Pictures of hyperbolic spaces

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(mpi-mis)

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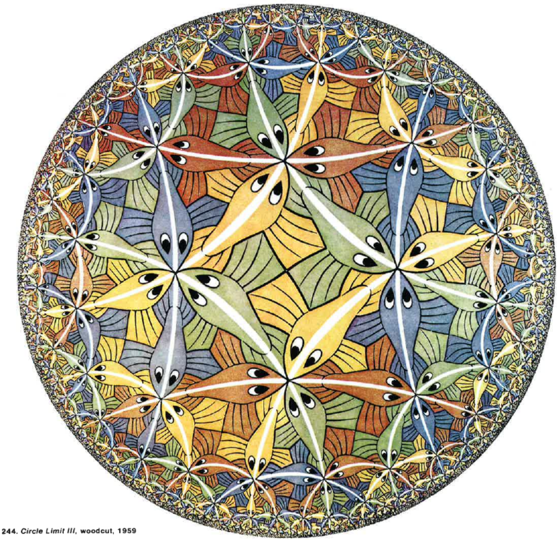
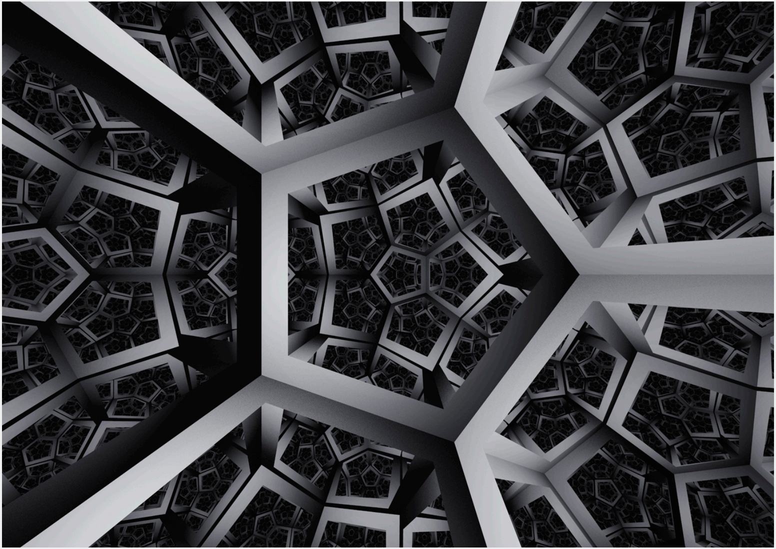
Hyperbolic 2-space, $\mathbb{H}^2$


244. *Circle Limit III*, woodcut, 1959

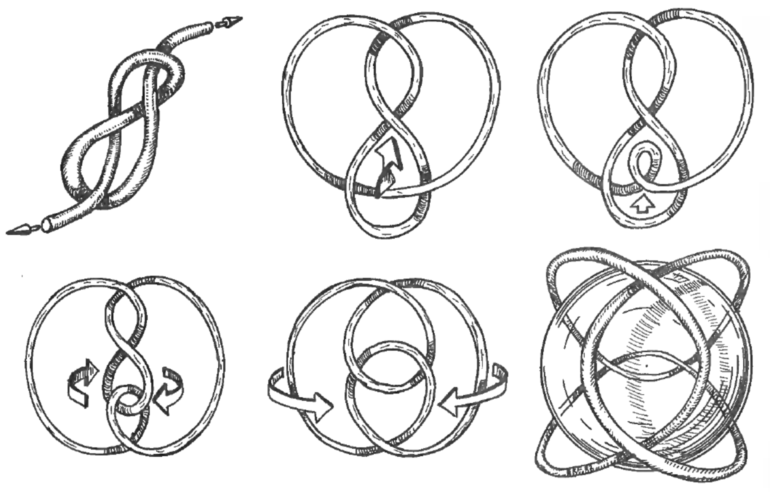
[M.C. Escher (1959)]

Hyperbolic 3-space, $\mathbb{H}^3$: tiling by dodecahedra



[Pierre Berger https://www.espaces-imaginaires.fr/works/ExpoEspacesImaginaires2.html]

THE FIGURE 8 KNOT $k(5/3)$



[Francis, p.150]

Theorem (Robert Riley (c.1974); William P. Thurston (c.1975))

The complement of the figure 8 knot,

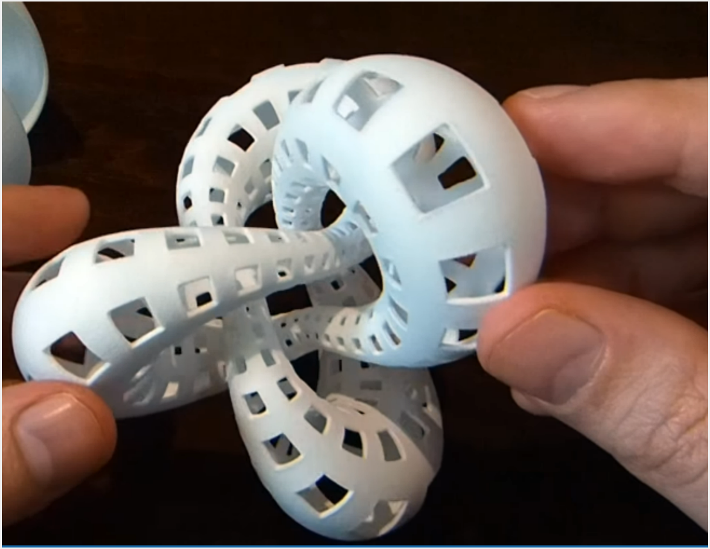
$$S^3 \setminus k(5/3),$$

admits a hyperbolic geometry.

Theorem (Thurston (c.1979))

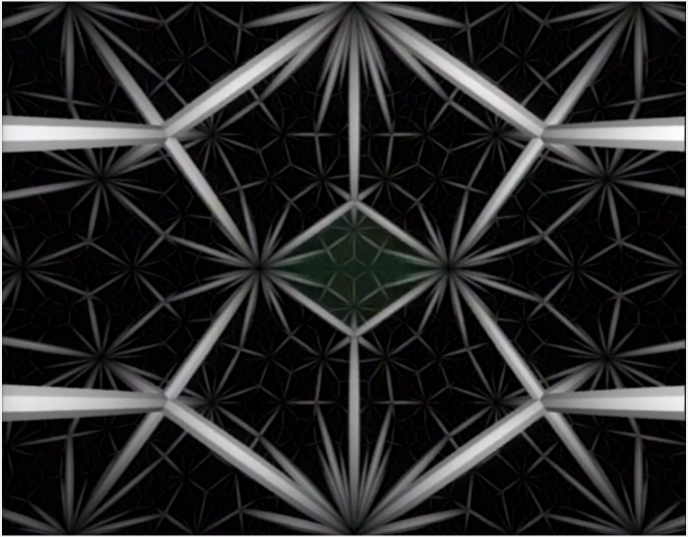
Almost every knot complement[1] *admits a hyperbolic geometry.*

[1] All but a small family of tabulated exceptions



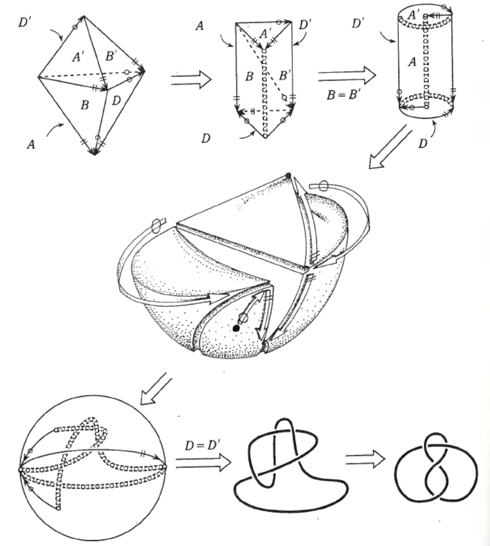
[Guéritaud/Segerman/Schleimer, https://youtu.be/xGf5jY_v5GE]

The Borromean rings complement



[Gunn/Maxwell, *Not Knot*]

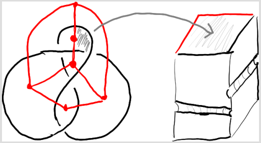
Idea: visualise knot complements by tiling



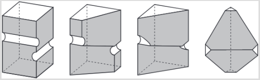
[Matsuzaki/Taniguchi, p.34]

SnapPea Algorithm (Jeff Weeks, c.1985)

1. Embed the knot in $S^2 \times [-1, 1]$ 'flatly' around $S^2 \times \{0\}$.
2. Cut straight down along the dual graph & the knot graph.



3. Collapse the quadrilateral slices to tetrahedra.



4. Glue four cusps onto these vertices to get spherical tetrahedra.
5. Do a bit of fiddling to get the hyperbolic geometry back.

This gives all hyperbolic manifolds

Theorem (Thurston, c.1979)

Every hyperbolic 3-manifold can be obtained by 'Dehn surgery' along some hyperbolic link.

Thus Weeks' algorithm triangulates every hyperbolic 3-manifold.

Theorem (Hyperbolic developing)

Let M be a hyperbolic orbifold. Then M is isometric to a orbifold of the form $\mathbb{H}^3/G$ for some discrete group G of hyperbolic isometries (called the ***holonomy group*** *of M). Conversely, given any discrete group $G \le \mathrm{Isom}^+(\mathbb{H}^3)$, $\mathbb{H}^3/G$ is a hyperbolic orbifold.*

Definition

A discrete group $G \le \mathrm{Isom}^+(\mathbb{H}^3)$ is called a **Kleinian group**.

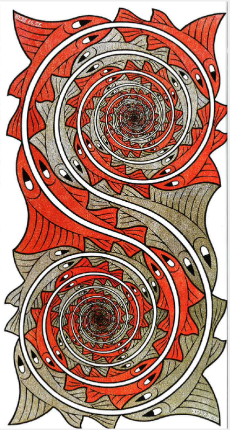
Theorem (Poincaré extension)

There is a natural isomorphism between the group $\mathrm{Isom}^+(\mathbb{H}^3)$ *of orientation-preserving hyperbolic isometries and the group* $\mathrm{PSL}(2,\mathbb{C})$ *of conformal maps on* $\partial\mathbb{H}^3 = \hat{\mathbb{C}}$.

Example (Robert Riley, c.1972)

The holonomy group of the figure 8 knot complement is

$$\left\langle \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -\exp(2\pi/3) & 1 \end{bmatrix} \right\rangle.$$



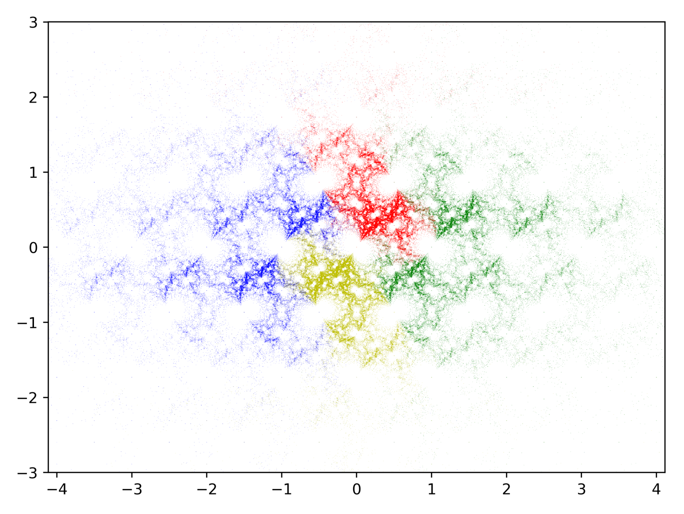
[M.C. Escher (1957)]

The dynamics of the action of a Kleinian group G on $\hat{\mathbb{C}}$ are complicated. There is a partition $\hat{\mathbb{C}} = \Omega(G) \cup \Lambda(G)$ similar to the partition between the Fatou and Julia sets of a holomorphic dynamical system.

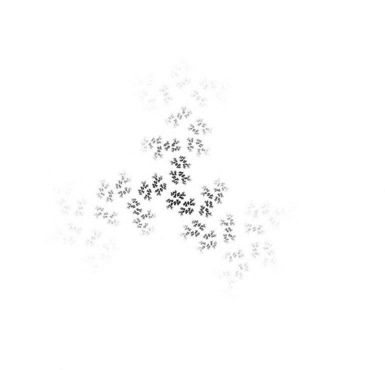
Definition

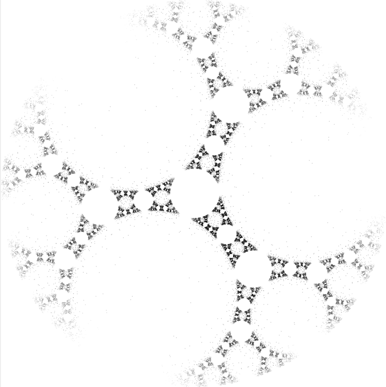
If G is non-elementary, then the **limit set** of G is the closure of the set of fixed points of elements of G.

Figure 8 knot group (dense in $\hat{\mathbb{C}}$)

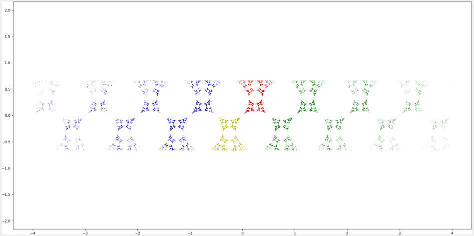


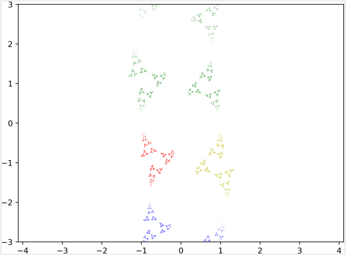
Elliptic Riley groups





Left: parabolic Riley group. Right: Indra's Necklace group.





Remark: why is this important?

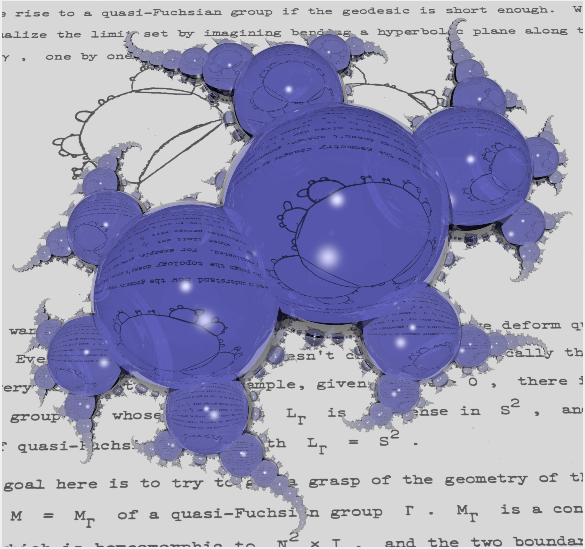
Theorem (Thurston (c.1979); The ending lamination theorem (Epstein/Marden/Minsky))

If G is non-degenerate[2] *then there is a strong deformation retract*

$$\frac{\mathbb{H}^3 \cup \Omega(G)}{G} \twoheadrightarrow \frac{\mathrm{h.conv}\,\Lambda(G)}{G}$$

and the 'folding structure' on the convex hull determines the hyperbolic geometry entirely.

[2] non-Fuchsian and non-elementary



[Jeffrey Brock and David Dumas, `https://www.dumas.io/poster/`]

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ALGORITHMS FOR COMPUTING LIMIT SETS

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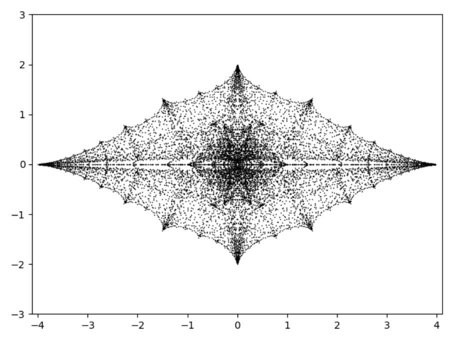
- So the problem is reduced to (1) enumerating 'lots of words', and (2) doing matrix products quickly.
- These are standard problems in computational combinatorial group theory.

A definition

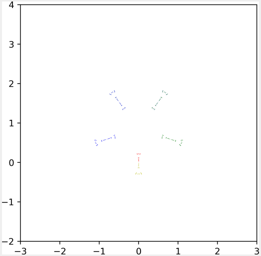
The (a, b)-**Riley slice**, $\mathcal{R}^{a,b}$, is the set of $\rho \in \mathbb{C}$ such that the limit set of the matrix group

$$\Gamma_\rho := \left\langle \begin{bmatrix} e^{\pi i/a} & 1 \\ 0 & e^{-\pi i/a} \end{bmatrix}, \begin{bmatrix} e^{\pi i/b} & 0 \\ \rho & e^{-\pi i/b} \end{bmatrix} \right\rangle$$

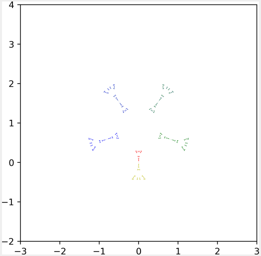
is *neither* dense in a circle packing, *nor* dense in $\hat{\mathbb{C}}$.



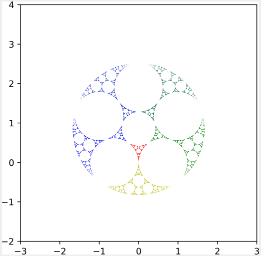
LIMIT SET DEFORMATIONS IN $\mathcal{R}^{5,\infty}$



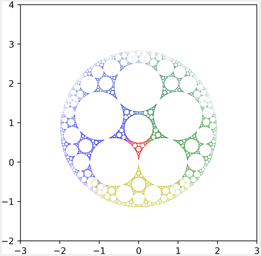
(a) $\rho = 4i$



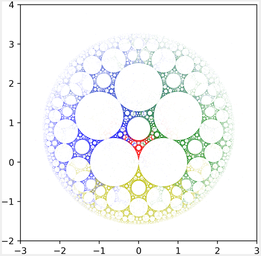
(b) $\rho = 3i$



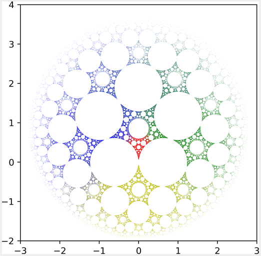
(c) $\rho = 2i$ (cusp group)



(d) $\rho = 1.7i$



(e) $\rho = 1.5i$



(f) $\rho = \sqrt{2}i$

- The definition I gave is hiding the term ‘quasiconformal deformation space’ behind the geometry of limit sets.
- Equivalent definition: $\mathcal{R}^{a,b}$ is the moduli space of hyperbolic orbifolds (...together with conformal boundary...) homeomorphic to a 3-ball with two cone arcs, one of order a and one of order b (whose boundary is a sphere with two a-cone points and two b-cone points).
- The closure $\overline{\mathcal{R}^{a,b}}$ is the moduli space of discrete groups, free on two elliptic generators (where we view parabolic elements as limiting cases of elliptic elements).
- Discrete groups in the exterior $\mathbb{C} \setminus \overline{\mathcal{R}^{a,b}}$ parameterise the 2-bridge link groups and their ‘untwistings’ (Heckoid groups).

The Keen–Series rational lamination

Theorem (Linda Keen & Caroline Series (1994); Yohei Komori & Series (1998); Elzenaar, Martin, Schillewaert (2022))

There exists a dense foliation of $\mathcal{R}^{a,b}$ by smooth analytic curves, indexed by $p/q \in \mathbb{Q}$, such that

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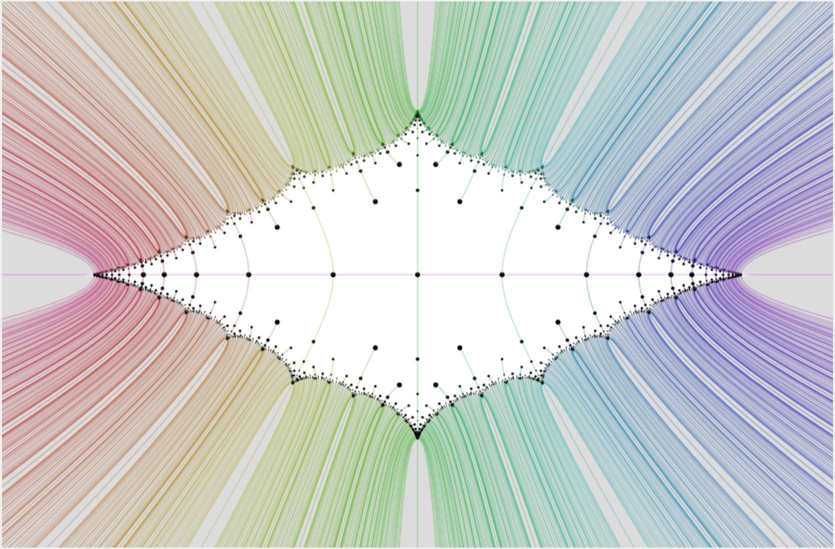
3. *The inverse images of* -2 *which lie at the ends of the curves (called* ***cusp points****) are dense in the boundary of* $\mathcal{R}^{a,b}$ *[Curtis McMullen, 1991]; they have circle packing limit sets and the points corresponding to different curves are distinct [Keen/Maskit/Series, 1991].*

THE KEEN–SERIES RATIONAL LAMINATION OF $\mathcal{R}^{\infty,\infty}$



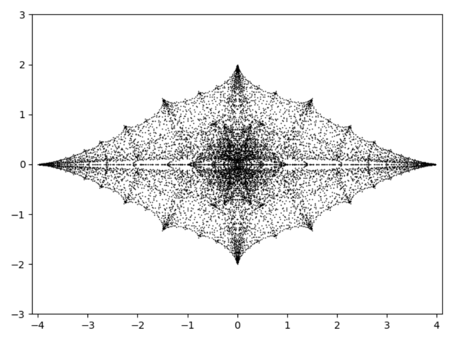
[Yasushi Yamashita, c.2007]

A method for drawing the Riley slice boundary and exterior

1. Compute all of the Farey polynomials. (*A priori*, each such computation requires $2q$ matrix multiplications. We have a recurrence relation that computes them all just with polynomial arithmetic [EMS22].)
2. Compute all of the inverse images $\Phi_{p/q}^{-1}(-2)$. (This is the computationally hard step. Geometric arguments show that even the inverse images which are not cusps lie in the exterior of the slice)

Conjecture

The points $\Phi_{p/q}^{-1}(-2)$ $(p/q \in \mathbb{Q})$ are dense in $\mathcal{R}^{a,b}$.



Some work has been done by Jane Gilman (2008) to explain the patterns of density that are visible (the 'parabolic dust').

- George K. Francis, *A topological picturebook* (Springer, 1987)
- William P. Thurston, *Geometry and topology of 3-manifolds* (unpublished lecture notes, c.1979)
- William P. Thurston, *Three-dimensional geometry and topology, Vol. 1* (Princeton, 1997)
- Jeff Weeks, “Computation of hyperbolic structures in knot theory”. In: *Handb. of Knot Theory* (Elsevier, 2005)
- David Mumford, Caroline Series, David Wright, *Indra’s pearls* (Cambridge, 2002)
- Albert Marden, *Hyperbolic manifolds* (Cambridge, 2015)
- Jessica Purcell, *Hyperbolic knot theory* (AMS, 2021)