Apocrypha and ephemera on the boundaries of moduli space

or, What I did on my holiday.

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Henry Moore: *Bronze Form* (1988). In situ, Wellington Botanic Garden ki Paekākā. Photo by author (2022)

Introduction and context

When Thurston reinvented hyperbolic geometry in the '70s and '80s, he brought numerous combinatorial ideas with him from his earlier work on foliations. We will survey some of these combinatorial ideas and their relation to work I have been doing for the past couple of years. One can view all of hyperbolic geometry post-Thurston as being essentially complicated combinatorics and geometry of circles on the sphere (the geometry being encoded by so-called *Kleinian groups*, which arose much earlier in conformal geometry).

Background material and useful references

These notes are essentially a compressed version of what was supposed to be the second third of my graduate seminar at the University of Auckland in 2021, which was sadly interrupted by the August lockdown. The three thirds were 'classical', i.e. pre-Thurston conformal geometry and complex dynamics; 'Thurston', i.e. hyperbolic geometry; and ‘arithmetic’, following [39]. Thus in some sense we *de jure* assume an awful lot of classical Kleinian group theory, from e.g. the books of Beardon [4] and Maskit [42]; but we *de facto* do not need it, since everything is highly intuitive.

We draw from a number of references in the theory of braid groups and mapping classes. Much of the relevant work is due to Joan Birman [6], including some joint work with Caroline Series [7], and much of this was surveyed in the excellent textbook of Benson Farb and Dan Margalit [26]; we also recommend Jessica Purcell’s knot theory book [49]. We will also deal with Thurston’s studies of curves on surfaces [25] and 3-manifolds [65, 64], much of which is found (with references to modern work) in the textbook of Marden [41]. For basic hyperbolic geometry one can see Thurston [64], Beardon [4], and Ratcliffe [50]. For the moduli theory of Kleinian groups, see Kapovich [31] and the survey article of Alex Elzenaar, Gaven Martin, and Jeroen Schillewaert [22]. Expositions of concrete special cases of this theory are originally due to Linda Keen and Caroline Series [35, 36] and we (that is, Elzenaar, Martin, and Schillewaert) have written several papers extending the Keen–Series theory in the special case of 3-orbifolds homeomorphic to a 3-ball with two disjoint cone arcs [22, 21, 23, 24].

Acknowledgements
Work on Kleinian groups and their moduli is joint with Gaven Martin and Jeroen Schillewaert. A brief workshop was held at the Max-Planck-Institut für Mathematik in den Naturwissenschaften in Leipzig in December 2023; thanks to Sam Fairchild, Ángel Ríos Ortiz, and Nick Early for helpful comments. A minicourse on this material was then taught at the University of Auckland in January 2023, following the NZMRI Summer Meeting on groups, geometry, and dynamics in Tāhunanui. The attendees asked very insightful questions and pointed out numerous problems, the answers to which and fixes for which have been incorporated into this version of the notes: hopefully all of the errors in particular have been corrected and replaced with a somewhat lower number of different errors (which the author takes full responsibility for, of course). Many thanks in particular to Gaven Martin, Jeroen Schillewaert, Ari Markowitz, and David Dijkema (as well as the other participants). Thanks also to Jeroen Schillewaert and the Clay Mathematics Institute for supporting the author’s attendence at the NZMRI meeting.

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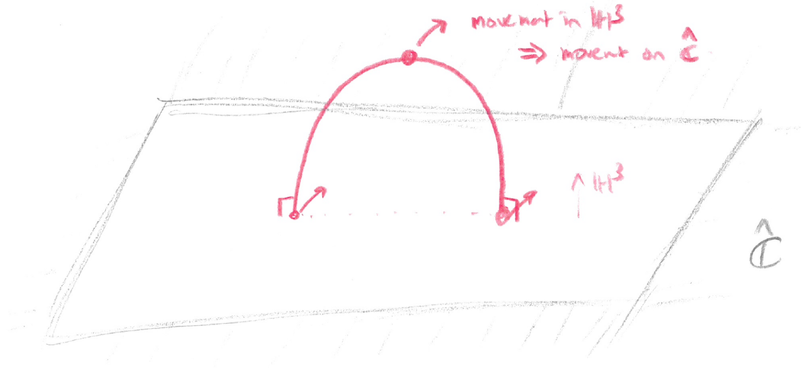


Figure 1: A geodesic in $\mathbb{H}^3$ lying on the visual boundary $\hat{\mathbb{C}}$.

§1. A crash course on Kleinian groups

As Kra [37] and Series [56] (among many others) have done before us, we will first have a crash course on Kleinian groups. But unlike Kra and Series, we will take a post-Thurston view immediately. For the basic material on hyperbolic metric spaces, we follow [9]. I will try to be consistent in notation with the lecture notes from my 2021 graduate course [20] which in turn are consistent with the books of Maskit [42] and Beardon [4], and the notes of Thurston [64]. A much nicer overview may be found in Thurston's famous survey paper [65].

We recall that **hyperbolic 3-space** is the unique Riemann manifold of constant curvature -1. We will primarily use the Poincaré model, supported on the open smooth manifold

$$\mathbb{H}^3 := \{x = (z, t) \in \mathbb{C} \times \mathbb{R} : t > 0\}$$

with Riemann metric given by

$$ds = \frac{|dz + dt|}{t}.$$

Geodesic lines in $\mathbb{H}^3$ are exactly the (Euclidean) semicircles and half-lines orthogonal to $\mathbb{C}$. There is a natural compactification of $\mathbb{H}^3$ given by adjoining the sphere at infinity, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. This is exactly the visual boundary of $\mathbb{H}^3$ as a CAT(0) space, and as such isometries of $\mathbb{H}^3$ extend to to conformal maps on the boundary (Fig. 1). In fact since the visual boundary is defined to be the space of geodesic half-rays modulo the relation of being eventually parallel, and because this relation is trivial in $\mathbb{H}^3$ by simple Euclidean geometry of circles, we get an isomorphism of topological groups $\mathrm{Isom}^+(\mathbb{H}^3) \simeq \mathrm{Conf}(\hat{\mathbb{C}})$. By standard results from undergraduate analysis we have further isomorphisms between $\mathrm{Conf}(\hat{\mathbb{C}})$ and the groups of Möbius transformations $\mathbb{M}$ and of fractional linear transformations $\mathrm{PSL}(2, \mathbb{C})$. As always a discrete subgroup of $\mathbb{M}$ (and by canonical identification, any of these groups) is called **Kleinian**.

We will now move to manifolds modelled on hyperbolic space, but we might as well be slightly more general and use Thurston's language of pseudogroups (essentially a version of sheaf theory). A **pseudogroup** on a topological space X is a set $\mathcal{G}$ of homeomorphisms defined on open subsets of X such that:

1. (Group-type axioms)
   a) If $f, g \in \mathcal{G}$ then $f \circ g \in \mathcal{G}$;
   b) If $f \in \mathcal{G}$ then $f^{-1} \in \mathcal{G}$;
2. (Sheaf-type axioms)
   a) The set $\{\operatorname{dom} f : f \in \mathcal{G}\}$ is an open cover of X;
   b) If $f \in \mathcal{G}$ and $V \subseteq \operatorname{dom} f$ then $f|_V \in \mathcal{G}$;
   c) Suppose $f \in \operatorname{Homeo}(X)$ and $(V_\alpha)_{\alpha \in A}$ is an open cover of $\operatorname{dom} f$. If $f|_{V_\alpha} \in \mathcal{G}$ for all $\alpha \in A$, then $f \in \mathcal{G}$.

Now a $\mathcal{G}$**-manifold**, for a pseudogroup $\mathcal{G}$ on some subset $X \subseteq \mathbb{R}^n$, is an n-manifold M such that the charts of M land in X and the transition maps of M lie in $\mathcal{G}$. For instance, a Riemann surface is a $\mathcal{C}$-manifold, where $\mathcal{C}$ is the pseudogroup of biholomorphic maps on subsets of $\mathbb{C} \simeq \mathbb{R}^2$.

1.1 Definition. An $\mathbb{H}^3$**-manifold** (or just **manifold** for us) is a $\mathcal{H}^3$-manifold, where $\mathcal{H}^3$ is the pseudogroup of hyperbolic isometries on $\mathbb{H}^3$.

Of course, one can replace $\mathbb{H}^3$ with one's favourite Thurston geometry (which is Sol).

It is natural to extend the definition of a manifold to allow for singularities. Let $\mathcal{G}$ be a pseudogroup on $X \subseteq \mathbb{R}^n$ and let O be a Hausdorff topological space. A $\mathcal{G}$**-orbifold structure** on O is given by the following data:

1. An open cover $\{V_i\}_{i \in I}$ of O;
2. For each i, a finite subpseudogroup $\Gamma_i \leq \mathcal{G}$ and a simply connected open set $X_i \subseteq X$ such that $\operatorname{dom} f \supseteq X_i$ and $f(X_i) \subseteq X_i$ for all $f \in \Gamma_i$ together with a continuous map $q_i : X_i \to V_i$ which descends to a homeomorphism $X_i/\Gamma_i \to V_i$;
3. For all $x_i \in X_i$ and $x_j \in X_j$ with $q_i(x_i) = q_j(x_j)$, a diffeomorphism h from an open connected neighbourhood W of x_i to a neighbourhood of x_j such that $q_j h = q_i|_W$.

Given a hyperbolic orbifold O, we can define a **developing map** via analytic continuation, following a version of the traditional construction of a universal cover in this way. The precise construction can be found in [50, §13.3], but the point is that the lifting of paths has to be compatible with the local group actions (Fig. 2). We say that O is **complete** if the development fills up the entirety of $\mathbb{H}^3$. Compare this with the following 2D example.

1.2 Example. Let G be the elementary group generated by $z \mapsto 2z$. A fundamental domain for this group is the annulus $\{z \in \mathbb{C} : 1 < |z| < 2\}$. The quotient orbifold is not complete since the tessellation by the fundamental domain misses 0 and ∞. It is an affine torus, by the way (Fig. 3) and the incompleteness is seen via the accumulation of the red curve in the quotient; see Exercise 3.1.10 of [66] for this example, and later on Example 3.3.4 for other affine manifolds including a second structure on the torus.

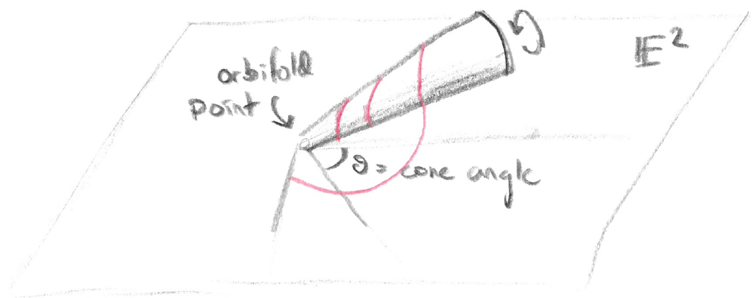


Figure 2: Unrolling a cone onto $\mathbb{E}^2$. The apex is a cone point: it is the centre of a rotation in the holonomy group (the symmetry group of the unrolling). See how the red path lifts, just like in the construction of a universal cover; but since we need to preserve geometry, some points might be missing.

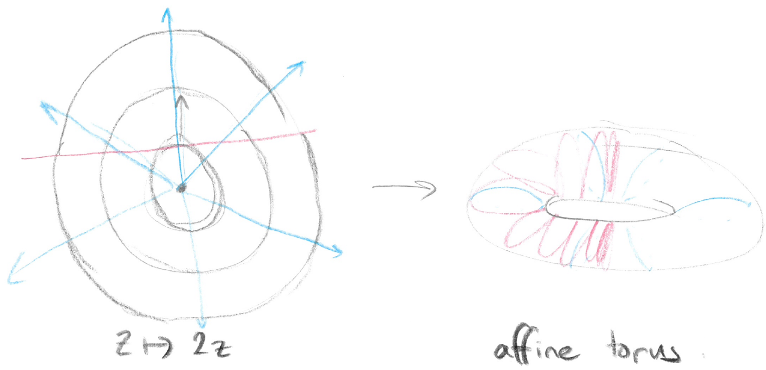


Figure 3: The affine torus produced as a quotient of $\mathbb{C} \setminus \{0\}$ by a dilation.

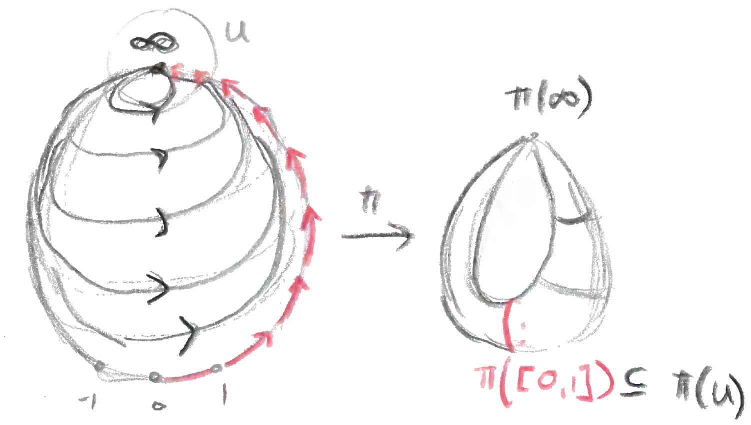


Figure 4: Attracting or repelling fixed points lead to non-Hausdorff quotients. Here, every neighbourhood of the projection of ∞ pulls back to an open set containing ∞ on the Riemann sphere; but every such neighbourhood contains infinitely many translates of the red segment $[0, 1]$ under the group action, and so every point of $\pi([0, 1])$ lies in every neighbourhood of $\pi(\infty)$. In fact, the only neighbourhood of $\pi(\infty)$ is the entire quotient!

Note, this definition of completeness coincides with the usual definition of completeness for a metric space (in the torus example, take any sequence in $\mathbb{C}$ converging to 0 which projects to a sequence of distinct points, then this is a non-converging Cauchy sequence).

1.3 Theorem. *If O is a connected complete $\mathbb{H}^3$-orbifold, then there is a Kleinian group Γ such that $O \simeq \mathbb{H}^3/O$. Further, Γ is isomorphic to the orbifold fundamental group $\pi_1^\circ(O)$. Conversely, if Γ is Kleinian then $\mathbb{H}^3/O$ is an $\mathbb{H}^3$-orbifold, in fact a hyperbolic $K(\Gamma, 1)$.*

The group Γ is called the **holonomy group** of O. Often rather than speaking about a group Γ we speak of a faithful discrete representation $\rho : \pi_1^\circ(O) \to \mathsf{PSL}(2, \mathbb{C})$, it amounts to the same thing.

Now a Kleinian group G does not just act on $\mathbb{H}^3$, it also acts on the sphere $\hat{\mathbb{C}} = \partial\mathbb{H}^3$. However, discreteness of G does *not* imply that the quotient $\hat{\mathbb{C}}/G$ is well behaved. For instance, take the group G of Example 1.2: the quotient $\hat{\mathbb{C}}/G$ is not Hausdorff at the projection of ∞—take the sequence 2^{-n}, all of these are identified by the group to a point ξ and every neighbourhood of 0 contains an element of the sequence so in the quotient every neighbourhood of 0 contains $\xi \neq 0$. For another example, take the group generated by the single translation $z \mapsto z + 1$ and look at the image of ∞ under the canonical projection map π (Fig. 4).

It turns out that all the bad points arise in this form (as fixed points and limits of fixed points), so we define the **limit set** of G to be $\Lambda(G) = \overline{\mathrm{Fix}(G)}$ where $\mathrm{Fix}(G)$ is the set of fixed points of non-torsion elements of G. The complement $\Omega(G) \coloneqq \hat{\mathbb{C}} \setminus \Lambda(G)$ has Hausdorff quotient under G, in fact $\Omega(G)/G$ is a Riemann surface (with marked orbifold points). This gives us two geometric objects associated to G:

- the **complete hyperbolic manifold without boundary** $N_G = \mathbb{H}^3/G$
- the **Riemann surface at infinity** $S_G = \Omega(G)/G$

Now one can play with some examples and see that the set $\Omega(G)$ is identified the set of lifts of points on the visual boundary of N_G: if g is a geodesic ray in $\mathbb{H}^3$ ending at a limit point, then it projects to a closed geodesic in the quotient. Hence we can identify S_G with the visual boundary of N_G, and the result is $O_G = (\mathbb{H}^3 \cup \Omega(G))/G$, the **Kleinian manifold** (although in general it is orbi not mani).

The main point of all of this discussion is that it can be reversed. This is known as the **Poincaré polyhedron theorem** [50, §13.5]: if you take a hyperbolic polyhedron P together with a set of isometries pairing the sides up, then (subject to some natural geometric conditions) the group G generated by the side-pairings is discrete and $\mathbb{H}^3/G$ is isometric to the space obtained by taking P and identifying the sides. This works even if $\overline{P} \cap \partial\mathbb{H}^3$ is non-empty, in which case the intersection forms a fundamental domain for the action of G as a surface group and the quotient surface is obtained by pairing arcs of circles [42, §VI.A.3].

Now for some examples. I want to talk about moduli, and there are two 'easy' moduli spaces (arguably), called the **Riley slice** and the **Maskit embedding**. So we will give some relevant groups.

1.4 Example (Figure 8 knot group)**.** Let $k \subseteq S^3 = \mathbb{R}^3 \cup \{\infty\}$ be the figure 8 knot (the best knot!!!). It is a result of Riley [52, Theorem 1] that $\pi_1(k)$ (recall, the fundamental group of a knot is the fundamental group of its complement in S^3) is isomorphic to the Kleinian group

$$\Gamma_{-\omega} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix} \right\rangle$$

where $\omega = \exp(2\pi i/3)$; the limit set is shown as Fig. 5. For a survey of the history see [22]. In any case, since $\Gamma_{-\omega}$ is non-elementary the quotient manifold $\mathbb{H}^3/\Gamma_{-\omega}$ is a hyperbolic manifold with fundamental group $\Gamma_{-\omega}$, and in fact it is homeomorphic to $S^3 \setminus k$. Nice pictures can be found in [44, p. 34] and in [27, p. 152]. This was in fact the example that led Thurston to the geometrisation conjecture (for various accounts of this history see [65, 51, 12] and the historical notes to Section 10.3 on p.504 of [50]).

1.5 Example (A Riley group)**.** Define the group

$$\Gamma_{3i} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 3i & 1 \end{bmatrix} \right\rangle.$$

See the limit set of Fig. 6. This group uniformises a manifold homeomorphic to the 3-ball minus two arcs, and hyperbolic metric coming from a braiding of the arcs of slope 2/1; see Fig. 7.

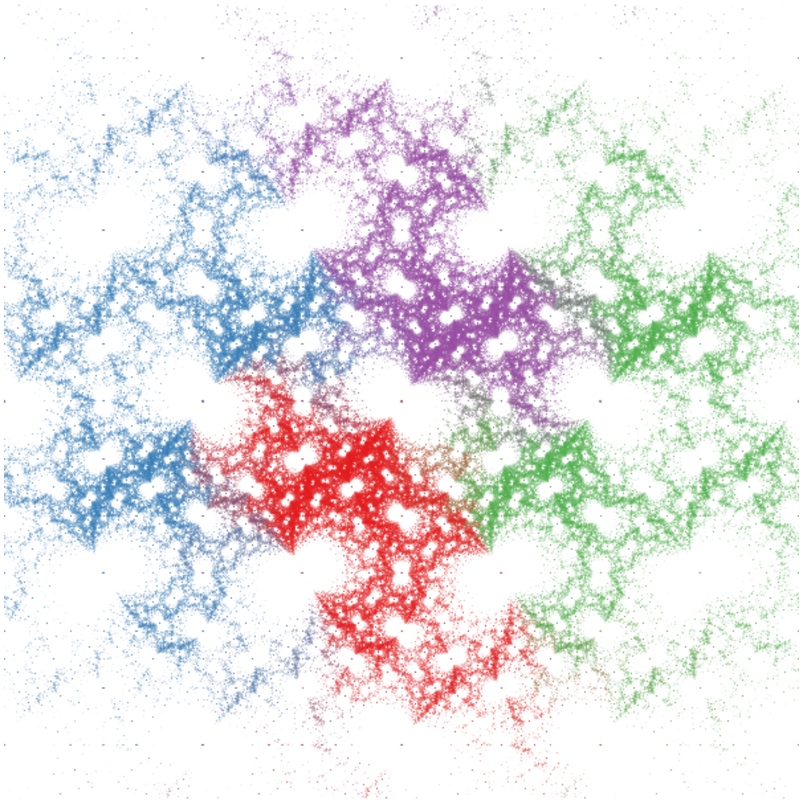


Figure 5: Limit set of the figure 8 knot group.

Figure 6: Limit set of the $\rho = 3i$ Riley group.

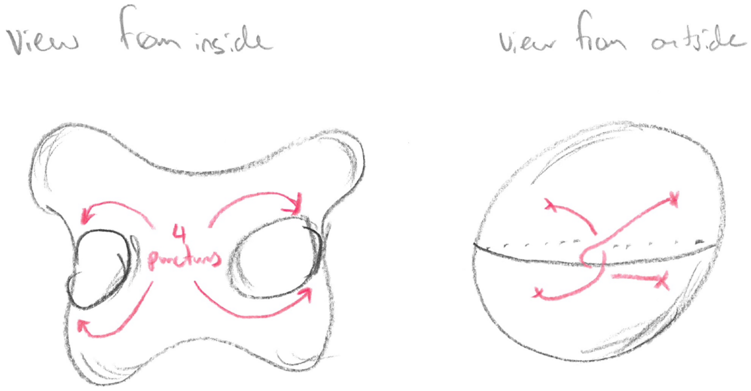


Figure 7: Two views of the quotient structures which arise from Γ_{3i} (Example 1.5). On the left, we see the hyperbolic structure on the Riemann surface at infinity (the quotient $\Omega(\Gamma_{3i})/\Gamma(3_i)$); observe that the punctures can be compactified by adding points 'infinitely far away' in the hyperbolic surface metric. On the right, we see the conformal structure at infinity from outside the three-manifold; the manifold is a ball with two arcs drilled out, and the arcs are twisted. We can untwist the arcs by isotopy in the 3-manifold, but tracking the movement of the 'fluid' we see that this will twist up the complex structure on the boundary.

Figure 8: Limit set of the $\mu = 3i$ Maskit group.

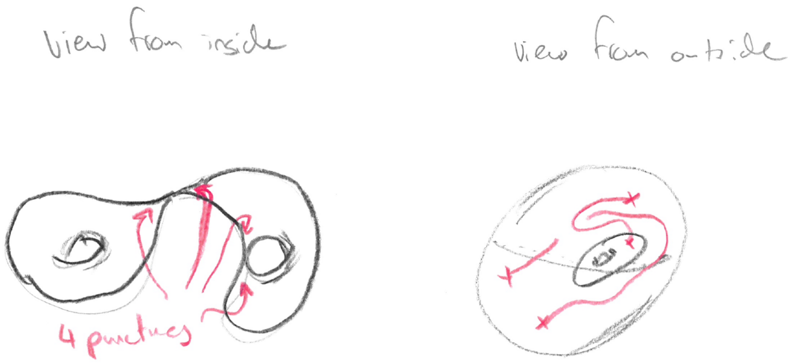


Figure 9: Two views of the quotient structures which arise from G_{3i} (Example 1.6). On the left, we see the hyperbolic structure on the Riemann surface at infinity (the quotient $\Omega(G_{3i})/G(3_i)$), and observe there is a similar kind of compactification as described in the caption of Fig. 7. On the right, we see the conformal structure at infinity from outside the three-manifold.

1.6 Example (A Maskit group)**.** Define the group

$$G_{3i} = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -i(3i) & i \\ i & 0 \end{bmatrix} \right\rangle.$$

See the limit set of Fig. 8. This group uniformises a manifold homeomorphic to a 3-ball with a deleted torus and two deleted arcs, see Fig. 9.

The Riley and Maskit groups are groups which lie on the boundary of the space of genus 2 Schottky groups.

1.7 Definition. A **Schottky group** is a finitely generated, free, purely loxodromic Kleinian group with $\Omega(G) \neq \varnothing$.

The easiest examples of Schottky groups are obtained from pairing disjoint circles; that is, take a hyperbolic polyhedron with no finite edges or vertices whose boundary at infinity consists of the common exterior of $2n$ disjoint round circles, and pair the circles up by arbitrary loxodromic transformations. The Maskit and Riley groups are obtained by 'pushing together' these circles until they become tangent; but we leave a more detailed discussion to the problem sheet.

Remark. There exist Schottky groups which do not admit a generating set which pair round circles—a non-constructive proof of this was first given by Marden, and an explicit example was found by Yamamoto [67]—but this can be fixed by allowing pairings of arbitrary topological loops [42, X.H] (in fact, quasicircles are enough [2, §9.3]).

Problems

1. **Isometric circles.** For this question fix $f \in \mathbb{M}$ such that $f(\infty) \neq \infty$.

a) Show that there exists some $r \in \mathbb{R}_{>0}$ such that if C is a circle of radius greater than r centred at $f^{-1}(\infty)$ then $f(C)$ is a circle of radius less than r about $f(\infty)$.

b) Improve the result of (a) enough that you can apply the intermediate value theorem to conclude the existence of a circle of radius r about $f^{-1}(\infty)$ that is mapped to a circle of the same radius about $f(\infty)$. Show that these are the only two circles of the same Euclidean radius paired by f. These are the **isometric circles** of f.

c) True or false: f is parabolic if and only if its isometric circles are tangent.

d) Give the most general theorem which you can that relates intersection properties of the isometric circles of f and the dynamical properties of f.

2. **Schottky groups.**

a) Let C_i for $i \in \{1,2\}$ be the circle of radius $\rho_i > 0$ about $x_i \in \mathbb{C}$. Suppose $C_1 \neq C_2$. Write down all transformations $f \in \mathbb{M}$ such that $f(C_1) = C_2$ and which map the exterior of C_1 (i.e. the component of $\hat{\mathbb{C}} \setminus C_1$ which contains ∞) to the interior of C_2. (Hint: pick three ordered points on C_1, map them to three points in the reversed order in C_2, what are the moduli?)

b) A **classical Schottky group** of rank n is given by the following data: (i) $2n$ disjoint circles, $C_1, \dots, C_n, C'_1, \dots, C'_n$, which bound a common exterior U; and (ii) for each i, a loxodromic transformation g_i which sends C_i to C'_i. Describe the homeomorphism class of the hyperbolic 3-manifold which it uniformises. Describe the conformal structure at infinity.

c) For $n = 2$ and $n = 3$, compute as many qualitatively different limit sets as possible for classical Schottky groups on $2n$ circles. How do the limit sets vary (qualitatively) as the coefficients vary?

d) Give an example, for arbitrary $n \in \mathbb{N}$, of a one-parameter family G_t ($t \in (0,1)$) of classical Schottky groups on $2n$ circles such that as $t \to 1$ the family converges (as a matrix group) to a group whose quotient surface is exactly a union of thrice-punctured spheres and as $t \to 0$ every generator is an involution in $\mathbb{M}$ (i.e. conjugate to $z \mapsto -1/z$).

3. **Fuchsian groups.** Let $\mathbb{H}^2 = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the hyperbolic plane.

a) Show that $A \in \mathsf{PSL}(2,\mathbb{C})$ preserves $\mathbb{H}^2$ iff $A \in \mathsf{PSL}(2,\mathbb{R})$.

b) Recall that the metric ϱ on $\mathbb{H}^2$ is given by $\cosh \varrho(w,z) = 1 + \frac{|w-z|^2}{2(\operatorname{Im} w)(\operatorname{Im} z)}$. Show that every element of $\mathsf{PSL}(2,\mathbb{R})$ is an isometry of $\mathbb{H}^2$. (Remark: the converse is also true, $\mathsf{PSL}(2,\mathbb{R}) = \operatorname{Isom}^+(\mathbb{H}^2)$.)

c) A Kleinian group which preserves $\mathbb{H}^2$ (i.e. a discrete group of isometries of $\mathbb{H}^2$) is called **Fuchsian**. Show that a discrete G is Fuchsian iff $\Lambda(G) \subseteq \mathbb{R}$.

4. **The (∞,∞,∞)-triangle group.** Let C_1, C_2, C_3, C_4 be four circles such that each C_i is tangent to C_{i-1} and C_{i+1} and there are no other intersection relations (all subscripts taken mod 4).

Figure 10: Rita Angus, *Growth*, 1968.

a) Show that the four intersection points lie on a fifth circle which is orthogonal to each C_i.

b) Give necessary and sufficient Eucidean-geometric conditions for the configuration to be $\mathbb{M}$-equivalent to the configuration given by the two vertical lines $\operatorname{Re} z = \pm 1$ and the two circles of radius 1 around $\pm 1/2$ respectively.

c) Show that the group G generated by the two elements $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ is discrete. Describe the homeomorphism class of the hyperbolic 3-manifold which it uniformises. Describe the conformal structure at infinity.

d) Show that G is an index two subgroup of a group generated by the reflections in an arbitrary (∞, ∞, ∞)-triangle. Reinterpret (b) in terms of this.

5. **The figure 8 knot.**

a) Draw a convincing picture or sequence of pictures to show that $\Gamma_{-\omega}$ has quotient manifold a figure eight knot.

b) Show that $\Gamma_{-\omega}$ has limit set equal to $\hat{\mathbb{C}}$ without appealing to 3-manifold geometry.

6. **Some more exotic manifolds.** We only dealt with hyperbolic space in the lecture but this works for all geometric spaces (suitably defined).

a) Give an affine structure on the punctured torus. Is it complete? (Of course not, but why not, and why is this an easy question to answer with no work?)

b) Let ρ be the isometry of $\mathbb{E}^3 = \mathbb{C} \times \mathbb{R}$ defined by $\rho(z, t) = (ze^{2\pi/3}, t)$. Describe the affine 3-orbifold $\mathbb{E}^3/\langle \rho \rangle$. Draw a picture of *Growth* (Fig. 10) as seen from behind the cone arc.

c) Recall that $\mathrm{SO}(4)$ is the group of rotations of S^3, where we view S^3 as embedded into $\mathbb{R}^4$ as a sphere centred at 0. The only subgroups of $\mathrm{SO}(4)$ which act freely on S^3 are the finite subgroups. Identify $\mathbb{R}^4 \simeq \mathbb{C}^2$, let ζ be a primitive pth root of unity (for some $p \in \mathbb{Z}$), let q be coprime to p, and let $\mathbb{Z}/p\mathbb{Z} \simeq \langle\zeta\rangle$ act on S^3 by

$$\zeta \cdot (w, z) := (\zeta w, \zeta^p z).$$

Give matrix representatives in $\mathsf{SO}(4)$ for this action (the group isomorphism class depends only on p, but the action depends on p and q); and a fundamental domain for the action.

This is the **Lens space** $L(p, q)$ [8, Example 7.4].

d) Show that the trefoil knot complement is diffeomorphic to $\mathsf{PSL}(2, \mathbb{R})/\,\mathsf{PSL}(2, \mathbb{Z})$ and hence the trefoil knot complement is a $\widetilde{\mathsf{PSL}}(2, \mathbb{R})$-manifold. (See for instance `https://math.stackexchange.com/a/3115852`.)

e) Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus.

   i. Show that the linear automorphism of $\mathbb{R}^2$ represented by $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ descends to T. The resulting map on the torus is the **Arnold's cat map** α.

   ii. Draw the mapping torus of α, $(T \times [0, 1])/((x, 1) \sim (\alpha(x), 0))$. This manifold is a Sol-manifold.

§2. Cutting sequences and geodesics

Let X be a topological space, and let $p : \hat{X} \to X$ be a covering of X. A **deck transformation** of p is a homeomorphism $f : \hat{X} \to \hat{X}$ such that $pf = p$; the set of deck transformations forms a group under composition, which we denote Aut p. We say that p is **regular** if the action of Aut p on each fibre of p is transitive. This is equivalent to asking that the group $p_*\pi_1(\hat{X}, \hat{x})$ is normal in $\pi_1(X, p(\hat{x}))$ for each $\hat{x} \in \hat{X}$. The automorphism group of the group can be computed from the fundamental groups of the covering and covered spaces [8, Corollary III.6.9]:

2.1 Proposition. *If $p : \hat{X} \to X$ is a regular covering map, with $\hat{x} \in \hat{X}$ and $x = p(\hat{x})$, then* $\mathrm{Aut}(p) \simeq \pi_1(X, x)/p_*\pi_1(\hat{X}, \hat{x})$.

If the quotient comes from a discontinuous torsion-free group action—in which we say that the action is **freely discontinuous**—then the covering satisfies the conditions of Proposition 2.1:

2.2 Proposition. *If a group G acts freely discontinuously on a path connected and locally path connected Hausdorff space X, then $p : X \to X/G$ is a regular covering map such that* $\mathrm{Aut}(p) = G$.

Remark. Proposition 2.1 and Proposition 2.2 can be improved to orbifolds—i.e. allowing torsion-free groups—with only a little additional work, see Section III.$\mathcal{G}$.3 of Bridson and Haefliger [9].

The final technical result which we need concerns homotopy classes of geodesics on hyperbolic surfaces. For the general theory of these surfaces, see Thurston [25].

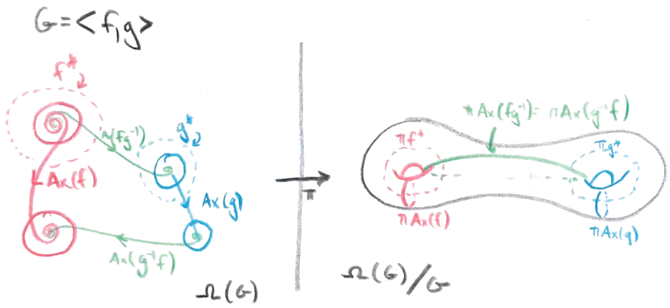


Figure 11: Lifts of geodesic curves with respect to a genus two classical Schottky group.

2.3 Proposition ([25, Lemma 3.4])**.** *Let S be a complete hyperbolic surface. If γ is a simple closed curve which is not boundary-parallel, then there exists a unique geodesic in the homotopy class $[\gamma]$.*

The primary application of these results is the following corollary.

2.4 Corollary. *Let G be a Kleinian group. Then the simple closed non-boundary-parallel hyperbolic geodesics on $S_G = \Omega(G)/G$ are of two kinds:*

1. *Geodesics which are homotopic to the projection in S of a homotopically nontrivial curve in $\Omega(G)$. (We call these curves **compression disc boundaries**.)*

2. *Geodesics which are homotopic to the projection in S of an arc α (a smooth curve) in $\Omega(G)$ with distinct endpoints on $\Lambda(G)$ such that there exists a loxodromic element $g \in G$ with $g(\alpha) = \alpha$. (We call these curves **long geodesics**.)*

Observe that the lifts are not unique, but they are unique modulo both (a) homotopy and (b) G-automorphism in $\Omega(G)$. If G is sufficiently nice (e.g. a Schottky group) then choosing a complete set of irredundant lifts should be straightforward.

2.5 Example. We give an example of Corollary 2.4 for a classical Schottky group on two loxodromic generators f and g; refer to Fig. 11. Here:

1. The axes of f and g, joining their fixed points, project to two meridians on the torus shown in solid blue and red respectively;

2. Loops concentric to the paired circles project to equators (which bound embedded discs in the 3-manifold) shown in dashed blue and red;

3. Other curves on the torus lift to more complicated curves in $\Omega(G)$. For instance, the green curve lifts to the axes of both fg^{-1} and gf^{-1}. Observe that these are conjugate in G—$g^{-1}(fg^{-1})g = g^{-1}f$—and that we need both lifts to give a lift of the entire green curve to the fundamental domain exterior to the paired circles.

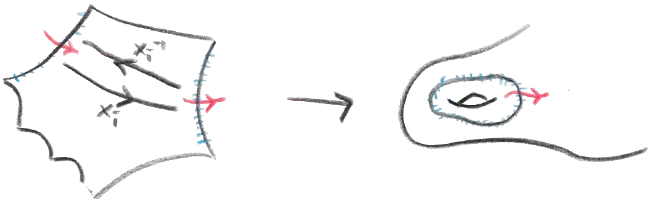


Figure 12: Orientation of arc germs with respect to a system of generators.

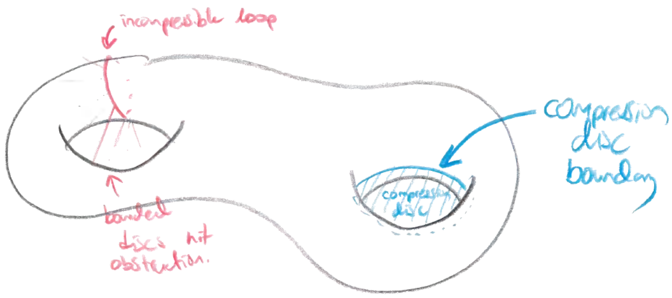


Figure 13: Curves bounding compression discs versus incompressible loops.

A systematic method for writing down a loxodromic element which represents a long geodesic follows from a circle of ideas dating all the way back to Dehn [16] (translated by Stillwell [15]). More modern work in the direction of Fuchsian groups was done by Birman and Series [7] and Series [57, 58]. The theory for more general Kleinian groups (which was also known to Series and used, for example, in work with Keen [36]) is obtained by a simple extension of the Fuchsian theory with the observation that there exist geodesic 'obstructions' coming from a multiply-connected covering space $\Omega(G)$.

2.6 Theorem (Dehn's algorithm for Kleinian groups)**.** *Suppose that G is a geometrically finite Kleinian group such that $S_G = \Omega(G)/G$ is a non-empty hyperbolic surface. Choose a finite-sided fundamental domain D for the action of G on $\Omega(G)$, and let $\langle X_1, X_2, \dots \rangle$ be the system of generators for G made up of the side-pairings for D. For each i, let X_i^* be the projection to S_G of the sides of D paired by X_i; we equip transverse germs of arcs to X_i^* with an orientation, so that lifts are oriented 'parallel' to the flow of X_i (Fig. 12). Alternatively, we view the orbit of D under G as a dual complex to the directed Cayley graph of G on the indicated generators, and orient arcs across sides of D in the direction of the dual edge of the Cayley graph.*

Let γ be a hyperbolic geodesic on S_G which is not boundary-parallel[1] and which does not bound a compression disc in O_G (i.e. there does not exist a topological disc embedded in N_G

[1]Actually, this is not a necessary assumption—one will just obtain parabolic or ellitpic elements, or with more general setup (like taking a G-invariant set $\Delta \subset \Omega(G)$ and looking at Δ/G) loxodromics representing boundary components.

with boundary equal to γ, Fig. 13).

Fix a basepoint on γ and an orientation of γ; perturb γ via small homotopies so that every intersection between γ and an X_i^* is transverse.[2] *Construct a word in G in the following way: walk along γ from the basepoint, and when you cross some X_i^* append X_i to the word (if the crossing is in the direction of the orientation of X_i^*) or X_i^{-1} (if the crossing is contrary to the orientation). By geometric finiteness, only finitely many generators are appended to this word by the time the walk returns to the basepoint.*

The resulting finite word is a loxodromic element in G, and the axis of this element projects to the homotopy class of γ.

§3. Sociology

We would like topological methods for detecting ghosts of finite structures on the boundary at infinity; the main result which we want to state is the so-called *ending lamination theorem*, Theorem 3.4. In order to understand the part of this theorem which deals with Riemann surfaces on the boundary of the 3-manifold, we need to understand how the geometry of the boundary is reflected in the core of the manifold.

Let $\text{h.conv}\,\Lambda(G)$ be the hyperbolic convex hull of $\Lambda(G)$, which looks something like Fig. 14, and set $M_G := (\text{h.conv}\,\Lambda(G))/G$. This manifold is the **convex core** of O_G; if G is non-elementary and non-Fuchsian, then M_G is a deformation retract of O_G. The boundary ∂M_G is a **pleated surface**; that is, there is a hyperbolic surface S together with an isometry $f : S \to \partial M_G$ (with respect to the intrinsic metric of ∂M_G) such that every point $s \in S$ lies in the interior of a geodesic in S which is mapped by f to a geodesic arc in ∂M_G, and such that f is homotopically incompressible (i.e. $f_* : \pi_1(S) \to \pi_1(M_G)$ has trivial kernel). The **pleating locus** of ∂M_G is the set of points through which there is exactly one geodesic in S which is mapped to a geodesic in M_G. The pleating locus is the canonical example of a measured geodesic lamination.

3.1 Definition. A **lamination** L on a surface M is a closed subset $A \subseteq M$ (the **support** of L) together with a local product structure on A: that is, there is a family of open sets $\{U_i\}$ of M which cover A, together with charts $\phi_i : U_i \to \mathbb{R} \times \mathbb{R}$ for each i, such that $\phi_i(A \cap U_i) = \mathbb{R} \times B$ ($B \subseteq \mathbb{R}$) for each i, and such that the transition maps are of the form

$$\phi_i \phi_j^{-1}(x, y) = (f_{i,j}(x, y), g_{i,j}(y))$$

for all i, j and for all $x \in \mathbb{R}, y \in B$. If the leaves are all geodesics, then L is a **geodesic lamination**.

A **transverse measure** on L is a regular measure defined on the set of embedded intervals in M which are transverse to every leaf that they meet. A lamination with a transverse measure is called a **measured lamination**. If S is a pleated surface in a manifold M, then S has a natural measure, namely the bending measure shown in Fig. 15.

Giving data of a geodesic lamination is equivalent to giving the data of **train tracks** on the manifold. Train tracks were introduced by Thurston (Fig. 16); the point is that one can conglomerate together parallel leaves of a lamination, as in Fig. 17, while keeping track of the measure additively. We give a formal definition following Kapovich [31] and Penner [48]. See also the classic notes inspired by Thurston's surface lectures [25].

[2] The resulting word is independent, up to conjugation and inversion in G, of all three of these choices.

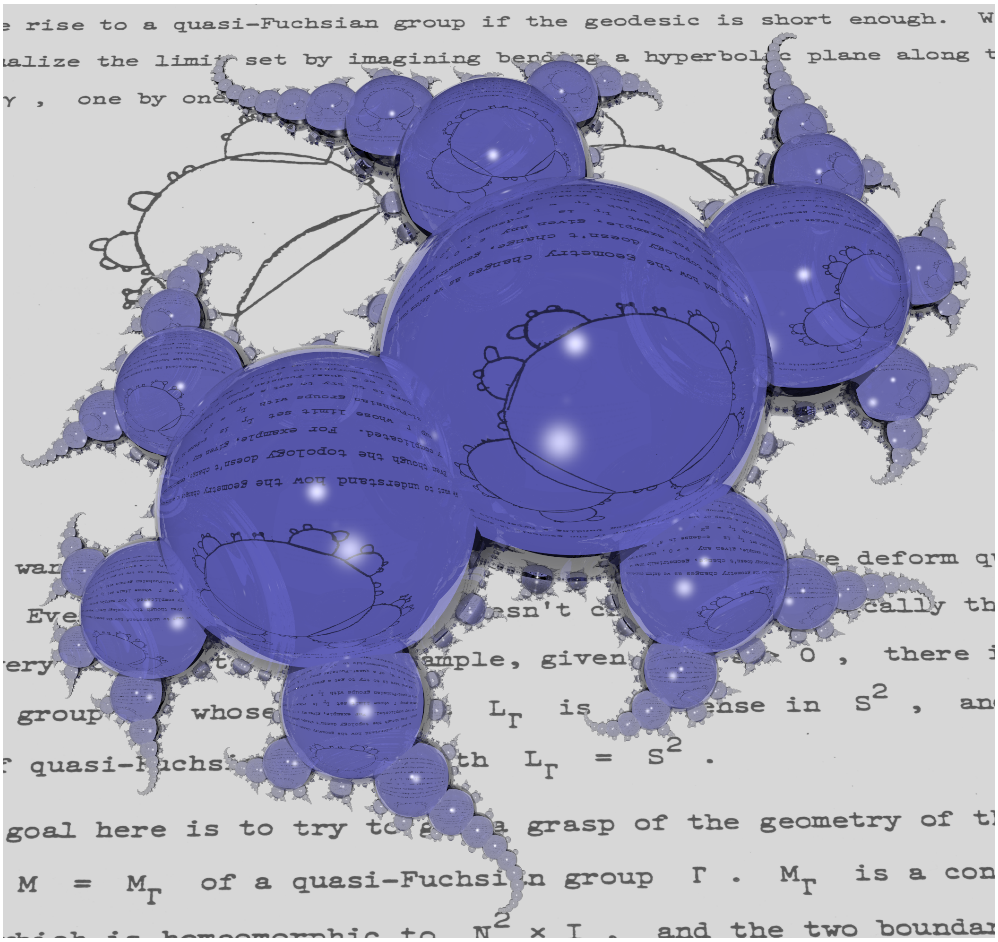


Figure 14: Jeffrey Brock and David Dumas, *Bug on notes of Thurston*. `https://www.dumas.io/poster/`.

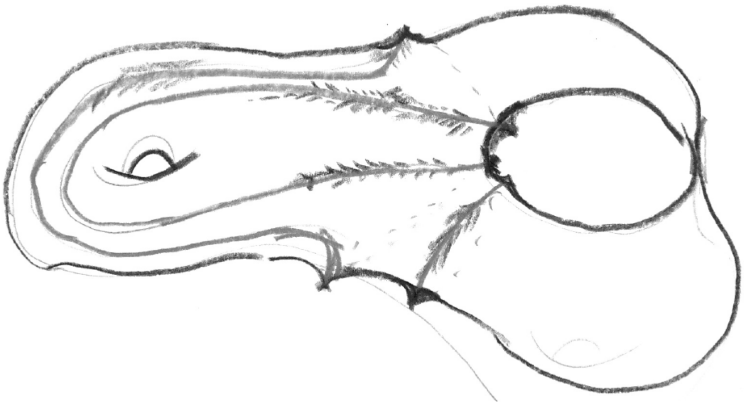


Figure 15: A geodesic measured lamination is a union of disjoint geodesics, where each geodesic (leaf of the lamination) is labelled with a weight that can be thought of as the 'angle of bending' as you walk across the leaf. The boundary of the convex core of a 3-manifold, $\partial\, \mathrm{h.conv}\, \Lambda(G)/G$, is really obtained as a physical pleated surface in an ambient space, and so in this case the angle is a real-life angle that you can measure with a protractor.

3.2 Definition. Let S be a hyperbolic surface. A set of **train tracks** on S is given by a trivalent graph G, together with an embedding $G \to S$ such that every edge is C^1 and such that at every vertex there is a distinguished tangent line. We call the edges **branches** and the vertices **junctions**[3] Isomorphism of train tracks is given by C^1 isotopy.[4]

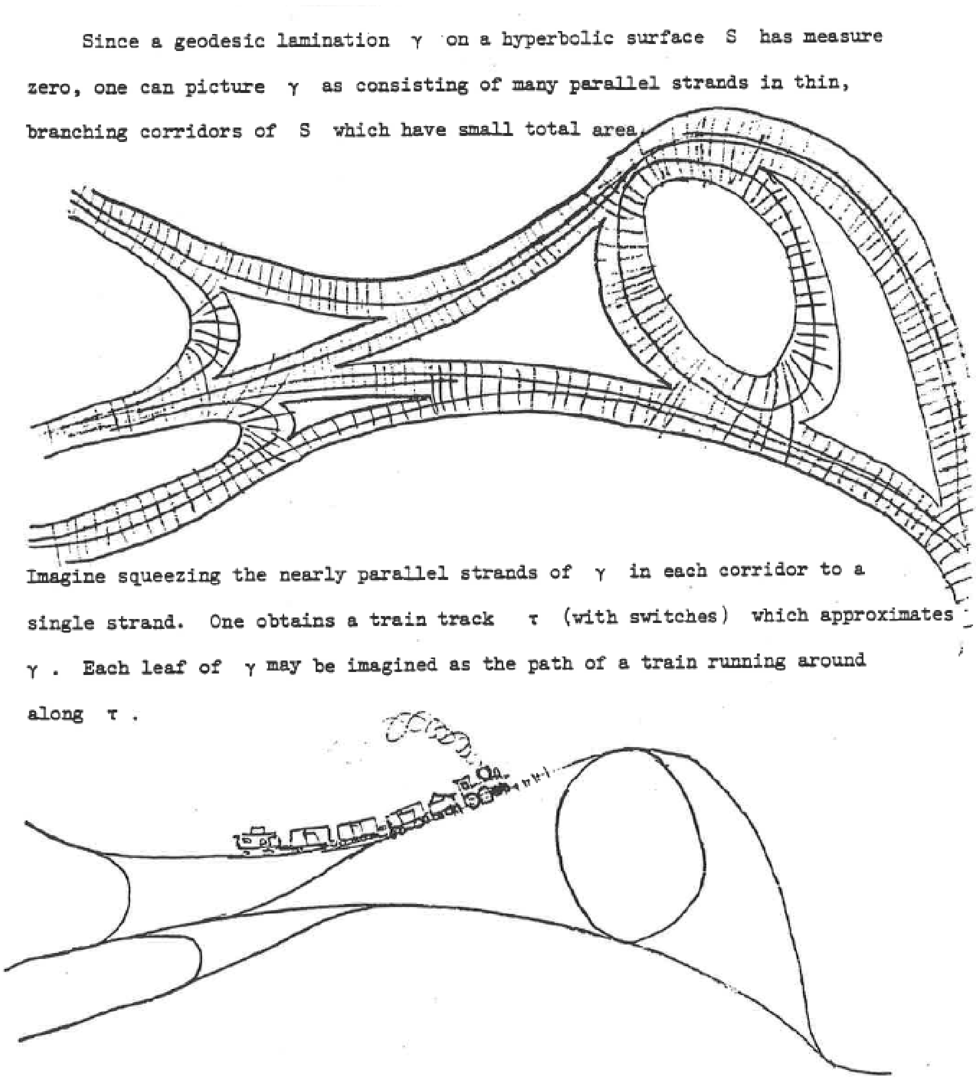
A **transverse measure** on a set of train tracks τ is an assignment of a non-negative real number to each branch of τ, such that if v is a vertex then the 'incoming' measure is the sum of the two 'outgoing' measures. The support of the measured train track is the subgraph of branches with positive measure.

A geodesic lamination is **maximal** if there is no geodesic lamination which it is strictly contained within (at the level of set inclusion of leaves).

3.3 Lemma. *Let S be a compact hyperbolic Riemann surface of genus g. A maximal geodesic lamination S consists of $3g-3$ geodesics, and defines a pair-of-pants decomposition of S if every leaf is nil-measure in S. If S is not compact with n paired cusps and no other boundary components, then a maximal geodesic lamination has $3g-3-n$ geodesics and these determine a pair-of-pants decomposition with some legs possibly length zero (at the punctures).*

[3]We now come to a terminological issue. In English, the apparatus by which a train is deflected from one track to another is called a *set of points*, while the point at which this deflection occurs is called a *junction*; in America, the apparatus is called a **switch** (and a junction is still a junction). Thus, in translating Thurston's work to English, one is tempted to replace 'switches' with 'points'; unfortunately, using the word 'point' would be ambiguous since we are doing geometry, and so we must settle for the slightly less correct translation of 'junction'.

[4]Actually slightly weaker isomorphism is needed for the equivalence with laminations, but this is not important.



Since a geodesic lamination γ on a hyperbolic surface S has measure zero, one can picture γ as consisting of many parallel strands in thin, branching corridors of S which have small total area.

Imagine squeezing the nearly parallel strands of γ in each corridor to a single strand. One obtains a train track τ (with switches) which approximates γ. Each leaf of γ may be imagined as the path of a train running around along τ.

Figure 16: Reproduction of page 8.51 of Thurston's notes [64].

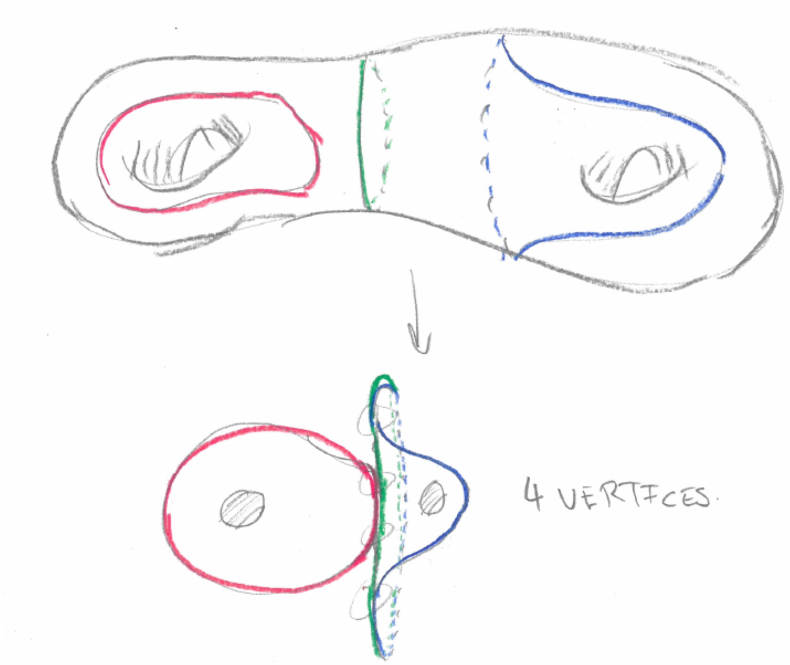


Figure 17: Compressing a (maximal) geodesic lamination to a train track in genus two.

We now have enough geometric background for the ending lamination theorem. For detailed information, see the exposition of Minsky [46]. I believe it was first conjectured by Thurston, for instance it appears in his 1982 list of problems [65].

3.4 Theorem (Ending lamination theorem)**.** *If M is a hyperbolic 3-manifold with finitely generated fundamental group, then M is determined up to isometry by its homeomorphism class and its end invariants.*

For completeness, we should say what we mean formally by 'end'. Let X be a topological space, and suppose that there is an ascending sequence

$$K_1 \subset K_2 \subset \cdots$$

of compact subsets such that $\bigcup_{i\in\mathbb{N}} \int X_i = X$. Then an **end** of X is a *descending* sequence of open sets

$$U_1 \supset U_2 \supset \cdots$$

such that each U_i is a connected component of the respective complement $X \setminus K_i$.

We can now say what we mean by 'end invariants'. There are three kinds, shown in Fig. 18.

1. If M is finite volume, then the ends of M are empty, or cusps. In this case, the end invariants are trivial.

2. If M is infinite volume but $\pi_1(M)$ is geometrically finite, then the end invariants are the complex structures on the Riemann surfaces ∂M.

3. If M is infinite volume and $\pi_1(M)$ is geometrically infinite, then the end invariants are complex structures on the components of the boundary (as in the geometrically finite case) together with certain laminations in the 3-manifold. What is going on here is that the group was obtained by taking a finite end and a dense lamination on that end, and pinching the lamination down to zero (i.e. the convex core bending measure becomes zero); the finite component of the end vanishes, but the ghostly lamination is left behind. More precisely there is sequence of embedded laminated surfaces in the manifold which vanish in the limit, and the union of these laminations gives a lamination of the neighbourhood of the end by surfaces transverse to those in the embedded sequence.

We will briefly give definitions and pictures for the quasi-sociology of Kleinian groups. We will begin with sociology, though: that is, the representation variety. Our primary reference is Kapovich [31, Chapter 8], for a change.

Let $(\rho_j)_{j\in\mathbb{N}}$ be a sequence of faithful discrete representations of a finitely generated group[5] G into $\mathbb{M} \simeq \mathsf{PSL}(2,\mathbb{C})$. We say that the sequence (ρ_j) **converges algebraically** to some $\rho \in \mathrm{Hom}(G,\mathbb{M})$ if we have $\rho_j(g) \to \rho(g)$ for all $g \in G$, with respect to the standard

[5]We recall, finitely generated does NOT imply geometrically finite (we shall see some examples, but the first were due to Greenberg [29]), but it DOES imply tameness on the level of 3-manifolds (this is Marden's tameness theorem) and it DOES imply analytic finiteness of the Riemann surface (this is Ahlfors' finiteness theoem).

Figure 18: Ends of a hyperbolic 3-manifold.

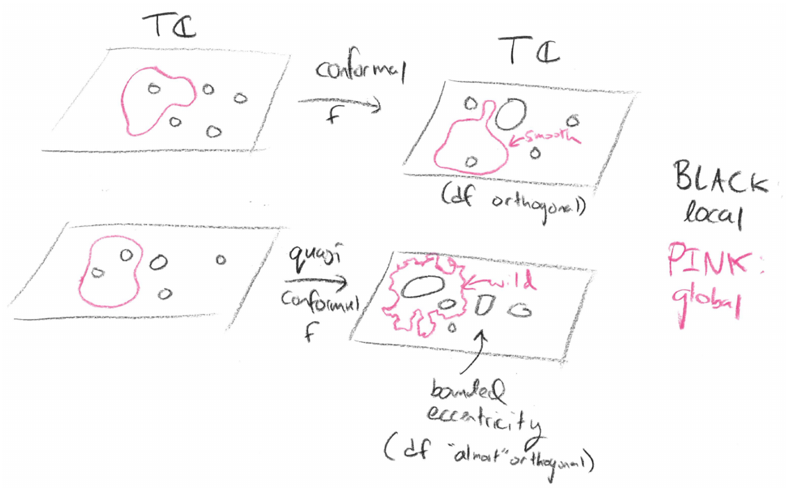


Figure 19: Local and global pictures of a quasiconformal map.

topology on $\mathbb{M}$. Let $D(G, \mathbb{M})$ be the space of discrete, faithful representations of G; then $D(G, \mathbb{M})$ is closed in the representation variety $R(G, \mathbb{M}) = \mathrm{Hom}(G, \mathbb{M})/\mathbb{M}$ (this is Chuckrow's theorem [31, Theorem 8.4]).

A second notion of convergence does not coincide with convergence of representations. Suppose that $G_j \leq \mathbb{M}$ is a sequence of subgroups ($j \in \mathbb{N}$). The **geometric limit** of the sequence is a group G_∞ such that

1. for each convergent sequence $(g_{j_i}) \leq G_j$, the limit $\lim_{i\to\infty} g_{j_i}$ lies in G_∞
2. for eacg $g \in G_\infty$ there is a sequence $g_j \in G_j$ such that $g_j \to g$.

The point is that geometric convergence of groups corresponds to quasi-isometric convergence of the 3-manifolds. The geometric limit might be strictly larger than the algebraic limit. If the Hausdorff limit of the sequence $\Lambda(G_j)$ is equal to the limit set of the geometric limit, then the algebraic and geometric limits coincide.

Recall before we continue that a quasiconformal map is a homeomorphism $f : U \to \hat{\mathbb{C}}$ such that the local dilation $\mu = (\partial f / \partial \overline{z}) \big/ (\partial f / \partial z)$ is globally bounded (μ is a bounded complex function on U, called the **Beltrami coefficient**). Locally on $T\mathbb{C}$ a quasiconformal map sends infinitesimal circles to infinitesimal ellipses of bounded eccentricity, but globally on $\mathbb{C}$ it can have fairly wild behaviour—see Fig. 19.

Fix a basepoint $\rho \in D(G, \mathbb{M})$. By the λ-lemma of Mañé, Sad, and Sullivan [40] as generalised by Slodkowski [62, 63] and Earle, Kra, and Krushkal' [17] (a modern textbook reference for the surrounding theory is Astala, Iwaniec, and Martin [2, Chapter 12] and a brief introductory survey appears in the author's paper [22]), small holomorphic deformations of the coefficients of ρ to new representations $\tilde{\rho}$ induce quasiconformal maps $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\tilde{\rho}(G) = \phi\rho(G)\phi^{-1}$. Conversely, if ϕ is a quasiconformal map with Beltrami coefficient μ satisfying $g^*\mu = \mu$ for all $g \in G$ then $\phi\rho\phi^{-1}$ is a discrete faithful representation. We define the **quasiconformal deformation space** of G to be the open subset of $D(G, \mathbb{M})$

$$\mathrm{QH}(G) = \frac{\{\rho \in \mathrm{Hom}(G, \mathbb{M}) : \exists \rho\text{-equivariant q.c. homeo. } \phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}\}}{\mathbb{M}}.$$

If $\Omega(G)/G$ is the disjoint union of the connected Riemann surfaces $\Sigma_1, ..., \Sigma_n$, then $\mathrm{QH}(G)$ is a quotient space of the space $\mathrm{Teich}(\Sigma_1) \times \cdots \mathrm{Teich}(\Sigma_n)$ by a subgroup $\widehat{\mathrm{Mod}}(G)$ of the mapping class group $\mathrm{Mod}(G)$; if G is geometrically finite, then $\widehat{\mathrm{Mod}}(G)$ is the group generated by Dehn twists along curves bounding compression discs in O_G.

Roughly speaking, this deformation space agrees with the natural algebraic deformation space.

3.5 Theorem. *Let $G : \mathbb{C}^r \to \mathsf{PSL}(2, \mathbb{C})$ be algebraic. (One should think of this as being a single group varying on r parameters.) Fix a basepoint $Z \in G$, and let* $\mathrm{Fam}(G, Z)$ *be the set of $\tilde{Z} \in \mathbb{C}^r$ such that $G(\tilde{Z})$ is group-isomorphic to $G(Z)$ and such that $\Omega(G(\tilde{Z}))/G(\tilde{Z})$ is homeomorphic to $\Omega(G(Z))/G(Z)$. Then* $\mathrm{Fam}(G, Z)$ *is biholomorphic to* $\mathrm{QH}(G(Z))$*, with the biholomorphism locally given by the λ-lemma and globally by the inclusion maps* $\mathrm{Fam}(G, Z) \hookrightarrow D(G(Z), \mathbb{M}) \hookleftarrow \mathrm{QH}(G(Z))$*, exactly when G hits every conjugacy class of $G(Z)$ in $\mathbb{M}$ exactly once.*

This theorem will be discussed in more detail in [19]. In particular, one can replace the final assumption with weaker ones and obtain more information.

We now do some quick examples.

3.6 Example (The Maskit slice, $\mathcal{M}$). The Teichmüller space of the group G_{3i} from Example 1.6 is naturally identified with (a natural quotient of) the set of $\mu \in \mathbb{C}$ such that

$$G_\mu = \left\langle \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} -i\mu & i \\ i & 0 \end{bmatrix} \right\rangle.$$

is discrete and has quotient surface homeomorphic to that of G_{3i}. See [47, 35].

A **Fuchsian group** is a pair (G, Δ) where Δ is a round open disc and G is a Kleinian group which preserves Δ and $\hat{\mathbb{C}} \setminus \Delta$. As an exercise one can show that $\Lambda(G) \subseteq \partial\Delta$. See

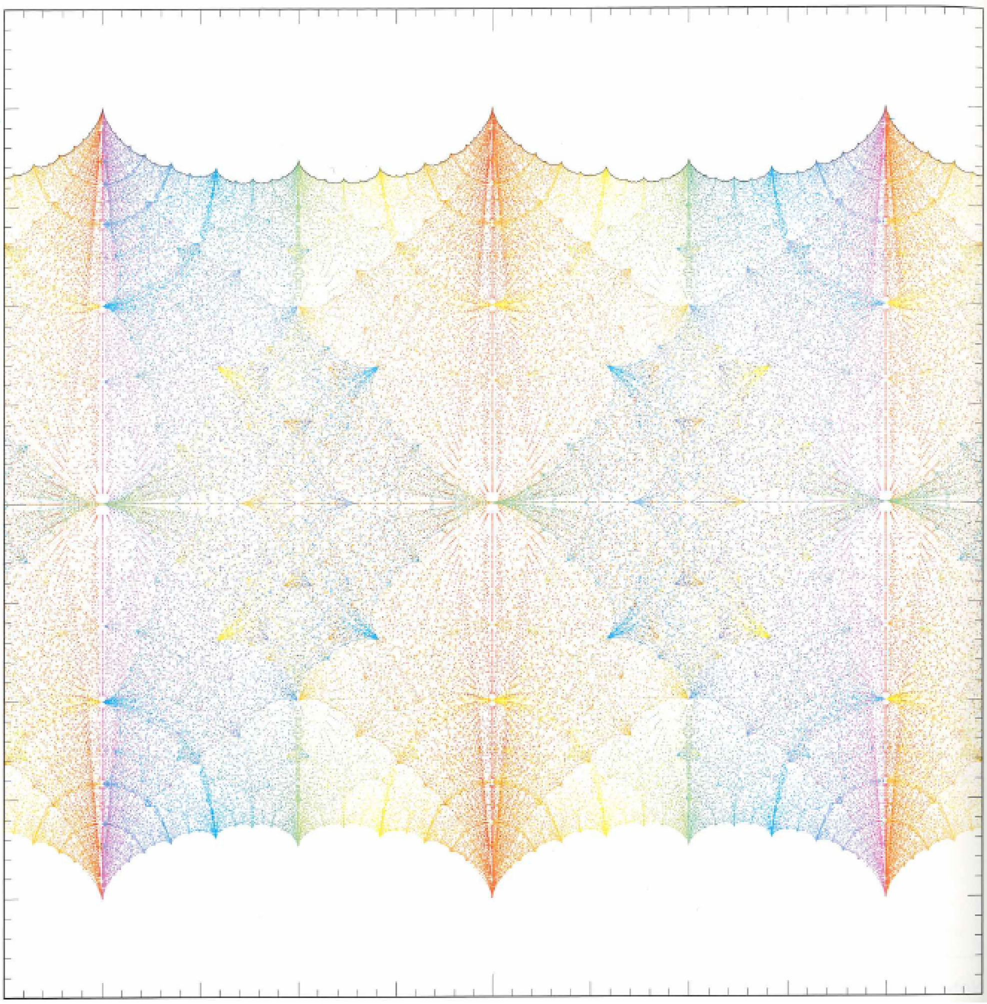


Figure 20: Maskit slice from [47, p. 288].

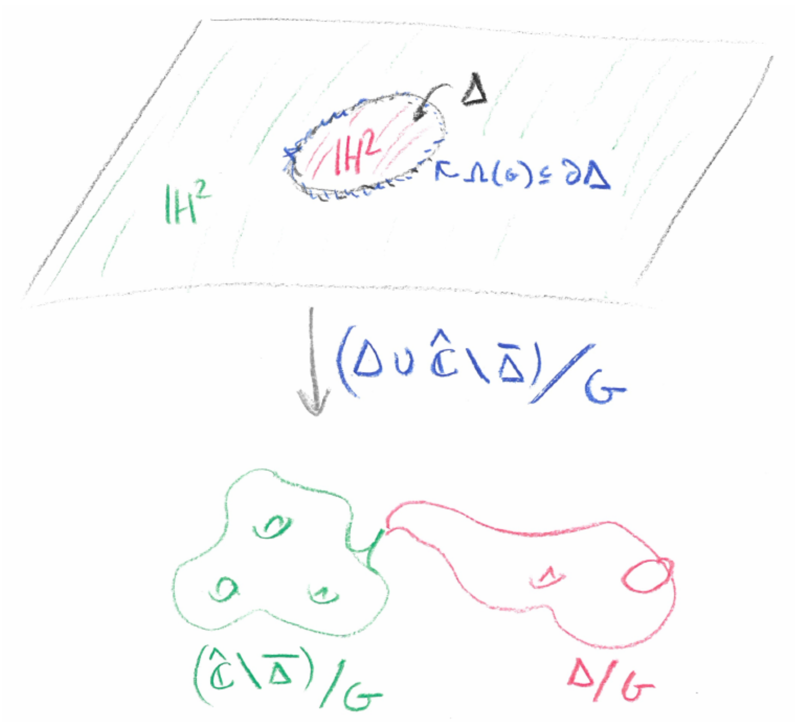


Figure 21: The action of a Fuchsian group produces two Riemann surfaces as quotients of two copies of $\mathbb{H}^2$; if the group has limit set dense in the circle then the two surfaces are disjoint—possibly joined by a shared cusp, as in this picture—or are glued together along deleted discs (this second situation is not pictured here, but easy examples are given by the Riley groups with real parameter).

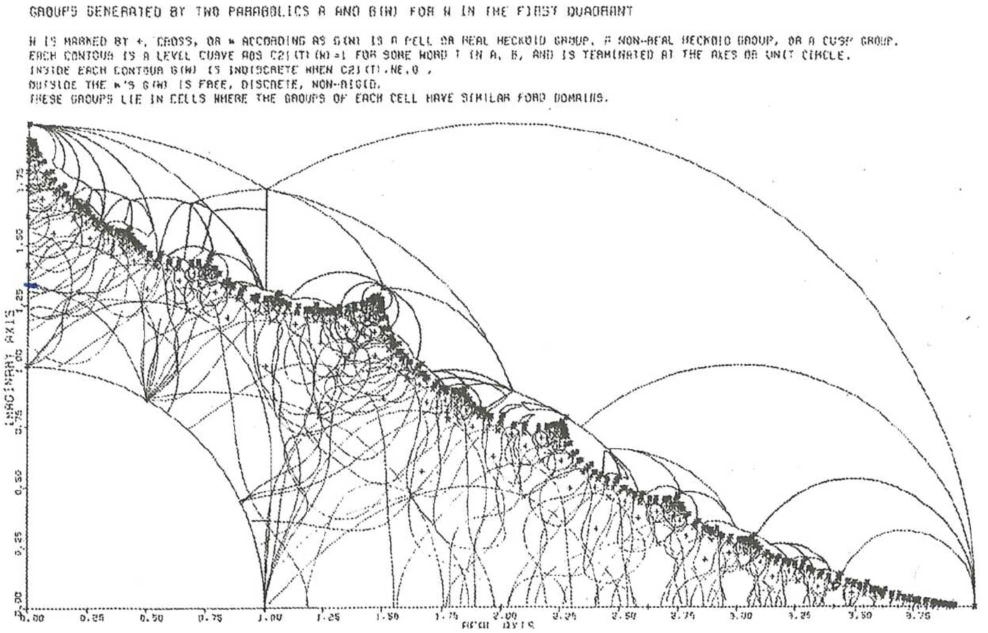


Figure 22: Riley's plot of two-bridge link groups in the $(+,+)$-quadrant of $\mathbb{C}$, taken from [1, Figure 0.2a].

Fig. 21 for a picture of the geometry. A **quasi-Fuchsian group** is a Kleinian group of the form $\tilde{G} = \phi G \phi^{-1}$ where G is Fuchsian and ϕ is a quasiconformal homeomorphism. In this case $\tilde{G}$ preserves the topological discs $\phi\Delta$ and $\phi(\hat{\mathbb{C}} \setminus \Delta)$ and $\Lambda(\tilde{G})$ is a subset of the topological circle bounding these quasidiscs. We say that $\tilde{G}$ is **of the first kind** if $\Lambda(\tilde{G}) = \partial\phi\Delta$. In this case, $\Omega(\tilde{G})/\tilde{G}$ is a disjoint union of two hyperbolic Riemann surfaces. A **Bers slice** is a subset of the moduli space of $\tilde{G}$ such that the complex structure of one component is held fixed. This gives an embedding of the Teichmüller space of the non-fixed component into the Teichmüller space of the group. A **Maskit slice** is a Bers slice such that the fixed component is a thrice-punctured sphere.

Here is an example for quasi-Fuchsian groups which are of the second kind (i.e. the limit set does not fill the quasicircle).

3.7 Example (The Riley slice, $\mathcal{R}$)**.** The Teichmüller space of the group Γ_{3i} from Example 1.5 is naturally identified with the set of $\rho \in \mathbb{C}$ such that

$$G_\mu = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle.$$

is discrete and has quotient surface homeomorphic to that of Γ_{3i}. See [36, 22, 21, 24]. See Fig. 22, Fig. 23, and Fig. 24.

Both Example 3.6 and Example 3.7 are slices through the boundary of genus 2 Schottky space; we can even pass naturally from one to the other via blowing up cusps (more

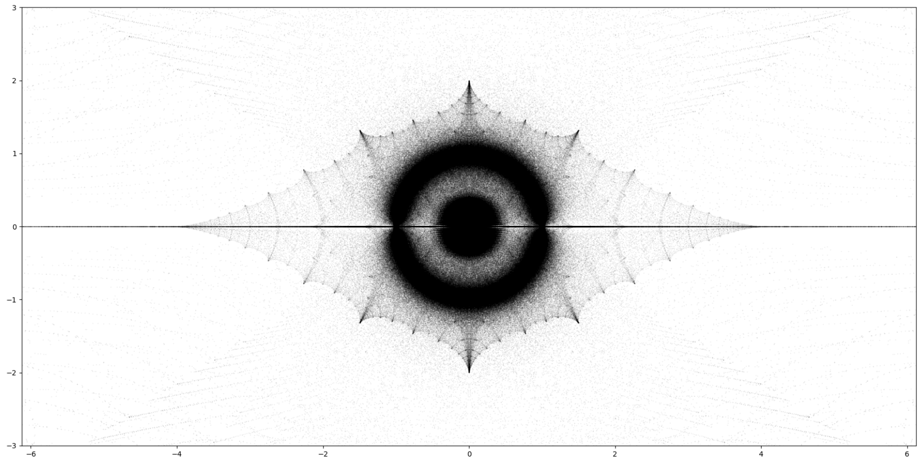


Figure 23: A higher resolution Riley slice.

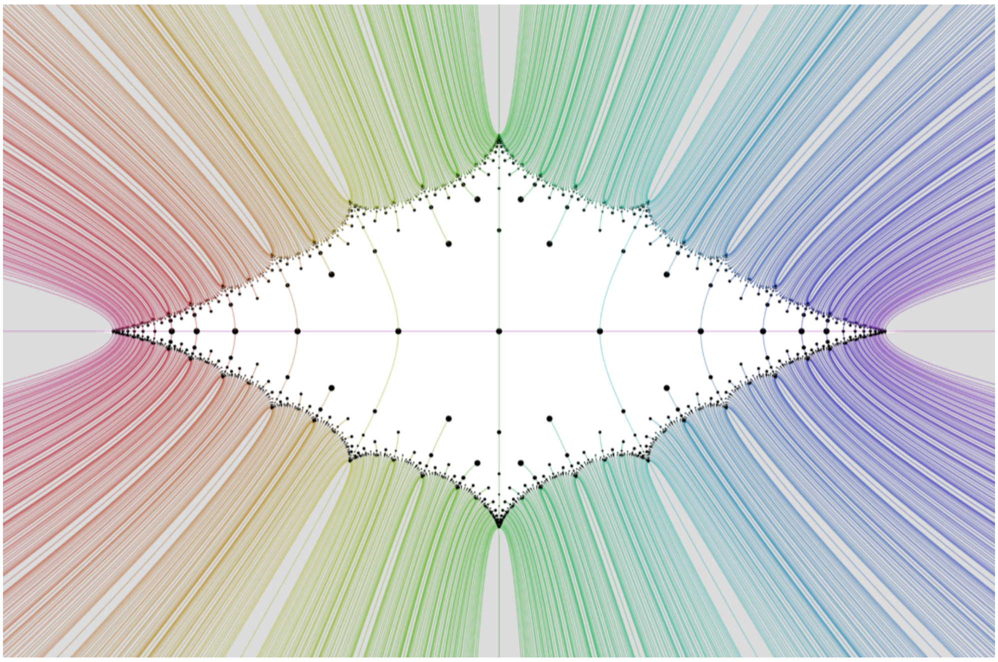


Figure 24: Riley slice with pleating rays, courtesy Y. Yamashita.

precisely, we use the Maskit combination theorems of [42, Chapter VII], see the lecture notes of Series [59] for a very comprehensive and explicit description of the combinatorial group theory).

Let $\mathcal{PML}(\Sigma)$ be the projective space of measured laminations on the surface Σ. Then $\mathcal{PML}(\Sigma) \cup \mathrm{Teich}(\Sigma)$ is the **Thurston compactification** of the Teichmüller space. It descends to the Teichmüller space of a group G since a Dehn twist along a compression disc can be undone by isotopy in the 3-manifold. (The point here is that the twisting is done through the 3-manifold, not intrinsically on the surface, and so it is necessary for there to be an ambient manifold.) Even better, if G is quasi-Fuchsian then there is a natural coordinate system on the Bers slice of G given by measured laminations [54]; and it is natural to conjecture that this construction extends to all geometrically finite groups—for instance the Riley slice theory verifies this for a class of quasi-Fuchsian groups of the second kind. In any case, the point is to assign to a group G the bending lamination of $\partial\,\mathrm{h.conv}\,\Lambda(G)/G$.

Remark. The main theme of the remainder of this note is the interplay between the different ways of keeping track of the canonical measured lamination.

The leaves and flat pieces are detected in the group structure via the concept of peripheral subgroups.

3.8 Definition. An **F-peripheral subgroup** in a Kleinian group G is a Fuchsian subgroup $F \leq G$ with invariant disc Δ such that $\Delta \subseteq \Omega(G)$.

If such a group (F, Δ) exists, then F acts as a group of hyperbolic isometries on the $\mathbb{H}^2$ embedded into $\mathbb{H}^3$ as the subsurface $\mathrm{h.conv}\,\partial\Delta$. This dome does not lie entirely in the boundary $\partial\,\mathrm{h.conv}\,\Lambda(G)$ except in very special groups, but F will preserve the subset of the dome which is contained in $\partial\,\mathrm{h.conv}\,\Lambda(G)$ (it preserves $\partial\,\mathrm{h.conv}\,\Lambda(G)$ and $\partial\,\mathrm{h.conv}\,\Lambda(F)$, so it preserves their intersection). We now observe that $\partial\,\mathrm{h.conv}\,\Lambda(G) \cap \partial\,\mathrm{h.conv}\,\Lambda(F)$ is a hyperbolic polygon in $\partial\,\mathrm{h.conv}\,\Lambda(G)$ and is in fact the Neilsen region of F [4, §8.5]. In particular,

3.9 Lemma. *Let F be a F-peripheral subgroup of a Kleinian group G; let S be the set of geodesics in $\mathbb{H}^3$ which bound the flat piece of $\partial\,\mathrm{h.conv}\,\Lambda(G)$ preserved by F. Each element of S descends to a leaf of the pleating locus of ∂M_G. For each each $s \in S$ let g_s be an element of G representing S, such that all the g_s are chosen compatibly with respect to a fixed generating set (Theorem 2.6). Then $F = \langle g_s \,:\, s \in S\rangle$ as a subgroup of G.*

Presumably not all flat pieces in the canonical pleated surface come from F-peripheral subgroups; one might imagine that some could come from quasi-Fuchsian groups, for instance:

3.10 Conjecture. *Let G be a geometrically finite Kleinian group. If S is a flat piece of ∂M_G bounded by three curves (possibly of length zero and represented by parabolics, Lemma 3.3) then those curves are represented by real-trace elements of G and those elements generate an quasi-Fuchsian peripheral subgroup of G uniformising S. The quasi-Fuchsian peripheral subgroups should come from small deformations of groups with Fuchsian peripheral groups.*

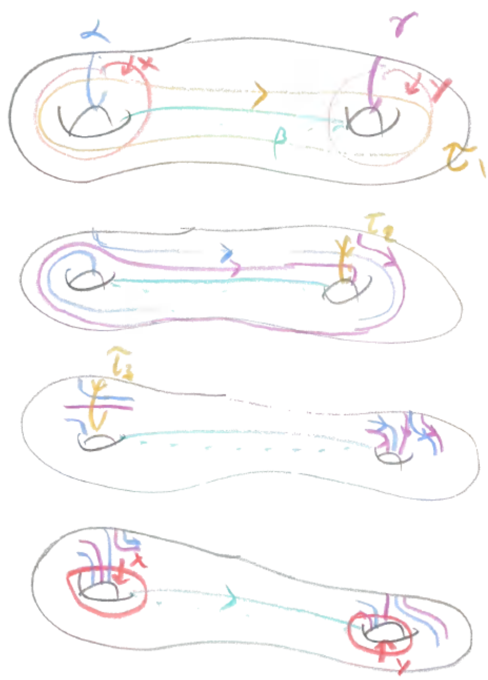


Figure 25: A more complicated maximal geodesic lamination on a genus two surface. We may read off that $W(\alpha) = X^2Y$, $W(\beta) = xY$, and $W(\gamma) = XY^2$ (with respect to the system of generators $\langle X, Y \rangle$ indicated).

3.11 Example. The general genus two Schottky group is of the form

$$G = \left\langle \begin{bmatrix} \mu & 1 \\ 0 & 1/\mu \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ \rho & 1/\lambda \end{bmatrix} \right\rangle$$

for suitable choices of $\mu, \lambda, \rho \in \mathbb{C}$. This group uniformises a 3-manifold with compact genus two surface at infinity. By taking the 'obvious' maximal geodesic lamination represented by X, Y, Xy and 'twisting' it via the three Dehn twists τ_1, τ_2, τ_3 indicated in Fig. 25, we obtain a more complicated system which is represented by $U = X^2Y$, $V = xY$, and $W = XY^2$ (where we use the convention that inversion of letter case indicates inversion of the generators, i.e. $x = X^{-1}$). This lamination will be the pleating lamination of ∂M_G whenever these three words are simultaneously hyperbolic (and the group is free and discrete).

Let us choose $\operatorname{tr} U = -3$, $\operatorname{tr} V = -4$, and $\operatorname{tr} W = -5$; taking a root with $|\rho|$ maximal (in analogy with the Riley slice situation, where we have conjectured in the past that the branch of the Farey polynomial which hits the boundary last as $\operatorname{tr} W_{p/q} \to -2$ is the pleating ray) we obtain the numerical approximation

$$\mu = 0.586304 + 2.06222i, \;\; \lambda = 0.190781 + 2.62721i, \;\; \text{and } \rho = 5.99998 + 0.0842261i.$$

The genus two situation (more generally, every surface for which a maximal lamination is made up of exactly three curves) is the only situation where multiple pairs of pants in the decomposition (equivalently multiple flat pieces) are bounded by exactly the same leaves. In particular, if we just apply the theory we have studied up until now we only have a single peripheral group F rather than two non-conjugate ones. To fix the problem we observe that every genus two surface (as well as the relevant degenerations, like the four-punctured sphere) admits a hyperelliptic involution; even better there exists an order 2 elliptic $\Phi \in \mathbb{M}$ such that $\Phi G = G$ and such that ΦF is an F-peripheral subgroup of G that is not conjugate in G to F. Actually writing down the matrix representing Φ is very straightforward, as observed by Keen [33] (see also the detailed geometric discussion in the special case that G is Fuchsian by Gilman and Keen [28]): one can take the Lie bracket $XY - YX$. This gives us

$$\Phi = \begin{bmatrix} \rho & 1/\lambda - \lambda \\ \rho(1/\mu - \mu) & -\rho \end{bmatrix}$$

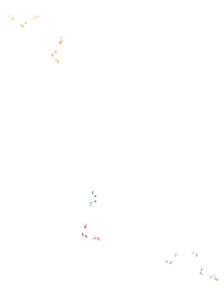
and hence we have the two peripheral groups $\langle W, V, U \rangle$ and $\langle \Phi W, \Phi V, \Phi U \rangle$ given numerically as:

$$F_1 = \left\langle \begin{bmatrix} -2.951 - 0.067i & 0.630 - 0.226i \\ -1.100 - 0.702i & -0.048 + 0.0669i \end{bmatrix}, \begin{bmatrix} -4.796 + 0.165i & -0.027 + 0.379i \\ 3.344 + 12.423i & 0.797 - 0.165i \end{bmatrix}, \begin{bmatrix} -4.972 - 0.061i & -0.142 - 0.0 \\ 6.204 + 1.221i & -0.028 + 0.0 \end{bmatrix} \right.$$

$$F_2 = \left\langle \begin{bmatrix} -19.635 + 2.7725i & 4.012 - 1.168i \\ 13.033 + 49.057i & -4.718 - 9.347i \end{bmatrix}, \begin{bmatrix} 7.999 - 11.493i & -0.824 - 0.099i \\ -4.333 - 2.785i & 0.993 + 0.378i \end{bmatrix}, \begin{bmatrix} -27.174 - 19.635i & -0.665 - \\ -25.413 + 67.413i & 0.218 + \end{bmatrix} \right.$$

The three limit sets are shown in Fig. 26.

(a) Limit set of G.



(b) Limit set of F_1.

(c) Limit set of F_2.

Figure 26: Limit sets associated to Example 3.11.

Problems

1. Let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the affine torus.

   a) Show that every geodesic on T is either (i) dense in T or (ii) the projection of a line of rational slope. Call the latter $\gamma(p/q)$ where $(p,q) = 1$ or $p/q \in \{1/0, 0/1\}$.

   b) Show that every geodesic lamination on T consists of a single geodesic.

   c) Show that there is a natural $\mathsf{PSL}(2,\mathbb{Z})$ action on non-dense geodesic laminations and this action preserves the simplicial structure with cells $(p/q, r/s, (p+r)/(q+s))$ where $|ps - qr| = 1$.

2. a) Find conditions on $\lambda, \mu \in \mathbb{R}$ such that the group G generated by the two elements $\left[\begin{smallmatrix} \lambda & -\lambda \\ \lambda & (1-\lambda^2)/2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mu & -\mu \\ -\mu & (1+\mu^2)/2 \end{smallmatrix}\right]$ is discrete and has quotient a disjoint union of two punctured tori.

   The group should be Fuchsian of the first kind and hence the hyperbolic metrics on $\mathbb{H}^2$ and $-\mathbb{H}^2$ descend. Let T^* be the quotient of $\mathbb{H}^2$.

   b) Show that every *closed* geodesic on T^* is either (i) dense in T^* minus some small open neighbourhood of the cusp or (ii) has homology class $p\gamma_0 + q\gamma_\infty$ for $(p,q) = 1$ or $p/q \in \{1/0, 0/1\}$ and with some fixed homology basis $(\gamma_0, \gamma_\infty)$. Even better, there is a bijection between points visible from the origin in $H_1(T^*) \simeq \mathbb{Z}^2$ and non-dense closed geodesics on T^*.

   c) Show that every geodesic lamination on T^* which does not meet the cusp end ($\iff$ has compact support) consists of a single geodesic.

   d) Show that there is a natural $\mathsf{PSL}(2,\mathbb{Z})$ action on non-dense geodesic laminations with compact support and that this action preserves the simplicial structure with cells $(p/q, r/s, (p+r)/(q+s))$ where $|ps - qr| = 1$.

3. (We saw the Maskit and Riley slices in the lecture. This problem contains both of them.) Let $G = G(\lambda, \mu, \rho)$ be the group generated by

$$X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda^{-1} \end{bmatrix}, Y = \begin{bmatrix} \mu & 0 \\ \rho & \mu^{-1} \end{bmatrix};$$

   the generic situation is that G is free and $\Omega(G)/G$ (it is a genus two Schottky group). Suppose $\lambda = 2$, then X represents the transformation $z \mapsto 4z + 2$ which sends the vertical line $\operatorname{Re} z = -1$ to the vertical line $\operatorname{Re} z = 1$. If $\mu = 2$ then the isometric circles of Y are the circles of radius $1/|\rho|$ centred around $-1/2\rho$ and $2/\rho$.

   a) Prove that these circles are contained strictly within the vertical strip $-1 < \operatorname{Re} z < 1$ iff the two following conditions hold (where $\rho = re^{i\theta}$):

$$r > \frac{1}{2}(\cos\theta + 2) < 1 \text{ and } r > (2\cos\theta + 1).$$

   That is, ρ lies in the mutual exterior of the two cardioids depicted in Fig. 27. This gives a rough bound on the family $\operatorname{Fam}(G(\lambda = 2, \mu = 2, \rho), |\rho| \gg 0)$, which is a one-dimensional slice through genus two Schottky space. (Compare

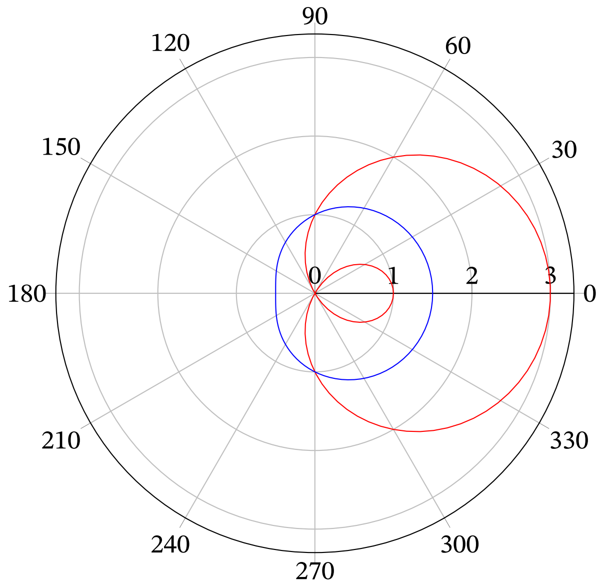


Figure 27: Bounds on the one-dimensional slice through genus two Schottky space.

this with the Riley slice, which is the slice $\lambda = \mu = 1$ through the boundary of this space.) Which points of the cardioids actually lie on the actual boundary of the deformation space?

b) Carry out an analysis similar to the torus questions for a genus two surface S. Such a theory should explain the following:

   i. Maximal geodesic laminations all have three geodesics, and there is a natural partition of the space of laminations given via density in non-zero measure subsets of the surface;

   ii. The geodesics can be viewed in some way as (isotopic to) projections of combinatorially defined curves with respect to the fundamental domain given in (a);

   iii. There is a bijection between the space of non-dense maximal geodesics and some natural subset of $H_1(S)$;

   iv. There is an action of some group like $\mathrm{PSL}(3, \mathbb{Z})$ on this subset.

c) If you take a maximal geodesic lamination with one lamination separating the surface into two tori-minus-discs (each with one of the other laminations) then this lamination can be pinched to a cusp. This reduces (topologically) to question 2 above. In particular, this explains why we ignore geodesics which hit the cusp. Check that your theory contains the theory of question 2.

4. (Open) Some open problems relating Kleinian groups to algebraic curves.

a) Give a strictly algebraic method for determining the canonical lamination on a Kleinian group.

b) What algebraic structure can be placed onto an algebraic curve in order to lift a complex structure to the structure of a visual boundary of a 3-manifold? (The answer should be more algebraic than 'a quadratic differential', for instance.)

c) What is the algebraic analogue of the procedure 'measured lamination → train track'?

§4. B-groups and other degeneracies

We will talk now about the kinds of groups which lie on the boundary. The original work on this is due to Bers [5] and Maskit [43]. A nice but slightly outdated survey was written by Canary [14].

Fix a geometrically finite Fuchsian group G, and let $\Gamma \in \partial\,\mathrm{QH}(G)$ (where the boundary is taken with respect to the algebraic topology). Automatically, Γ is group-isomorphic to G. There are several cases; let $\rho : G \to \Gamma \leq \mathbb{M}$ be a corresponding representation in $D(G, \mathbb{M})$.

1. If there exists some $\eta \in G$ such that η is loxodromic but $\rho\eta$ is parabolic, then Γ is called a **cusp group**. It is a famous result of McMullen [45] that such groups are dense in the boundary. In this case, Γ is geometrically finite and $\Lambda(\Gamma)$ is a nil-measure set. If every component of $\Omega(\Gamma)/\Gamma$ is a thrice-punctured sphere, then Γ is a **maximal cusp**. Such groups are dense in the boundary of every Schottky space (again by McMullen). We have already seen that complex structure is determined by the limit set; cusp groups have circle-packing limit sets, and are geometrically finite [34].

2. If Γ has a simply connected invariant component, then it is called a **B-group**. These were introduced by Bers [5], and by his results every group on the boundary of a quasi-Fuchsian space is a B-group. A B-group is called **degenerate** if it has exactly one component. These exist. In fact, most boundary groups of quasi-Fuchsian spaces of the first kind are degenerate [5, Theorem 14]. One can think of degenerate groups in this case being 'irrational', and cusp groups being 'rational'. All degenerate groups are geometrically infinite, by the previously cited result of Greenberg [29]. It is an open conjecture of Bers that every degenerate group does in fact arise on the boundary of some quasi-Fuchsian space.

3. On the boundary of quasi-Fuchsian spaces of the second kind, the equivalent of a degenerate group is a group with dense limit set in $\hat{\mathbb{C}}$ (rather than dense in a disc). These may be geometrically finite in special cases, but I believe in general they can also be geometrically infinite. (Example?)

The point, compared with the previous section, is the following picture.

4.1 Conjecture. *Let G be a geometrically finite Kleinian group which supports quasiconformal deformations, and let $Q = \mathrm{QH}(G)$ be its deformation space. Then:*

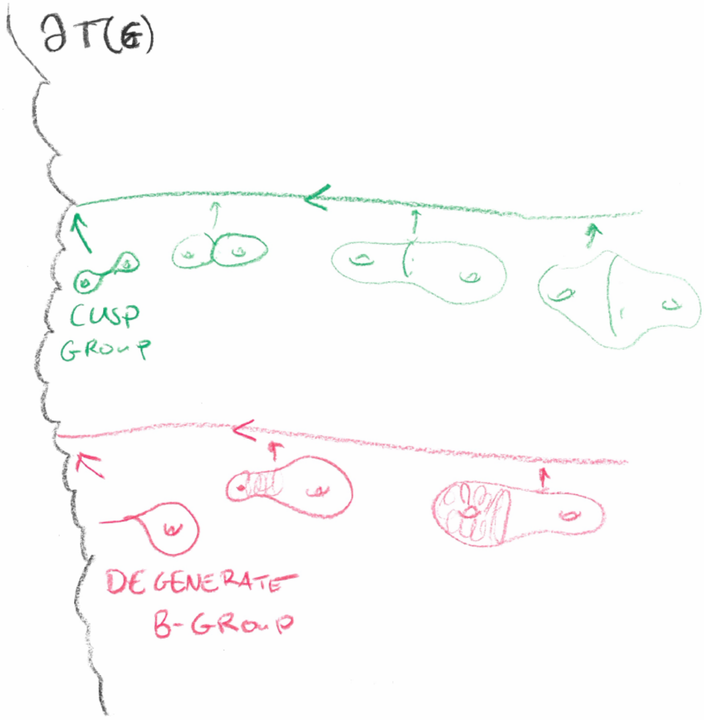


Figure 28: Here we see two deformations to the boundary of the Teichmüller space of groups uniformising genus two compact surfaces. The green degeneration is a pinching along a lamination with measure zero, which produces a new pair of cusps and no virtual ends, while the red end pinches down a geodesic with nonzero area and produces an unpaired cusp and the ghost of a virtual end.

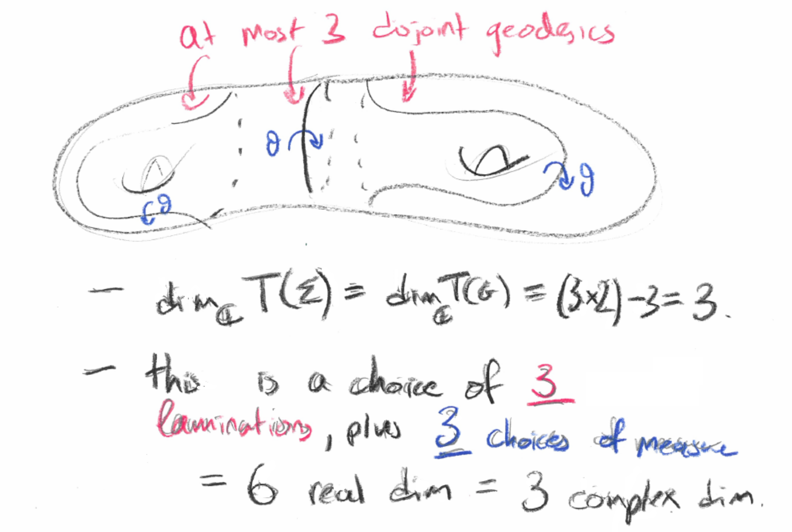


Figure 29: We see that the dimension of QH(G) comes from a choice of a maximal system of geodesics and a measure on each.

1. *There exists a geometric coordinate system on Q, namely* $Q = \prod_{s=1}^{n} \mathcal{ML}(S_i)$ *where* $S_1, \ldots, S_n$ *are the components of* $\Omega(G)/G$ *(c.f. Fig. 29).*[6]
2. *The boundary* ∂Q *admits a natural partition into cusp groups and non-cusp groups, where the non-cusp groups have limit sets of positive measure. Cusp groups are dense in* ∂Q*, but most groups (in the sense of category) on the boundary are non-cusp groups. Cusp groups correspond to choosing a lamination on the surface which is not dense and pinching a corresponding loxodromic element (or system of loxodromic elements) to parabolics, while non-cusp groups correspond to picking a dense lamination on a component of the surface and shrinking down the corresponding piece of the convex core to zero volume (so the component vanishes). See Fig. 28 for an explicit picture of this on a higher dimension Teichmüller space.*
3. *Topologically, Q is homeomorphic to a sphere of dimension equal to that of the Teichmüller space* Teich($\Omega(G)/G$)*.*

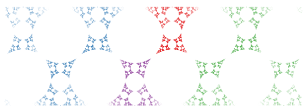
We can give some rough pictures based on our extensions to the theory of Keen and Series which appear in [23]: see Fig. 30. This gives a boundary group where the Riemann surface vanishes. (The corresponding 3-manifold will be related to some wild knot, which is not usually treated in textbooks; we recommend [32][7] and [53] which do deal with wild objects.)

[6]This is not quite true, and we actually need to take a quotient here—for instance, in the Riley slice, we have to quotient out by the geodesic γ_∞. But it is true enough.

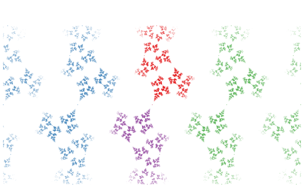
[7]This book is worth a look at, just for the rather whimsical pictures.

(a) 1/1

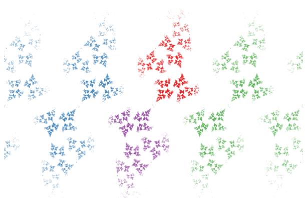
(b) 1/2



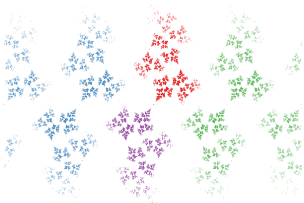
(c) 2/3



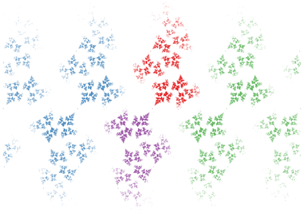
(d) 3/5



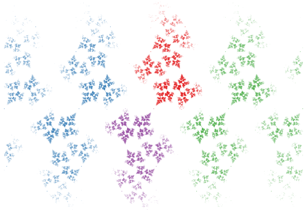
(e) 5/8



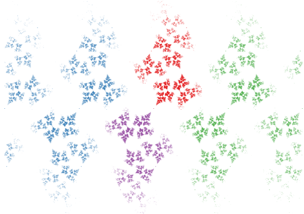
(f) 8/13



(g) 13/21



(h) 21/34



(i) 34/55

Figure 30: Approximations to the $1/\phi = 2/(1+\sqrt{5})$ boundary group along cusp groups. Observe the apparent breakdown of technology! For computational details see [23, 18].

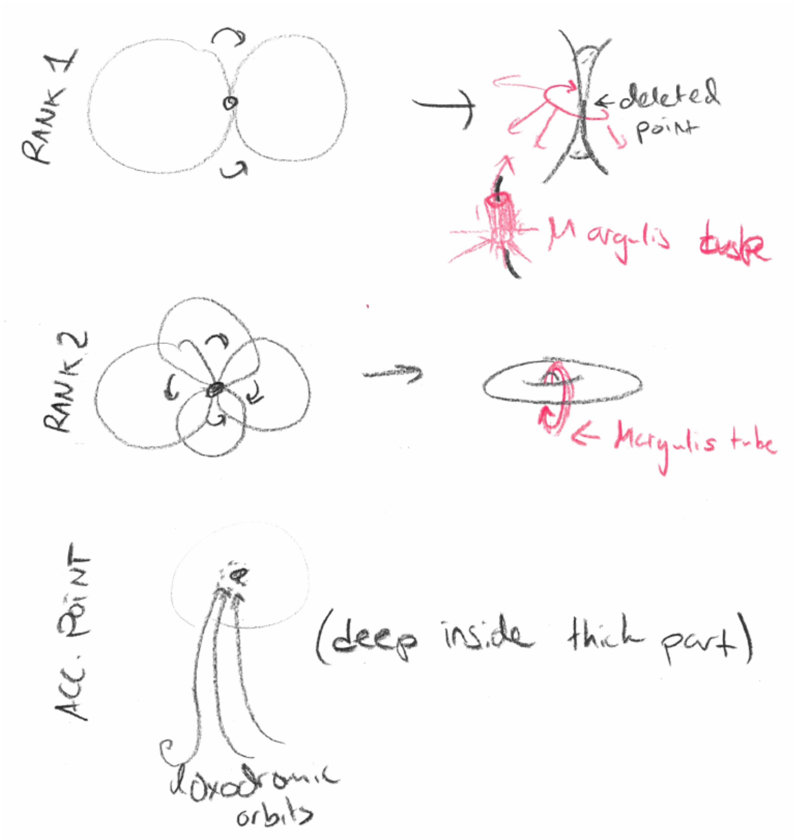


Figure 31: The three kinds of limit point.

Let us restrict ourselves to cusp groups; in fact, maximally cusped groups for the time being. If Γ is geometrically finite, then its limit points are of three kinds (Fig. 31).

1. rank 1 cusps (fixed points of cyclic maximal parabolic subgroups);
2. rank 2 cusps (fixed points of rank 2 maximal parabolic subgroups); and
3. points of approximation (accumulation points of isometric circles).

In addition, all rank 1 cusps are doubly cusped. The corresponding cusp structures in the 3-manifold are Margulis cusps (rank 1) and Margulis tubes (rank 2).

The proof of the following is just by looking at the picture.

4.2 Lemma. *Consider a Riemann surface consisting of a disjoint union of an even number of thrice-punctured spheres, $R = R_1 \cup \cdots \cup R_k$, and suppose that we additionally specify a matching structure on the punctures (i.e. for every puncture p let p' be a puncture distinct*

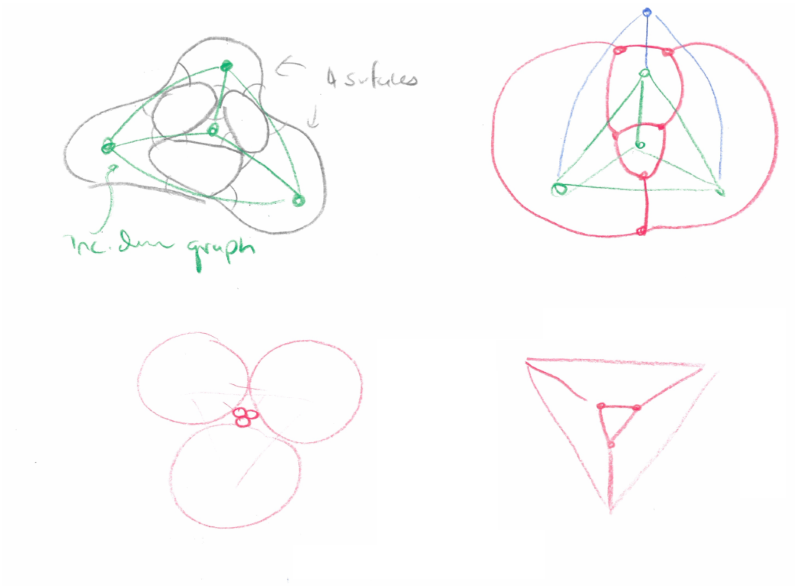


Figure 32: The construction of a cusp group realising a graph curve. Clockwise from top left: (i) the incidence data for the curve, and the surface as a union of thrice-punctured spheres; (ii) the construction of the doubled dual $\hat{G}$; (iii) the tangency data for circles; (iv) a sufficiently symmetric choice of tangent circles realising this incidence.

from p, such that $(p')' = p$). Then there exists a 3-manifold realising this matching structure at infinity.

Note, giving a matching structure in this way is equivalent to giving a trivalent graph, where *we allow loops*. (Note, $2|e| = 3|v|$ so $|v|$ is even as expected.) If G is a trivalent graph, then we can construct explicitly a Schottky-type group which realises the arrangement G. We will deal with the planar case in this section. (Note, $K_{3,3}$ is the simplest example of a non-planar trivalent graph, it is a very instructive example to see what kinds of problems arise.) We define a 'doubled dual' graph $\hat{G}$: glue two copies of G to each other along the bounding edges, and take the dual graph of the result, keeping track of the quotient $q : \hat{G} \to G$. Now construct an arrangement of tangent circles with the following properties.

1. the tangency relation of the circles is given by $\hat{G}$;
2. if $q(C) = q(C')$, $q(D) = q(D')$, and $C \sim C'$ (hence $D \sim D'$) then the line segment l between the centres of C and C' is equal in length to the line segment m between the centres of D and D', and there are circles K through C and D and K' through C' and D' such that both l and m are orthogonal to K and K'.

Existence of an arrangement of tangent circles if $\hat{G}$ is constructed from a planar trivalent graph G as described follows from an extension of the Koebe-Andreev-Thurston circle packing theorem due to Brightwell and Scheinerman [11].

A **graph curve** is a connected projective algebraic curve which is a union of projective lines, each meeting exactly three others and with all intersections transverse [3]. There is a bijection between graph curves and trivalent graphs without loops. Topologically, a graph curve is a union of 2-spheres, such that each sphere is tangent to exactly three others (c.f. [10, pp. 142–144].) There is a natural construction called **plumbing** [38] (see also [41, Exercise 4.28]) which allows us to replace paired cusps with algebraic singularities on the level of abstract surfaces without losing any information.

Putting all this together, we have a construction which allows us to realise any planar trivalent graph as a maximally cusped Riemann surface via a Kleinian group, and since thrice-punctured spheres have trivial Teichmüller space the complex structure is exactly that of the corresponding graph curve. We can now ask two natural questions:

1. What about non-planar arrangements?
2. Can we classify *all* of the maximally cusped groups which realise a particular graph curve?

§5. Braids, links, and mapping class groups: living in a post-Birman world

We have already seen the mapping class group arise in the study of moduli, but we only mentioned it as an aside. In this section, we will use it more extensively. If S is a Riemann surface, then the **mapping class group** or **modular group** of S is the group[8]

$$\mathrm{Mod}(S) = \pi_0(\mathrm{Homeo}^+(S, \partial S))$$

where $\mathrm{Homeo}^+(S, \partial S)$ is the group of orientation-preserving self-homeomorphisms of S which preserve ∂S pointwise.

5.1 Theorem. *The mapping class group is generated by Dehn twists.*

Suppose that S is genus g with n punctures, and $\overline{S}$ is the canonical compactification. One of the most important results for us is the **Birman exact sequence** [6],

$$1 \longrightarrow K \hookrightarrow \pi_1(C(S,n)) \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(\overline{S}) \to 0.$$

We can even write down K explicitly in most cases; for instance in the case of the four-punctured sphere, we explain the whole thing in the expository section of [23]. This allows us to give combinatorial paths through the moduli space: we walk from one projective lamination fibre to another by mapping class actions, and then the only thing to do is choose the measure (which, as we have seen, is just a choice of arbitrary real numbers on the train track). Again in special cases this is worked out very explicitly [7, 57, 35, 36, 23]—in addition I highly recommend the exposition of Series in [60, 61].

[8] Alternatively one sees $\mathrm{MCG}(S)$ or, in an instance of terrible compromise, $\mathrm{Mcg}(S)$...

§5.1. Example: Riley groups, again; adapted from [23]

Let S be a 2-sphere with four marked points, two (resp. X and x) labelled with an integer a and the other two (resp. Y and y) labelled with an integer b (here, $0 < a, b \leq \infty$). We view S as the Riemann surface at infinity of some hyperbolic 3-orbifold O homeomorphic to an open 3-ball and with two singular arcs, one of order a joining X to x and one of order b joining Y to y. That is, there are a pair of homotopically distinct and nontrivial loops in the 3-orbifold—represented by elliptic (or parabolic) elements γ_1, γ_2 of respective orders a and b of the holonomy group of O (for the definition see e.g. [66, §3.4])—which each bound singular arcs in O of respective orders a and b whose four endpoints are the marked points of S. The remainder of this section will describe some models for the moduli space of hyperbolic metrics which are induced on O by different arrangements of the arcs in 3-space, and the moduli space of complex structures which are induced on S when it is viewed as the horizon of O.

Given our surface S with four marked points, we write $\mathrm{Homeo}^+(S)$ for the group of orientation-preserving homeomorphisms of S which preserves both the set $\mathrm{Sing}(S) = \{X, x, Y, y\}$ and its complement and which acts on this 4-set in such a way as to preserve the marking integers—that is, a homeomorphism f is an element of $\mathrm{Homeo}^+(S)$ only if the integral label of z matches that of $f(z)$ for all $z \in \mathrm{Sing}(S)$. (We allow the two integers a and b to be equal, in which case every orientation-preserving homeomorphism which preserves $\mathrm{Sing}(S)$ is allowed even if it permutes the X's with the Y's). The (marked) **mapping class group** of S is the group $\mathrm{Mod}(S) \coloneqq \mathrm{Homeo}^+(S)/\sim$, where $f \sim g$ whenever f and g are isotopic via an isotopy which also preserves $\mathrm{Sing}(S)$ and its complement while respecting integral labels [26, §2.1].

The labelling structure may be precisely modelled in the following way: let $C^{\mathrm{ord}}(S, 2, 2)$ be the set of 4 distinct ordered points on the sphere S (that is, $C^{\mathrm{ord}}(S, 2, 2)$ is the set $S^{\times 2} \times S^{\times 2} \setminus \mathrm{BigDiag}(S^{\times 4})$ where $\mathrm{BigDiag}(S^{\times 4})$ is the 'big diagonal' of 4-tuples of points where at least two of the points are repeated); there is a natural action of a subgroup $\mathcal{S}$ of $\mathrm{Syn}(4)$ on $C^{\mathrm{ord}}(S, 2, 2)$ which depends on whether $a = b$: if $a \neq b$ then $\mathcal{S}$ is the Klein 4-group $\mathrm{Syn}(2) \times \mathrm{Syn}(2)$ permuting the coordinates of each factor separately, and if $a = b$ then $\mathcal{S}$ is the whole $\mathrm{Syn}(4)$ and is allowed to permute all four points. The **pairwise configuration space** of four points on S is the quotient

$$C(S, 2, 2) = C^{\mathrm{ord}}(S, 2, 2)/\mathcal{S}.$$

The geometric interpretation is supposed to be clear: the first $S^{\times 2}$ component keeps track of the order a points and the second component keeps track of the order b points.

The homotopy group $\pi_1(C(S, 2, 2))$ (that is, the possible paths up to homotopy traced out by the four points without colliding, where paths must connect points with the same label) is called the **spherical braid group on** 4 **strands** [26, §9.1].

The main point which forms the basis of the Birman–Keen–Series theory of the mapping class groups of four-times marked spheres is the following. Let S be a sphere with four marked points which bounds a 3-orbifold O in such a way that the marked points are identified in pairs, as above; assume that $a \neq b$, so the pairs are distinguishable from each other. Then the Birman exact sequence becomes

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \hookrightarrow \pi_1(C(S, 2, 2)) \xrightarrow{\mathcal{P}ush} \mathrm{Mod}(S) \longrightarrow 1$$

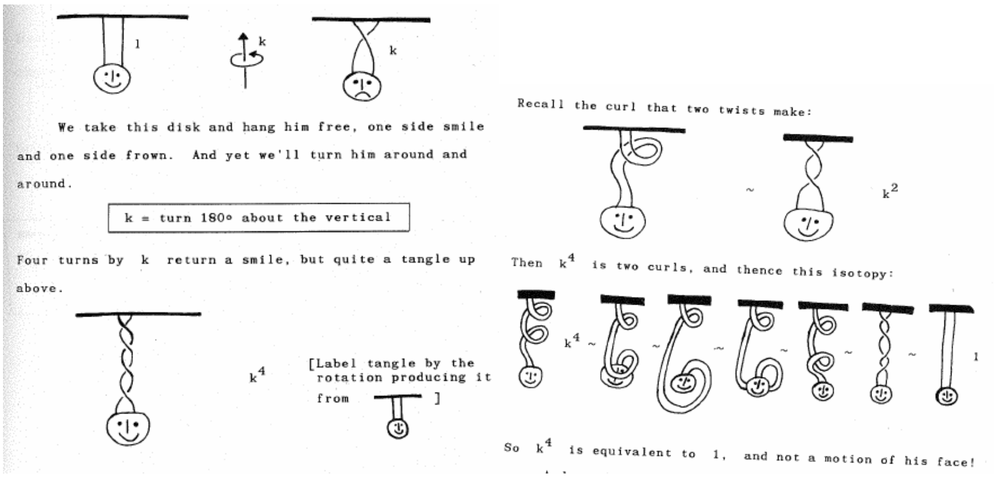


Figure 33: The belt trick [32, §VI.1].

where

$$\mathrm{Mod}(S) = \mathrm{PSL}(2,\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathrm{PSL}(2,\mathbb{Z}) \rtimes (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) = \mathrm{Mod}(S_{0,4}).$$

Here, we use $S_{0,4}$ to denote the sphere with four indistinguishable marked points; the map $\mathrm{Mod}(S) \to \mathrm{Mod}(S_{0,4})$ is the evident inclusion map (every mapping class which respects the marking structure in pairs is also a mapping class when that pair-structure is forgotten), the map $\mathcal{Push}$ is the map which sends a braid $\beta : [0,1] \to C(S,2,2)$ to the induced homeomorphism $\beta(0) \to \beta(1)$, and where $\ker \mathcal{Push} \simeq \mathbb{Z}/2\mathbb{Z}$ is generated by a homeomorphism Θ which corresponds to a 2π rotation of the four marked points (equivalently, a single 2π twist added to the end of the braid); the image of this in the mapping class group is trivial (the twist can be undone by rotating the 'back' of the sphere via an isotopy without moving the points) and it is an involution in the braid group by the belt trick Fig. 33.

Let α_1 and α_2 be two disjoint paths on the sphere which join the points in pairs, preserving the integral labelling. If α_1 and α_2 are pushed slightly into the interior of the sphere without passing through each other, the resulting arrangement is called a **rational tangle**. Every two-bridge knot comes from taking such a rational tangle and pairing the four marked points on the sphere with two disjoint paths outside the sphere (which may not preserve the integer labels) which contract onto the sphere via an isotopy such that the images do not cross (this gluing is the so-called **numerator closure**).

There is a natural way to enumerate the rational tangles, essentially due to Schubert [55] and described in [13, §12.B] or [49, Chapter 10]. First, take a sequence $a_0, \dots, a_m$ of integers. Every rational tangle is obtained by laying out four parallel strands (two labelled with the integer a and two with the integer b) and then alternatingly braiding the two leftmost strands and then the two middle strands with plaits of $a_0, a_1, \dots$ crossings (where the sign of each a_i denotes the direction of twisting to produce each braiding cluster); this

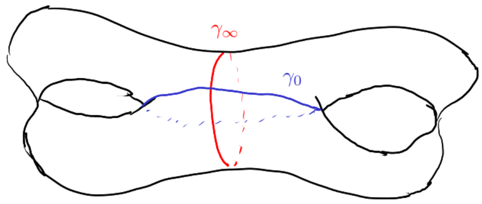


Figure 34: The first homology group of the 4-punctured sphere is isomorphic to $\mathbb{Z}^2$; one possible basis is formed by the two cycles γ_0 and γ_∞ depicted.

produces a plait made up of four strands, which are joined like-labelling-to-like at one end to form two plaited cords, one labelled with a and one with b; the numerator closure is then the capping of the four remaining ends.

We now recall a standard fact from classical number theory. If $(a_0, a_1, \dots, a_k)$ is a finite sequence of integers, we define the **simple continued fraction**

$$[a_0; a_1, \dots, a_k] \coloneqq a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}}}.$$

Every rational number can be expressed as a finite simple continued fraction in exactly two ways, one with an even and one with an odd number of convergents (number of sequence elements a_n) [30, Theorem 162]. These can be computed efficiently [30, §10.9]. This relationship gives a bijection between the space of 2-bridge links and the set of rational numbers; the rational number associated to a given 2-bridge link is called the **rational form** or **Schubert normal form** for the link.

By the theory above, for every 2-bridge knot we obtain an element of the braid group $\pi_1(C(S, 2, 2))$, and hence an element of the mapping class group. (In fact, for each tangle we obtain an element of the braid group, but the image of a tangle under Θ gives the same knot up to isotopy, and so we only really get a well-defined element of the mapping class group.) Note that the involutions ι_1 and ι_2 both preserve the knot structure (whether the two strands/four marked points are indistinguishable or not), and so we in fact have an injection ϕ : {2-bridge knots} $\to \mathsf{PSL}(2, \mathbb{Z})$. Observe next that the twisting sequence of a 2-bridge link k is in fact coding a sequence of Dehn twists needed to twist the unbraid into the rational tangle whose closure is k: if γ_0 and γ_∞ are the two curves marked in Fig. 34 (which together form a basis for $H_1(S, \mathbb{Z}) \simeq \mathbb{Z}^2$) and τ_0 and τ_∞ are the respective Dehn twists, then a rational tangle whose closure is k is represented by the element

$$\tau_0^{a_0} \tau_\infty^{a_1} \tau_0^{a_2} \tau_\infty^{a_3} \cdots$$

of the mapping class group. The action of the $\mathsf{PSL}(2, \mathbb{Z})$ semidirect multiplicand of $\mathrm{Mod}(S)$ on S can be identified with the usual matrix group action of $\mathsf{PSL}(2, \mathbb{Z})$ on $H_1(S, \mathbb{Z})$ after we choose this basis; take $\gamma_0 = (0, 1)^t$ and $\gamma_\infty = (1, 0)^t$, and write the standard generating set

$$\mathsf{PSL}(2,\mathbb{Z}) = \left\langle R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle \le \mathsf{PSL}(2,\mathbb{C})$$

(c.f. the case of the punctured torus in [60]). If A is an element of $\mathsf{PSL}(2,\mathbb{Z})$, say

$$A = \begin{bmatrix} p & r \\ q & s \end{bmatrix},$$

then A sends $\gamma_\infty \mapsto p\gamma_\infty + q\gamma_0$ and $\gamma_0 \mapsto r\gamma_\infty + s\gamma_0$. Observe that this gives a map from $\mathsf{PSL}(2,\mathbb{Z})$ to the space of ordered singular $\mathbb{Z}$-homology bases of S (where the ordering is given by $p\gamma_\infty + q\gamma_0 \preceq r\gamma_\infty + s\gamma_0$ if $p/q \le r/s$).

Now let $h : \pi_1(S) \to H_1(S)$ be the usual Abelianisation projection. Define the **geometric intersection** of a pair of homology classes $\alpha, \beta \in H_1(S)$ in the usual way, namely $i(\alpha,\beta)$ is the infimum, over all of the choices of σ and γ in the free homotopy classes of all curves in $h^{-1}(\alpha)$ and $h^{-1}(\beta)$ respectively, of $|\sigma \cap \gamma|$.

The following result is standard:

5.2 Proposition. *Suppose $\alpha = p\gamma_\infty + q\gamma_0$ and $\beta = r\gamma_\infty + s\gamma_0$ are arbitrary homology classes.*

1. $i(\alpha,\beta) = 2\left|\det\begin{bmatrix} p & r \\ q & s \end{bmatrix}\right|$.
2. *If* $\gcd(p,q) \neq 1$*, then* $i(\alpha,\alpha) > 0$ *(i.e. there is no simple closed curve on S in the homology class of α).*
3. *If* $\gcd(p,q) = 1$*, then* $i(\alpha,\alpha) = 0$*; and further, there is exactly one non-freely-homotopic* geodesic *(with respect to any chosen hyperbolic metric on S) simple closed curve on S which projects to α under h. In this case, both α and $-\alpha$ correspond to the two orientations on this geodesic, and so the geodesic can be identified by the rational number $p/q = (-p)/(-q) \in \hat{\mathbb{Q}} \coloneqq \mathbb{Q} \cup \{\infty\}$; we write $\gamma_{p/q}$ for this geodesic.*

Now it is an easy exercise to check that

$$\mathsf{PSL}(2,\mathbb{Z}) = \langle R, L \rangle \rtimes \langle Q \rangle \tag{5.3}$$

where L is the matrix

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

(the key observation is that Q conjugates L to R^{-1}). Let $\Gamma_1 \coloneqq \langle L, R\rangle$ be the orientation-preserving (with respect to the action on the upper half-plane) part of $\mathsf{PSL}(2,\mathbb{Z})$; then Γ_1 is in bijection with the space of unordered bases of the subspace of $H_1(S,\mathbb{Z})$ of non-boundary-parallel curves on S. By direct computation we see now that the action of Γ_1 as a subset of the mapping class group is

$$\begin{aligned} R.\gamma_\infty &= \gamma_\infty, & R.\gamma_0 &= \gamma_{1/1} \\ L.\gamma_\infty &= \gamma_{1/1}, & L.\gamma_0 &= \gamma_0; \end{aligned}$$

that is, R acts as τ_∞ and L acts as τ_0. In particular, we have a bijection between the space of two-bridge knots and the group Γ_1 such that the knot with rational form $p/q = [a_0; a_1, \dots, a_N]$ is represented by the matrix

$$L^{a_0} R^{a_1} L^{a_2} \cdots .$$

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