THE ACTION OF $\mathsf{PSL}(2, \mathbb{C})$ ON CIRCLES

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Abstract. One of the most fundamental properties of Möbius transformations is that they preserve the inversive product. The literature includes formulae for the actions of basic elements like reflections; this note supplies explicit formulae for the action of arbitrary Möbius transformations on the set of circles in the Riemann sphere by utilising the standard decomposition into two reflections (three reflections, if the transform is orientation-reversing) and a rotation.

1. Introductory remarks

It is possible [1, Theorem 3.2.3] (also [2, Chapter 20]) to view the space of circles (including lines) in Euclidean n-space as a projective $(n+1)$-space, and Mobius transformations act as projectivities in this space. In the case of interest to us $(n = 2)$ we have a map $\mathsf{PSL}(2, \mathbb{C}) \to \mathsf{PGL}(4, \mathbb{R})$.

Circles in $\mathbb{R}^2$ are represented as 4-tuples of coefficients (a_0, a_1, a_2, a_3) such that the circle is the locus of z such that $a_0|z|^2 - 2(a_1, a_2)z + a_3 = 0$—where here (a_1, a_2) is the normal dot product in $\mathbb{R}^2$.

We describe the very simple algorithm implemented in [3] as the function `cayley.action_on_circles` that takes a 2×2 matrix M in $\mathsf{PSL}(2, \mathbb{C})$ and determines a corresponding linear map on $\mathbb{R}^4$ corresponding to the action on the space of circles. The outcome will be an explicit description of this representation,

$$\varrho : \widetilde{\mathsf{PSL}(2}, \mathbb{C}) \to \mathsf{GL}(4, \mathbb{R}),$$

where $\widetilde{\mathsf{PSL}(2}, \mathbb{C})$ indicates the group of possibly orientation-reversing Möbius transformations of the Riemann sphere $\hat{\mathbb{C}}$.

2. Elementary actions

We will need the matrices of various elementary Möbius transformations. These matrices may be found in the proof of [1, Theorem 3.2.3].

(1) Let $A \in \mathsf{PO}(2)$. Then

$$\varrho(A) = \begin{bmatrix} 1 & & \\ & A & \\ & & 1 \end{bmatrix}.$$

In particular, some useful orthogonal matrices are:

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Date: December 19, 2024.
2020 *Mathematics Subject Classification.* 30C10, 51M09.
Key words and phrases. hyperbolic geometry, conformal geometry.

(a) if $\theta \in [0, 2\pi]$ then the rotation by angle θ about 0 is sent to

$$\varrho(W_\theta) = \begin{bmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{bmatrix}.$$

(b) if J is reflection in the x axis (i.e. complex conjugation), then

$$\varrho(J) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & 0 & -1 & \\ & & & 1 \end{bmatrix}.$$

(2) Let $D_k(z) = kz$ be a dilation, for $k \in \mathbb{R} \setminus \{\pm 1, 0\}$. Then

$$\varrho(D_k) = \begin{bmatrix} 1 & & & \\ & k & & \\ & & k & \\ & & & k^2 \end{bmatrix}.$$

(3) Let $M_\omega(z) = \omega z$ be complex multiplication, so $\omega \in \mathbb{C}$. then

$$\varrho(M_\omega) = \varrho(D_{|\omega|} W_{\arg\omega}) = \begin{bmatrix} 1 & & & \\ & \Re\omega & -\Im\omega & \\ & \Im\omega & \Re\omega & \\ & & & |\omega|^2 \end{bmatrix}.$$

(4) Let $R_\circ$ be reflection in the unit circle. Then

$$\varrho(R_\circ) = \begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{bmatrix}.$$

(5) Let $J_{p,q}$ be reflection in the perpendicular bisector of the segment $[p, q]$ for $p, q \in \mathbb{C}$. Set $\theta = \pi/2 + \arg(p-q)$, and for convenience define $\mathcal{C}(p, q) = \Re(p-q)/|p-q| = \cos(\arg(p-q))$ and $\mathcal{S}(p, q) = \Im(p-q)/|p, q| = \sin(\arg(p-q))$. Then

$$\begin{aligned} \varrho(J_{p,q}) &= \varrho(T_{(p+q)/2} J W_{-2\theta} T_{-(p+q)/2}) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}\begin{pmatrix} \Im(p+q)\mathcal{C}(p,q) \\ \quad + \Re(p+q)(\mathcal{S}(p,q)+1) \end{pmatrix} & -\mathcal{S}(p,q) & -\mathcal{C}(p,q) & 0 \\ \frac{1}{2}\begin{pmatrix} \Re(p+q)\mathcal{C}(p,q) \\ \quad - \Im(p+q)(\mathcal{S}(p,q)-1) \end{pmatrix} & -\mathcal{C}(p,q) & \mathcal{S}(p,q) & 0 \\ \frac{1}{4}\begin{pmatrix} 2|p+q|^2 \\ \quad + ((p^*+q^*)^2 + (p+q)^2)\,\mathcal{S}(p,q) \\ \quad\quad + 4\Im(p+q)\Re(p+q)\mathcal{C}(p,q) \end{pmatrix} & -\begin{pmatrix} \Im(p+q)\mathcal{C}(p,q) \\ \quad + \Re(p+q)(\mathcal{S}(p,q)+1) \end{pmatrix} & \begin{pmatrix} \Im(p+q)(\mathcal{S}(p,q)-1) \\ \quad - \Re(p+q)\mathcal{C}(p,q) \end{pmatrix} & 1 \end{bmatrix} \end{aligned}$$

(6) Let $T_u(z) = z + u$ for $u \in \mathbb{C}$ be translation. Then

$$\varrho(T_u) = \begin{bmatrix} 1 & & & \\ \Re u & 1 & & \\ \Im u & & 1 & \\ u\overline{u} & 2\Re u & 2\Im u & 1 \end{bmatrix}.$$

(7) Let $R_{\zeta,r}$ be reflection in the circle centred at $\zeta \in \mathbb{C}$ of radius $r > 0$. Then

$$\begin{aligned}\varrho(R_{\zeta,r}) &= \varrho(T_\zeta D_r R_\circ D_{1/r} T_{-\zeta}) \\ &= \begin{bmatrix} r^2|u|^2 & 2r^2\Re(u) & 2r^2\Im(u) & r^2 \\ \Re(u)\left(1-r^2|u|^2\right) & 1-2r^2\Re(u)^2 & -2r^2\Im(u)\Re(u) & -r^2\Re(u) \\ \Im(u)\left(1-r^2|u|^2\right) & -2r^2\Im(u)\Re(u) & 1-2r^2\Im(u)^2 & -r^2\Im(u) \\ r^2|u|^4-2|u|^2+\frac{1}{r^2} & 2\Re(u)\left(r^2|u|^2-1\right) & 2\Im(u)\left(r^2|u|^2-1\right) & r^2|u|^2 \end{bmatrix}\end{aligned}$$

3. The algorithm

In the following, we have as given a matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that } \det M = 1$$

together with a boolean flag $\mathcal{OP}$ which is true when we should interpret M as the orientation-preserving Möbius transformation

$$M(z) = \frac{az+b}{cz+d} \in \mathsf{PSL}(2,\mathbb{C})$$

and false when we should interpret it as the orientation-reversing map

$$\tilde{M}(z) = \frac{a\bar{z}+b}{c\bar{z}+d} \in \widetilde{\mathsf{PSL}(2,\mathbb{C})} \setminus \mathsf{PSL}(2,\mathbb{C})$$

We will decompose M as a product of elementary actions, using [4, §I.C.2]. There are two cases, depending on whether c is zero (so M acts as a Euclidean motion) or not.

3.1. The case that $c = 0$. The transformation is the Euclidean motion $z \mapsto (az+b)/d$, possibly with a conjugation. If $\mathcal{OP}$ is True, then

$$\varrho(M) = \varrho(T_{b/d} M_{a/d}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \Re\left(\frac{b}{d}\right) & \Re\left(\frac{a}{d}\right) & -\Im\left(\frac{a}{d}\right) & 0 \\ \Im\left(\frac{b}{d}\right) & \Im\left(\frac{a}{d}\right) & \Re\left(\frac{a}{d}\right) & 0 \\ \frac{bb^*}{|d|^2} & \frac{ba^*+ab^*}{|d|^2} & \frac{i(ab^*-ba^*)}{|d|^2} & 1 \end{bmatrix}.$$

If $\mathcal{OP}$ is False then we must precompose with J, obtaining

$$\varrho(\tilde{M}) = \varrho(T_{b/d} M_{a/d}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \Re\left(\frac{b}{d}\right) & \Re\left(\frac{a}{d}\right) & \Im\left(\frac{a}{d}\right) & 0 \\ \Im\left(\frac{b}{d}\right) & \Im\left(\frac{a}{d}\right) & -\Re\left(\frac{a}{d}\right) & 0 \\ \frac{bb^*}{|d|^2} & \frac{ba^*+ab^*}{|d|^2} & \frac{i(ba^*-ab^*)}{|d|^2} & 1 \end{bmatrix}.$$

3.2. The case that $c \neq 0$. The meat of the procedure is when M is not a Euclidean motion. We will use boldface letters $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ to denote the matrices p, q, and r from [4, §I.C.2] in order to avoid clashes with other notation.

(1) Compute isometric circle data: set $\alpha = -d/c$, $\beta = a/c$, and $r = 1/|c|$.
(2) Define $\mathbf{p} = \varrho(R_{\alpha,r})$.
(3) If $\alpha = \beta$ then set $\mathbf{q} = I_4$ otherwise set $\mathbf{q} = \varrho(J_{\alpha,\beta})$.
(4) Compute the holonomy angle θ of M:
   (a) If $\alpha = \beta$, set $x = y = \alpha + ir$. Otherwise set

$$\begin{aligned} x &= \frac{r}{|\alpha-\beta|}\alpha + \left(1 - \frac{r}{|\alpha-\beta|}\right)\beta \\ y &= \frac{r}{|\alpha-\beta|}\beta + \left(1 - \frac{r}{|\alpha-\beta|}\right)\alpha. \end{aligned}$$

(b) Then let

$$\theta = \arg\left(\frac{(ax+b)/(cx+d)-\beta}{y-\beta}\right).$$

(5) If $\mathcal{OP}$ is TRUE then set

$$\mathbf{r} = \varrho(T_\beta W_{\theta-\pi} J T_{-\beta})$$

otherwise if $\mathcal{OP}$ is FALSE then set

$$\mathbf{r} = \varrho(T_\beta W_\theta T_{-\beta}).$$

(6) The image of the representation of M (if $\mathcal{OP}$ is TRUE) or $\tilde{M}$ (if $\mathcal{OP}$ is FALSE) is then $\mathbf{rqp}$.

4. Explicit matrices

In this section, $\mathcal{OP}$ is always TRUE and $\alpha \neq \beta$. We set

$$\mu = \frac{1}{(a+d)^2(1-\frac{1}{|a+d|})^2},$$
$$\mathcal{S} = \sin\arg\mu, \mathcal{C} = \cos\arg\mu,$$
$$\mathcal{S}' = \sin\left(\arg\left(-\frac{a+d}{c}\right)\right), \mathcal{C}' = \cos\left(\arg\left(-\frac{a+d}{c}\right)\right).$$

Then $\mathbf{r}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \Im\left(\frac{a}{c}\right)\mathcal{S} + \Re\left(\frac{a}{c}\right)(\mathcal{C}+1) & -\mathcal{C} & -\mathcal{S} & 0 \\ \Re\left(\frac{a}{c}\right)\mathcal{S} - \Im\left(\frac{a}{c}\right)(\mathcal{C}-1) & -\mathcal{S} & \mathcal{C} & 0 \\ \begin{pmatrix} \frac{2|a|^2}{|c|^2} + 4\Im\left(\frac{a}{c}\right)\Re\left(\frac{a}{c}\right)\mathcal{S} \\ + 2\left(\Re\left(\frac{a}{c}\right)^2 - \Im\left(\frac{a}{c}\right)^2\right)\mathcal{C} \end{pmatrix} & -2\begin{pmatrix} \Im\left(\frac{a}{c}\right)\mathcal{S} \\ + \Re\left(\frac{a}{c}\right)(\mathcal{C}+1) \end{pmatrix} & 2\begin{pmatrix} \Im\left(\frac{a}{c}\right)(\mathcal{C}-1) \\ - \Re\left(\frac{a}{c}\right)\mathcal{S} \end{pmatrix} & 1 \end{bmatrix}$$

$\mathbf{q}$ is the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \begin{pmatrix} \Im\left(\frac{a-d}{c}\right)\mathcal{S}'\mathcal{C}' \\ + \frac{1}{2}\Re\left(\frac{a-d}{c}\right)(\mathcal{C}'^2 - \mathcal{S}'^2 + 1) \end{pmatrix} & \mathcal{S}'^2 - \mathcal{C}'^2 & -2\mathcal{S}'\mathcal{C}' & 0 \\ \begin{pmatrix} \Re\left(\frac{a-d}{c}\right)\mathcal{S}'\mathcal{C}' \\ + \frac{1}{2}\Im\left(\frac{a-d}{c}\right)(\mathcal{S}'^2 - \mathcal{C}'^2 + 1) \end{pmatrix} & -2\mathcal{S}'\mathcal{C}' & \mathcal{C}'^2 - \mathcal{S}'^2 & 0 \\ \frac{1}{2}\begin{pmatrix} \left|\frac{(a-d)^2}{c^2}\right| + 4\Im\left(\frac{a-d}{c}\right)\Re\left(\frac{a-d}{c}\right)\mathcal{S}'\mathcal{C}' \\ + \mathcal{C}'^2\left(\Re\left(\frac{a-d}{c}\right)^2 - \Im\left(\frac{a-d}{c}\right)^2\right) \\ + \mathcal{S}'^2\left(\Im\left(\frac{a-d}{c}\right)^2 - \Re\left(\frac{a-d}{c}\right)^2\right) \end{pmatrix} & \begin{pmatrix} -2\Im\left(\frac{a-d}{c}\right)\mathcal{S}'\mathcal{C}' \\ + \Re\left(\frac{a-d}{c}\right)(\mathcal{S}'^2 - \mathcal{C}'^2 - 1) \end{pmatrix} & \begin{pmatrix} -2\Re\left(\frac{a-d}{c}\right)\mathcal{S}'\mathcal{C}' \\ + \Im\left(\frac{a-d}{c}\right)(\mathcal{C}'^2 - \mathcal{S}'^2 - 1) \end{pmatrix} & 1 \end{bmatrix}$$

and $\mathbf{p}$ is the matrix

$$\begin{bmatrix} |d|^2 & dc^* + cd^* & 2|c|^2\Im\left(\frac{d}{c}\right) & |c|^2 \\ -\left(|d|^2 - 1\right)\Re\left(\frac{d}{c}\right) & 1 - 2|c|^2\Re\left(\frac{d}{c}\right)^2 & -2|c|^2\Im\left(\frac{d}{c}\right)\Re\left(\frac{d}{c}\right) & -|c|^2\Re\left(\frac{d}{c}\right) \\ -\left(|d|^2 - 1\right)\Im\left(\frac{d}{c}\right) & -2|c|^2\Im\left(\frac{d}{c}\right)\Re\left(\frac{d}{c}\right) & 1 - 2|c|^2\Im\left(\frac{d}{c}\right)^2 & -|c|^2\Im\left(\frac{d}{c}\right) \\ \frac{|d|^4 - 2|d|^2 + 1}{|c|^2} & 2\left(|d|^2 - 1\right)\Re\left(\frac{d}{c}\right) & 2\left(|d|^2 - 1\right)\Im\left(\frac{d}{c}\right) & |d|^2 \end{bmatrix}.$$

One may compute the product $\varrho(M) = \mathbf{rqp}$ using MATHEMATICA, see Listing 1. The final expression has `LeafCount` of 4215, and is far too complicated to include here.

```
W[\[Theta]_] := {{1,0,0,0},
                 {0,Cos[\[Theta]],-Sin[\[Theta]],0},
                 {0,Sin[\[Theta]],Cos[\[Theta]],0},
                 {0,0,0,1}};
J[] := {{1,0,0,0}, {0,1,0,0}, {0,0,-1,0}, {0,0,0,1}};
T[u_] := {{1,0,0,0},
          {Re[u],1,0,0},
          {Im[u],0,1,0},
          {Abs[u]^2, 2 Re[u], 2 Im[u] ,1}};
L[k_] := {{1,0,0,0}, {0,k,0,0}, {0,0,k,0}, {0,0,0,k^2}};
M[z_] := {{1,0,0,0},
          {0,Re[z],-Im[z],0},
          {0,Im[z],Re[z],0},
          {0,0,0,1}};
R[] := {{0,0,0,1}, {0,1,0,0}, {0,0,1,0}, {1,0,0,0}};
J[p_,q_] := T[(p+q)/2].J[].W[-2(Pi/2+Arg[p-q])].T[-(p+q)/2];
R[z_,r_] := T[z].L[r].R[].L[1/r].T[-z];

\[Alpha] = -d/c;
\[Beta] = a/c;
r = Abs[1/c];
p = FullSimplify[R[\[Alpha], r]];
q = TrigExpand@FullSimplify[J[\[Alpha], \[Beta]]];
x = FullSimplify[\[Alpha]*r/Abs[\[Alpha]-\[Beta]] + \[Beta]*(1-r/Abs[\[Alpha]-\[Beta]])];
y = FullSimplify[\[Beta]*r/Abs[\[Alpha]-\[Beta]] + \[Alpha]*(1-r/Abs[\[Alpha]-\[Beta]])];
\[\Theta] = FullSimplify[Arg[((a*x+b)/(c*x+d) - \[Beta])/(y - \[Beta])], {a*d-b*c==1}];
r = TrigExpand@FullSimplify[T[\[Beta]].FullSimplify[W[\[Theta]-Pi].J[]]. T[-\[Beta]]];

finalM = FullSimplify[ Simplify[ Simplify[r.q].p, {a*d-b*c==1}, TimeConstraint->.1], {a*d-b*c==1}, TimeConstraint->.1]
```

Listing 1. Mathematica implementation of the representation. Note, we use `L` to denote dilation as `D` is reserved by the language.

References

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