Strange circles

The Riley slice of quasi-Fuchsian space

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(mpi-mis)

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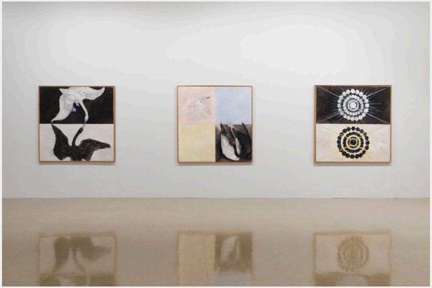

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Ambient space

Spaces 'of type' X

Moduli spaces

$$\begin{tikzcd} X \arrow[d, two heads] & & \Gamma \arrow[ll, squiggly, "\text{acts on}"'] \\ X/G & & G \arrow[u, hook] \end{tikzcd}$$

$$\{X/H : X/G \approx X/H\} \longleftrightarrow \{H : G \sim H\}$$

Geometric group theory

$$
\begin{tikzcd}
\text{Hyperbolic space} & \mathbb{H}^3 \arrow[d, two heads] & \mathrm{PSL}(2,\mathbb{C}) \arrow[l, squiggly, "\text{acts on}"'] \\
\text{Hyperbolic manifolds} & \mathbb{H}^3/G & G \text{ discrete} \arrow[u, hook] \\
\text{Moduli spaces} & \left\{\begin{matrix}\text{isometric}\\ \text{manifolds}\end{matrix}\right\} \arrow[r, leftrightarrow] & ???
\end{tikzcd}
$$

Kleinian groups

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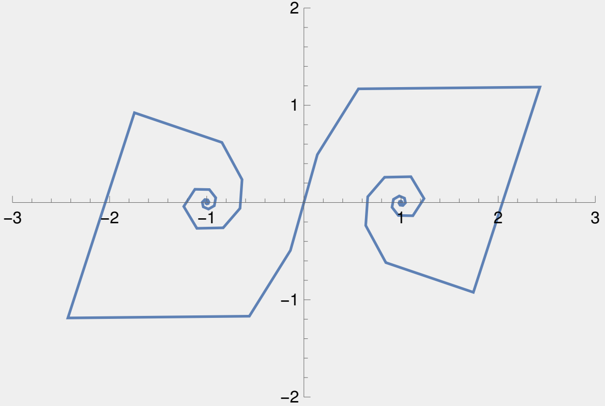
- Discreteness is enough to make sure that $\mathbb{H}^3/G$ is an orbifold.
- It is **not** enough for $\hat{\mathbb{C}}/G$ to be a Riemann surface.

Limit sets

Discrete subgroups of $PSL(2, \mathbb{C})$ have orbits which accumulate at 'limit points'.

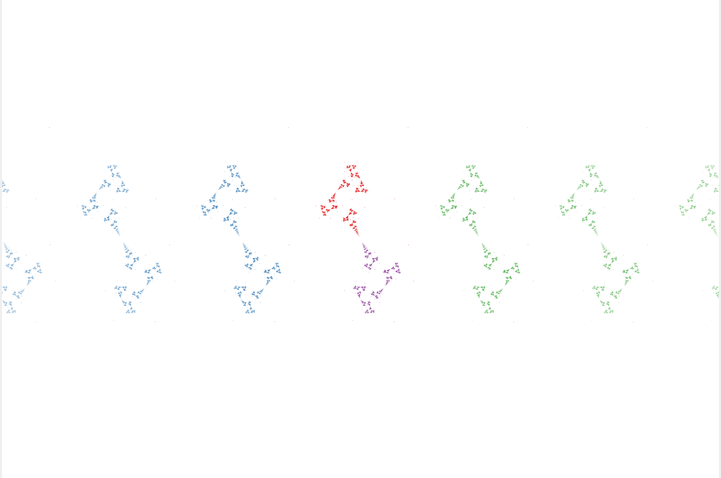


Orbit of 0 under $\begin{bmatrix} 0.9 + 0.05i & 0.1 + 0.45i \\ 0.1 + 0.45i & 0.9 + 0.05i \end{bmatrix}$.

Source at −1, sink at 1; orbit lies on a log-spiral.

An example

$$\Gamma_{1+2i} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+2i & 1 \end{bmatrix} \right\rangle$$



An example, cont.

$$\Gamma_{1+2i} = \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1+2i & 1 \end{bmatrix} \right\rangle$$

Theorem

Γ_{1+2i} is discrete and free on these two generators.

Theorem

The Riemann surface coming from Γ_{1+2i} is a 4-times punctured sphere.

These results are not supposed to be obvious!!!

The moduli space associated to the example

Question

What happens when you wiggle the parameter $1 + 2i$ around in $\mathbb{C}$?

'Wiggle' means 'move holomorphically'.

Theorem (Extended λ-lemma: Mañé, Sad, Sullivan (1983); Słodkowski (1991); Earle, Kra, Krushkal' (1994))

If a matrix coefficient in a generator of a Kleinian group moves holomorphically, then the limit set deforms ***quasiconformally*** *until it collides with itself.*
If Γ_ρ is a group with parameter ρ, then $\Gamma_{\tilde{\rho}} = f^{-1}\Gamma_\rho f$ for some quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

The Riley slice is the quasiconformal moduli space of the example.

Definition

The **Riley slice** $\mathcal{R}$ is the set of $\rho \in \mathbb{C}$ such that

$$\Gamma_\rho \coloneqq \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle \leq \mathrm{PSL}(2, \mathbb{C})$$

is quasiconformally conjugate to Γ_{1+2i}.

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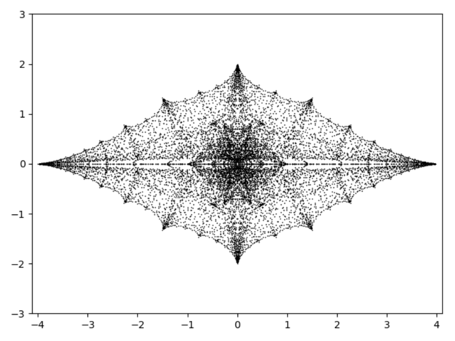

Big theorem 1

Theorem (Lyubich, Suvorov (1988); Maskit, Swarup (1989); Ohshika, Miyachi (2010); Aigner (conj. 2001); Akiyoshi, Ohshika, Parker, Sakuma, Yoshima (2020); Aimi, Lee, Sakai, Sakuma (2020))

The Riley slice $\mathcal{R}$ is equivalently:

1. *The set of $\rho \in \mathbb{C}$ such that*

$$\Gamma_\rho := \left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ \rho & 1 \end{bmatrix} \right\rangle \leq \mathrm{PSL}(2,\mathbb{C})$$

   is free and discrete, and such that the associated Riemann surface is homeomorphic to a 4-times punctured sphere.
2. *The interior of the set of ρ such that Γ_ρ is free and discrete.*

Proof is extremely non-trivial (especially part 2).

Big theorem 2

Theorem (Lyubich, Suvorov (1988); Maskit, Swarup (1989); application of Bers, Greenberg, Marden (1971))

The Riley slice is homeomorphic to an annulus. More precisely,

$$\mathcal{R} \simeq \frac{\mathbb{H}^2}{\langle \omega \rangle}$$

where $\mathbb{H}^2$ is the Teichmüller space of 4-times punctured spheres and where $\omega \in \mathrm{Mod}(S_{0,4})$ is explicit enough that the quotient is 'obviously' an annulus.

Proof is an exercise in applying Teichmüller theory.

Big theorem 3

Theorem (Keen, Series (1994); Komori, Series (1998); generalised E., Martin, Schillewaert (2021-22))

The Riley slice is homeomorphic to an annulus. More precisely,

$$\mathcal{R} \simeq S^1 \times \mathbb{R}_{>0}$$

where the first coordinate chooses a distinguished geodesic on the Riemann surface, and where the second coordinate gives its measure.

This is a concrete case of the Ahlfors-Bers-Maskit deformation/parabolicity theory for Kleinian groups.

What about the 'algebraic' moduli space of 4-punctured spheres?

$$
\begin{tikzcd}
\mathrm{Teich}(S_{0,4}) \simeq \mathbb{H}^2 \arrow[d, "\mathrm{Mod}(S_{0,4})"'] \arrow[dr, "\langle \omega \rangle"] & \\
& \mathcal{R} \simeq \mathbb{R}_{>0} \times \mathbb{R}/2\mathbb{Z} \arrow[dl, "?"] \\
\mathcal{M}_{0,4} \simeq S_{0,3} &
\end{tikzcd}
$$

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 & \mathcal{R} \simeq \mathbb{R}_{>0} \times \mathbb{R}/2\mathbb{Z} \arrow[dl, "\text{knot theory}"] \\
\mathcal{M}_{0,4} \simeq S_{0,3} &
\end{tikzcd}$$

- Very introductory survey for people in other fields: A.J.E., G.M., J.S. "Concrete one complex dimensional moduli spaces of hyperbolic manifolds and orbifolds". In: *2021-22 MATRIX annals* (Springer, to appear; arXiv 2022).
- More detailed introduction: A.J.E. *Deformation spaces of Kleinian groups.* MSc thesis. The Uni. of Auckland (2022).
- Our recent work: A.J.E., G.M., J.S. "Approximations of the Riley slice" (arXiv 2021), "The combinatorics of Farey words and their traces" (arXiv 2022), "The Riley slice and its elliptic cousins" (to appear soon).
- Title picture: Hilma af Klint, *The SUW series, group IX*, 1915, oil on canvas. City Gallery Wellington (2021–2022), exhibition courtesy of the Hilma af Klint Foundation. Photo: Cheska Brown.