 be four circles such that each C_i is tangent to C_{i-1} and C_{i+1} and there are no other intersection relations (all subscripts taken mod 4).

(a) Show that the four intersection points lie on a fifth circle which is orthogonal to each C_i.

(b) Give necessary and sufficient Eucidean-geometric conditions for the configuration to be $\mathbb{M}$-equivalent to the configuration given by the two vertical lines $\operatorname{Re} z = \pm 1$ and the two circles of radius 1 around $\pm 1/2$ respectively.

(c) Show that the group G generated by the two elements $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ is discrete. Describe the homeomorphism class of the hyperbolic 3-manifold which it uniformises. Describe the conformal structure at infinity.

(d) Show that G is an index two subgroup of a group generated by the reflections in an arbitrary (∞, ∞, ∞)-triangle. Reinterpret (b) in terms of this.

Figure 1: Rita Angus, *Growth*, 1968.

Question 5: The figure 8 knot

(a) Draw a convincing picture or sequence of pictures to show that $\Gamma_{-\omega}$ has quotient manifold a figure eight knot.

(b) Show that $\Gamma_{-\omega}$ has limit set equal to $\hat{\mathbb{C}}$ without appealing to 3-manifold geometry.

Question 6: Some funner manifolds

We only dealt with hyperbolic space in the lecture but this works for all geometric spaces (suitably defined).

(a) Give an affine structure on the punctured torus. Is it complete? (Of course not, but why not, and why is this an easy question to answer with no work?)

(b) Let ρ be the isometry of $\mathbb{E}^3 = \mathbb{C} \times \mathbb{R}$ defined by $\rho(z,t) = (ze^{2\pi/3}, t)$. Describe the affine 3-orbifold $\mathbb{E}^3/\langle\rho\rangle$. Draw a picture of *Growth* (fig. 1) as seen from behind the cone arc.

(c) Recall that $\mathrm{SO}(4)$ is the group of rotations of S^3, where we view S^3 as embedded into $\mathbb{R}^4$ as a sphere centred at 0. The only subgroups of $\mathrm{SO}(4)$ which act freely on S^3 are the finite subgroups. Identify $\mathbb{R}^4 \simeq \mathbb{C}^2$, let ζ be a primitive pth root of unity (for some $p \in \mathbb{Z}$), let q be coprime to p, and let $\mathbb{Z}/p\mathbb{Z} \simeq \langle\zeta\rangle$ act on S^3 by

$$\zeta \cdot (w, z) := (\zeta w, \zeta^p z).$$

Give matrix representatives in $\mathrm{SO}(4)$ for this action (the group isomorphism class depends only on p, but the action depends on p and q); and a fundamental domain for the action.

This is the **Lens space** $L(p,q)$ (Bredon, example 7.4).

(d) Show that the trefoil knot complement is diffeomorphic to $\mathrm{PSL}(2,\mathbb{R})/\mathrm{PSL}(2,\mathbb{Z})$ and hence the trefoil knot complement is a $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$-manifold. (See for instance `https://math.stackexchange.com/a/3115852`.)

(e) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus.

i. Show that the linear automorphism of $\mathbb{R}^2$ represented by $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ descends to T. The resulting map on the torus is the **Arnold's cat map** α.

ii. Draw the mapping torus of α, $(T \times [0,1])/((x,1) \sim (\alpha(x), 0))$. This manifold is a Sol-manifold.