Tropical geometry and buildings

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October 6, 2020

Abstract

The goal of these three talks is to cover the material of a 2007 paper by Jowsig, Sturmfels, and Yu [JSY07] that applies tropical geometry to the problem of computation in the Bruhat-Tits building of $\mathrm{SL}(n, K)$, where K is a field with non-trivial valuation. The first two talks are introductory talks leading up to the content of the paper; the first also includes some material on tropical algebraic geometry, following e.g. [IMS07], [MS15], [Mac11], or [RST03], and the second includes a basic introduction to buildings, following e.g. [AB08]. The talks should be accessible to anyone with a basic understanding of algebraic geometry (i.e. there is no dependence on any of the theory of toric varieties, and no prior knowledge of buildings should be needed). In addition the first and second talks are mutually independent (though both are needed for the third talk).

Contents

Talk I: Tropical geometry

According to [IMS07], 'tropical geometry first appeared as a subject on its own in 2002'; the name of the subject honours the Brazillian mathematician Imre Simon who used the tropical semiring in the 1980s to study optimisation theory, while the algebro-geometric connection may be traced back to work by George

M. Bergman in the 1970s. Tropical geometry is intimately related to the study of toric varieties and to the fields of combinatorial geometry and combinatorial commutative algebra; our interest in it here is primarily the study of tropicalisations of convex sets, since such objects are precisely the images under a valuation of convex sets in a field — we will use them to study convex hulls in Bruhat-Tits buildings.

I.1 Tropicalisation

Let k be a (possibly not algebraically closed) field. The philosophy of tropical geometry is the existence of a map

$$\mathsf{trop} : \{\text{deformable classical structures over } k\} \to \{\text{combinatorial structures over } \mathbb{R}\} \tag{1}$$

such that

$$\mathsf{trop}\, f = \lim_{\varepsilon \to 0} -\log_\varepsilon \left| f(\varepsilon) \right| \tag{2}$$

(where $f(\varepsilon)$ is the deformation of f by ε).

Of course $-\log|\cdot|$ is only defined if k is something like $\mathbb{R}$ or $\mathbb{C}$. Thus we need to provide an analogous structure on k for this philosophy to be of any use. The most useful '$k \to \mathbb{R}$ maps' turn out to be valuations:

I.1 Definition. A **valuation** on k is a map $v : k \to \mathbb{R} \cup \{\infty\}$ such that for all $a, b \in k$:

1. $v(ab) = v(a) + v(b)$ (motivation: $-\log|ab| = -\log|a| - \log|b|$)
2. $v(a + b) \leq \min\{v(a), v(b)\}$ (motivation: $-\log|x + y| \leq \min\{-\log|x|, -\log|y|\}$)
3. $v(a) = \infty$ iff $a = 0$. (motivation: $-\log|x| = \infty$ iff $x = 0$)

We will also require our valuations to be **anti-compatible** with any ordering $\leq$ on k; namely, $a \leq b \implies v(a) \geq v(b)$. (motivation: $-\log$ is monotone decreasing)

I.2 Lemma. *1.* $v(1) = 0$

2. $v(-a) = v(a)$ *for all* $a \in k$

3. *if* $v(a) \neq v(b)$ *then* $v(a + b) = \min\{v(a), v(b)\}$

Note that we obtain, based on these properties, a natural semigroup structure on $\mathbb{R}$.

I.3 Definition. The **tropical semiring** $\mathbb{T}$ is the semiring supported on $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ with addition $a \oplus b := \min\{a, b\}$ and multiplication $a \odot b := a + b$. The semiring is endowed with a total ordering given by extending that of $\mathbb{R}$ in the obvious way.

Note that some authors (e.g. [IMS07]) take $a \oplus b := \max\{a, b\}$. This makes sense if one replaces $-\log|\cdot|$ in Eq. (1) with $\log|\cdot|$, as $\log(x + y) \approx \max\{\log|x|, \log|y|\}$ whenever x and y are of different magnitude (c.f. part 3 of Lemma I.2). Of course the two structures are equivalent; but we prefer the min formulation as it is what has become standard (despite the max formulation being older, indeed it was studied as early as 1971 by Bergman — c.f. [MS15, section 1.4]). For a more comprehensive discusion of this, see [Jos20, section 1.3].

We will usually define a map trop as in Eq. (1) by replacing every $+$ with $\oplus$, every $\cdot$ with $\odot$, and every field element with its valuation.

I.4 Definition. Let $f \in k[X_1, ..., X_r]$. Then, if $[X_1^{\alpha_1} \cdots X_r^{\alpha_r}]f$ denotes the coefficient of $X_1^{\alpha_1} \cdots X_r^{\alpha_r}$ in f so

$$f = \sum_{X_1^{\alpha_1} \cdots X_r^{\alpha_r} \in \operatorname{supp} f} [X_1^{\alpha_1} \cdots X_r^{\alpha_r}] X_1^{\alpha_1} \cdots X_r^{\alpha_r}$$

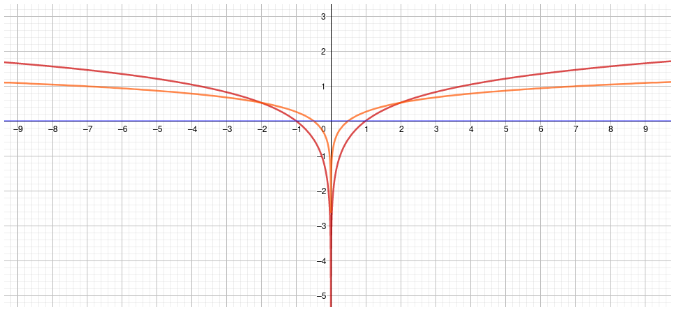

Figure 1: red: $Y = -2\log_\varepsilon |X|$; orange: $Y = \log_\varepsilon |2| + \log_\varepsilon |X|$; blue: $Y = \log_\varepsilon |1|$.

we define the **tropicalisation** trop $f \in \mathbb{T}[X_1, ..., X_r]$ by

$$\operatorname{trop} f := \bigoplus_{X_1^{\alpha_1} \cdots X_r^{\alpha_r} \in \operatorname{supp} f} \nu([X_1^{\alpha_1} \cdots X_r^{\alpha_r}]) \odot X_1^{\alpha_1} \odot \cdots \odot X_r^{\alpha_r}.$$

Note that a tropical polynomial $f \in \mathbb{T}[X_1, ..., X_r]$ defines a continuous finite-piecewise linear concave function $\mathbb{R}^r \to \mathbb{R}$ with integer coefficients for the non-constant terms; in fact the converse is also true, in the sense that every continuous finite-piecewise linear concave function $\mathbb{R}^r \to \mathbb{R}$ with integer coefficients for the non-constant terms is equal to the function defined by some tropical polynomial.

I.2 Tropical varieties

Finding a 'zero' of a tropical polynomial does not always make sense since there is no subtraction: consider $x \oplus -1 = 0$. Thus we need a tropical analogue of the zero locus of a polynomial.

I.5 Example. Recall that our motivation was taking logs of varieties. Consider $f = (X-1)^2 \in \mathbb{C}[X]$, so $\mathbf{Z}(f) = \{1\}$. Then

$$-\log_\varepsilon |f| = -\log_\varepsilon (X^2 - 2X + 1) \approx \min\{-2\log_\varepsilon |X|, \log_\varepsilon |2| + \log_\varepsilon |X|, \log_\varepsilon |1|\};$$

if we graph this for $\varepsilon \to 0$ (Fig. 1) we see that, for sufficiently small ε (in fact $\varepsilon < 1$) we have that, at the points $X = \pm 1$ (i.e. the zeros of f), the minimum is attained twice. The reason for this is that when f has a root at x, $-\log_\varepsilon |f(x)| = -\infty$ so the approximation $\approx$ must fail badly at x; i.e. two of the summands must be of similar magnitude and thus something must be cancelling, so the minimum is attained twice (Note that $\approx$ in the above display means 'approximately, when ε is small and $|f|$ is non-zero'.)

Based on this motivation* and on the 'niceness' of the results we obtain, the tropical analogue of a zero is the following.

I.6 Definition. Let $f \in \mathbb{T}[X]$ be a tropical polynomial. We say that $x \in \mathbb{T}^r$ is a **root** of f if the piecewise function defined by f is non-differentiable at x. The set $\mathbf{Z}(f)$ of roots of f is the **zero locus** or **non-differentiablity locus** of f.

An example of such a 'nice' result is the following.

*which goes deeper: indeed, it is a general philosophy in real algebraic geometry that the number of zeros depends on the number of terms, not the degree, so we would expect a 'realification' of a variety to depend roughly on the number of terms of the original polynomial

I.7 Theorem (Tropical fundamental theorem of algebra)**.** *Let $f \in \mathbb{T}[X]$ be of degree n. Then the function defined by f is equal to the function defined by $(x \oplus \alpha_1) \odot \cdots \odot (x \oplus \alpha_n)$ for some $\alpha_1, ..., \alpha_n \in \bar{\mathbb{R}}$; and this factorisation is unique.*

I.8 Example. Unique factorisation does not hold in more than one variable:

$$(x \oplus 0) \odot (y \oplus 0) \odot (x \odot y \oplus 0) = (x \odot y \oplus x \oplus 0) \odot (x \odot y \oplus y \oplus 0)$$

where all the bracketed factors are irreducible.

In general, if $f \in \mathbb{T}[X_1, ..., X_r]$ is a tropical polynomial. we define $\mathbf{Z}(f)$ to be the non-differentiablity locus of the function defined by f.

I.9 Definition. Let Y be a variety in k^r, and let $\mathfrak{a} \subseteq k[X_1, ..., X_r]$ be the corresponding ideal. We define

$$\mathsf{trop}\, Y := \bigcap_{f \in \mathfrak{a}} \mathbf{Z}(\mathsf{trop}\, f).$$

Note that $\mathsf{trop}\, Y$ is in fact determined by $Y \cap (k^*)^r = Y \cap \mathrm{m{-}Spec}\, k[X_1^{\pm 1}, ..., X_r^{\pm 1}]$, since $\mathsf{trop}(X^m f) = \mathsf{trop}\, f$ (indeed, $\mathsf{trop}(X^m f) = X^m \odot \mathsf{trop}\, f = X^m + \mathsf{trop}\, f$, and $\mathsf{trop}\, f$ is non-differentiable at a point iff $X^m + \mathsf{trop}\, f$ is non-differerentiable there). Thus we will nearly always care only about subvarieties of the torus.

I.10 Example. Tropicalisation of varieties does not commute with intersections. In particular, if $S = \{f_1, ..., f_n\}$ is a finite generating set for $\mathfrak{a}$, it is not necessarily true that $\mathsf{trop}\, Y = \bigcap_{i=1}^n \mathbf{Z}(\mathsf{trop}\, f_i)$. If equality *does* hold, we call S a **tropical basis** for $\mathfrak{a}$. The study of such things belongs to the theory of Gröbner bases over $\mathbb{Z}$ and can be used to study the presentations of discrete groups [MS15, section 1.6].

We may describe the structure of tropical varieties fairly easily:-

I.11 Theorem (Fundamental theorem of tropical algebraic geometry)**.** *Let K be an algebraically closed field with nontrivial valuation v; let $\mathfrak{p} \subseteq K[X_1^{\pm 1}, ..., X_n^{\pm 1}]$ be a prime ideal; and let $Y \subseteq (K^*)^n$ be the variety of $\mathfrak{p}$. Then the following coincide:*

1. *the tropical variety* $\mathsf{trop}\, Y$*;*
2. *the closure in $\mathbb{R}^n$ of the set*

$$v(Y) := \{(v(x_1), ..., v(x_n)) : (x_1, ..., x_n) \in Y\}.$$

Further, if w is any point in $v(K^*)^n \cap \mathsf{trop}\, Y$, then $v^{-1}(w) \cap Y$ is dense in Y.

Discussion of proof. The full details of the proof may be found as [MS15, theorem 3.2.3]. The idea is to prove the result for the hypersurface case (which is itself non-trivial, see [MS15, theorem 3.1.3] which uses Gröbner basis theory for the hypersurface proof) and then to use a projection to the hypersurface case to prove the general result.

A tropical variety is a particular kind of polyhedral complex.

I.12 Definition. A **polyhedron** Δ is the intersection of finitely many halfspaces; in particular, it is a closed convex set (so the notion of faces makes sense). We will say that a polyhedron is **Λ-rational** for a lattice Λ embedded in the ambient space $\mathbb{R}^n$ if the vertices of Δ lie in Λ.

A **polyhedral complex** is a collection Σ of polyhedra such that:

1. If $\sigma \in \Sigma$ and $\tau \preceq \sigma$ then $\tau \in \Sigma$.

2. If $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau$ is a face of τ and σ.

A **facet** of Σ is a polyhedron $\sigma \in \Sigma$ that is maximal with respect to $\preceq$. The complex is **pure of dimension** d if every facet of Σ has dimension d. We shall denote the set of d-dimensional polyhedra by $F_d(\Sigma)$. It is **connected through codimension 1** if, for every two d-dimensional polytopes $\sigma, \sigma' \in \Sigma$, there is a chain $\sigma = \sigma_1, ..., \sigma_m = \sigma'$ of d-dimensional polytopes of Σ such that $\sigma_i \cap \sigma_{i+1}$ is a facet of both σ_i and σ_{i+1} $(1 \leq i < m)$.

At each polyhedron $\tau \in \Sigma$ the **star of** Σ **at** τ, $\mathrm{star}_\Sigma(\tau)$, is the smallest subcomplex of Σ consisting of all the members of Σ with τ as a face; we place a fan structure on it by taking the cone corresponding to $\sigma \in \mathrm{star}_\Sigma(\tau)$ to be $\hat{\sigma} = \mathbb{R}_{\geq 0}\{x - y : x \in \sigma, y \in \rho\}$.

Our tropical varieties shall have some additional structure: they will turn out to be polyhedral complexes, pure of dimension d, such that each facet is assigned a weight in a global way.

I.13 Definition. Let Σ be a Λ-rational fan in $\mathbb{R}^n$, pure of dimension d; the fan is **weighted** if it is endowed with a function $m : F_d(\tau) \to \mathbb{N}$. If $\tau \in F_{(d-1)}(\Sigma)$ then let $L = \operatorname{span} \tau$; L is a $(d-1)$-dimensional subspace, and so $L \cap \Lambda$ is a lattice of degree $d - 1$ and $N(\tau) \cap \Lambda = \Lambda/(L \cap \Lambda)$ is a lattice of degree $n - d + 1$.

Let σ be a facet of Σ including τ as a face; then $(\sigma + L)/L$ is a lattice cone of dimension 1; let x_σ be the first lattice point on this ray. Then Σ is **balanced at** τ if

$$\sum_{\tau \preceq \sigma} m(\sigma) x_\sigma = 0$$

and Σ is **balanced** if it is balanced at all $\tau \in F_{(d-1)}(\Sigma)$. A polyhedral complex Σ is **balanced** if the fan $\mathrm{star}_\Sigma(\tau)$ is balanced for all $\tau \in F_{(d-1)}(\Sigma)$.

I.14 Theorem (Structure theorem of tropical varieties)**.** *Let Y be a subvariety of $(K^*)^n$ of dimension d. Then* $\mathsf{trop}\, Y$ *is the support of a balanced, weighted, $v(K^*)$-rational polyhedral complex, pure of dimension d and connected through codimension 1.*

Discussion of proof. The full details occupy sections 3.3 to 3.5 of [MS15]. We will briefly mention what weighting function we need in order to make $\mathsf{trop}\, Y$ balanced; the idea is that if Y is irreducible of dimension d and σ is a maximal ‘ of the complex Σ that $\mathsf{trop}\, Y$ is supported on, then the vanishing set of all the initial terms of $\mathfrak{a}$ is a d-dimensional toric variety and hence (by toric algebraic geometry) is a union of d-dimensional toric orbits; then the weight $m(\sigma)$ is the number of such orbits. (This is the content of [MS15, lemma 3.4.7].)

I.3 Tropical convexity

Let k be a field with a valuation $v : k \to \mathbb{R}$ that is *surjective*.

Recall that a **cone** in k^n (for an ordered field k) is a set C such that C is closed under addition, and such that $k_{\geq 0} C \subseteq C$. For any set S we write $\operatorname{pos} S$ for the smallest cone containing S. A cone C is **polyhedral** if $C = \operatorname{pos} S$ for $|S| < \infty$. A cone is **strongly convex** if it does not contain any line.

I.15 Lemma. *Let $x_1, ..., x_r \in k^n$; a linear combination $\lambda_1 x_1 + \cdots + \lambda_r x_r$ is **positive** if each $\lambda_i \in k_{\geq 0}$. If $S \neq \emptyset$, then* $\operatorname{pos} S$ *is the set of all positive combinations of finitely many elements of S.*

These characterisations may be tropicalised.

I.16 Definition. A set $C \subseteq \mathbb{T}^n$ is a **tropical cone** if it is closed under tropical addition and tropical scalar multiplication. If $S \subseteq \mathbb{T}^n$ is any subset, we write $\operatorname{tpos} S$ for the set of all tropical linear combinations $\lambda_1 \odot x_1 \oplus \cdots \oplus \lambda_r \odot x_r$ where the $x_i \in S$ and the $\lambda_i \in \mathbb{T}$. A tropical cone C is **polyhedral** if $C = \operatorname{tpos} S$ for $|S| < \infty$

Note that we are replacing $k_{\geq 0}$ with $\mathbb{T}$, not $\mathbb{T}_{\geq 0}$. This is because $v(k_{\geq 0})$ hits every element of $\mathbb{T}$ (indeed, $v(-x) = v(x)$ for all $x \in k$).

I.17 Lemma. *For every cone $C \subseteq k^n$, $v(C)$ is a cone in $\mathbb{T}^n$ and conversely every cone arises in this way. C is polyhedral iff $v(C)$ is polyhedral.*

Now we characterise convexity. A set $X \subseteq \mathbb{P}V$ for V a finite dimensional vector space over k (here $\mathbb{P}V$ denotes the projective space obtained from V, i.e. $\mathbb{P}V = V \setminus \{\mathbf{0}\}/\sim$ where $a \sim b$ iff $a = \lambda b$ for some $\lambda \in k$, and $\dim \mathbb{P}V = \dim V - 1$) is **convex** if its inverse image in V is convex. Note that the inverse image will be a cone; if the cone is polyhedral then X is called a **polytope**.

I.18 Definition. We define **tropical projective space** of dimension n to be the set $\mathbb{P}\mathbb{T}^n := \mathbb{T}^{n+1} \setminus \{\infty\}/\sim$ where $a \sim b$ iff $a = \lambda \odot b$ for some $\lambda \in \mathbb{T}$.

Note that $\lambda \odot b = \lambda\mathbf{1} + b$; hence as a set we have that $\mathbb{P}\mathbb{T}^n = \mathbb{T}^{n+1} \setminus \{\infty\}/\mathbb{R}\mathbf{1}$, equipped with the canonical projection $v \mapsto v + \mathbb{R}\mathbf{1}$ and the standard basis $(e_1, ..., e_{n+1})$ given by $e_i = (\infty, ..., \infty, 0, \infty, ..., \infty)$ where the 0 is in the ith position.

Remark. The tropical projective space $\mathbb{P}\mathbb{T}^n$ is a compactification of $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ such that the pair of spaces $(\mathbb{P}\mathbb{T}^n, \mathbb{R}^{n+1}/\mathbb{R}\mathbf{1})$ is homeomorphic to the pair of spaces $(\Delta_n, \operatorname{int} \Delta_n)$ ($\Delta_n = \operatorname{conv}\{e_1, ..., e_{n+1}\}$ the standard n-simplex in $\mathbb{R}^n + 1$). [Jos20, proposition 5.3]

If $v \in \mathbb{P}\mathbb{T}^n$, the vector of **canonical coordinates** for v is the unique $w \in \mathbb{T}^{n+1}$ such that each component of w is non-negative and has at least one zero coordinate. We write $|v|$ for this vector.

I.19 Definition. A subset $X \subseteq \mathbb{P}\mathbb{T}^n$ is **tropically convex** if it arises as the image of a cone under the canonical projection $\mathbb{T}^{n+1} \to \mathbb{P}\mathbb{T}^n$. If $S \subseteq \mathbb{P}\mathbb{T}^n$ is any subset, we define the **convex hull** $\operatorname{tconv} S$ to be the smallest tropically convex set containing S; it is the image of the set of tropical positive combinations of elements of X under the canonical projection:

$$\operatorname{tconv} M := \{\lambda_1 \odot |x_1| \oplus \cdots \oplus \lambda_r \odot |x_r| + \mathbb{R}\mathbf{1} : r \in \mathbb{N}, \lambda_1, ..., \lambda_r \in \mathbb{T}, x_1, ..., x_r \in \mathbb{T}^{n+1}\}$$

I.20 Theorem (Properties of tropically convex sets)**.**

1. *Intersections of tropically convex sets are tropically convex.*
2. *(Carathéodory's theorem) If $x \in K = \operatorname{tconv}\{x_1, ..., x_r\}$ in $\mathbb{P}\mathbb{T}^n$ with $r > n$, then x is a tropical linear combination of $n - 1$ of the x_i. [Jos20, theorem 5.37]*
3. *Tropical linear spaces are tropically convex. [MS15, proposition 5.2.8]*
4. *Tropical polytopes are compact.*
5. *The tropical line segment $\mathsf{trop}[p, q] := \operatorname{tconv}\{p, q\}$ in $\mathbb{T}\mathbb{P}^n$ is the union of at most n classical line segments considered as a subset of $\mathbb{R}^n$. [Jos20, proposition 5.11]*
6. *(Farkas' theorem / Point separation lemma) Let P be a tropical polytope. For all $x = (x_1, ..., x_{n+1}) \in \mathbb{P}\mathbb{T}^n$, if x does not lie in V then there is a tropical hyperplane defined by $\mathbb{Z}(a_1 \odot X_1 + \cdots + a_{n+1} \odot X_{n+1})$ such that for some k we have $a_k + x_k = \min\{a_1 + x_1, ..., a_{n+1} + x_{n+1}\}$ (i.e. the minimum is attained at least once) and such that for all points of P, the minimum is never attained. [MS15, proposition 5.2.10]*

I.21 Example. We will compute $\mathsf{trop}[p, q]$ in $\mathbb{P}\mathbb{T}^2$ for $p = (0, 1, 0)$ and $q = (0, 4, 1)$. A point x lies in $\mathsf{trop}[p, q]$ iff $x = \alpha \odot (0, 1, 0) \oplus \beta \odot (0, 4, 1)$; i.e. $x = (\min\{\alpha, \beta\}, \min\{\alpha + 1, \beta + 4\}, \min\{\alpha, \beta + 1\})$. Note

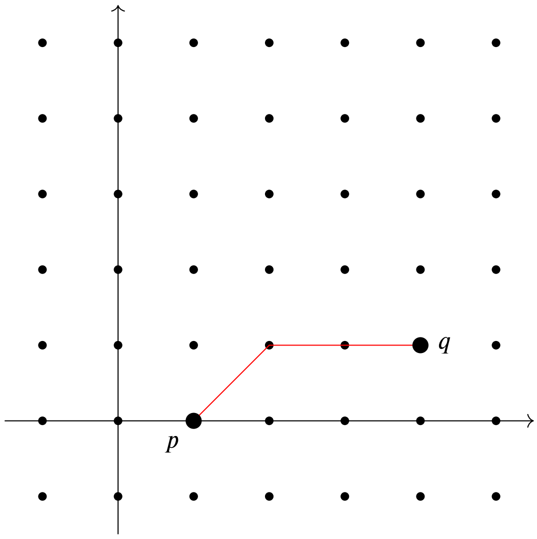

Figure 2: The line segment $\mathsf{trop}[p,q]$.

$x+\mathbb{R}\mathbf{1} = x-\beta\mathbf{1}+\mathbb{R}\mathbf{1}$, so setting $\delta = \alpha - \beta$ we have $x \sim (\min\{\delta,0\}, \min\{\delta+1,4\}, \min\{\delta,1\})$. Now:

range of δ	**generic point of** $\mathsf{trop}[p,q]$	**corresponding segment**
$\delta \le 0$	$(\min\{\delta,0\}, \min\{\delta+1,4\}, \min\{\delta,1\}) = (0,1,0)$	p
$0 \le \delta \le 1$	$(\min\{\delta,0\}, \min\{\delta+1,4\}, \min\{\delta,1\}) = (0,\delta+1,\delta)$	$[p,(0,2,1)]$
$1 \le \delta \le 3$	$(\min\{\delta,0\}, \min\{\delta+1,4\}, \min\{\delta,1\}) = (0,\delta+1,1)$	$[(0,2,1),q]$
$3 \le \delta$	$(\min\{\delta,0\}, \min\{\delta+1,4\}, \min\{\delta,1\}) = (0,4,1)$	q.

See Fig. 2.

I.22 Example. Let $v_1 = (0,1,0), v_2 = (0,4,1), v_3 = (0,3,3), v_4 = (0,0,2) \in \mathbb{PT}^2$. We compute the tropical polytope $\mathrm{tconv}\{v_1,...,v_4\}$ to be the shaded area of Fig. 3.

Talk II: Buildings

Recall from my earlier talk that a simplicial complex Δ on a finite set X is a collection of subsets of X (**simplices**) closed under taking subsets. To extend this to infinite vertex sets, we will ask that each simplex is finite. To avoid redundancy we will usually ensure that the set X contains only as many elements as are needed (i.e. delete any elements from X that do not, as singletons, appear as faces).

II.1 A straight line to a definition

The following definition appears in [Cox73, chapter XI] in the context of the study of kaleidoscopes.

II.1 Definition. A **Coxeter group of rank** n is a group of the form

$$W = \langle x_1,...,x_n | \forall_{i,j} (x_i x_j)^{m(i,j)} = 1 \rangle$$

where each $m(i,j) = m(j,i) \in \mathbb{Z} \cup \infty$ (the relation $(x_i x_j)^\infty$ meaning that $x_i x_j$ has infinite order) and $m(i,i) = 1$ for all i.

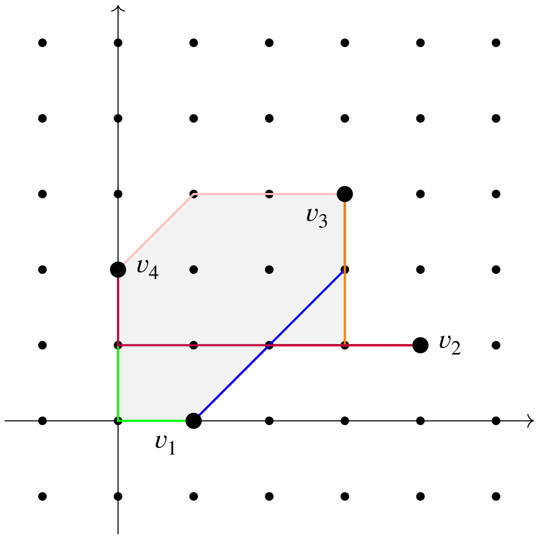

Figure 3: The line segments $\mathsf{trop}[v_1, v_2]$ (red), $\mathsf{trop}[v_1, v_3]$ (blue), $\mathsf{trop}[v_1, v_4]$ (green), $\mathsf{trop}[v_2, v_3]$ (orange), $\mathsf{trop}[v_2, v_4]$ (purple), $\mathsf{trop}[v_3, v_4]$ (pink).

The **Coxeter system** is the pair (W, S) where $S = \{x_1, ..., x_n\}$ is the distinguished set of generators of W.

For example, the dihedral groups are generated by the reflection elements, and the products of distinct pairs of reflections are rotations.

II.2 Theorem. *Let W be a group generated by a subset S of elements of order 2. The following are equivalent:*

1. *(W, S) is a Coxeter system.*
2. *There is an action of W on $S^W \times \{\pm 1\}$ such that a generator $s \in S$ acts as*

$$s(t, \varepsilon) = \begin{cases} (t^s, \varepsilon) & t \neq s \\ (s, -\varepsilon) & t = s. \end{cases}$$

   The elements of S^W are the ***reflections*** *of W; the idea is that every reflection $t \in S^W$ determines a 'wall' (i.e. the 'reflecting hyperplane') and two halfspaces $(t, +)$ and $(t, -)$; a reflection acts on its own halfspaces by exchanging them, and acts on other halfspaces by 'rotation or translation' (i.e. movement).*

3. *For an element $w \in W$, let $l(w)$ denote the length of the shortest possible representation of w as a word in S. If $w = s_1 \cdots s_m$ with $m > l(w)$, then there are indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_m$ (where a hatted symbol is omitted).*

   Intuitively, every non-geodesic 'reflection path' can be reduced by cancelling pairs of reflections.

4. *For any $w \in W$, $s \in S$, and representation $w = s_1 \cdots s_d$ of w as a* reduced *word in S, either $l(sw) = d + 1$ or there is some i such that $w = ss_1 \cdots \hat{s}_i \cdots s_d$.*

   Intuitively, extending a path by a reflection either gives a cell that is minimally a single reflection further ('reflecting further away from the centre'), or a cell that is the same length (i.e. we 'reflect across the centre').

5. *For any $w \in W$ and $s, t \in S$ such that $l(sw) = l(w)+1$ and $l(wt) = l(w)+1$, either $l(swt) = l(w)+2$ or $swt = w$.*

   Intuitively, extending a path in both directions either does nothing (the added reflections cancel) or gives a path two steps longer (a pair of reflections that do not cancel have total constructive interference).

Proof. [AB08, theorem 2.49]

A Coxeter system (W, S) is just a generalised reflecton group; indeed, it has a canonical representation in a vector space.

II.3 Theorem. *Let V be the vector space with basis $(e_s : s \in S)$ over $\mathbb{R}$. Define a symmetric bilinear form B on V given on the basis by*

$$B(e_s, e_t) = -\cos \frac{\pi}{m(s,t)}.$$

Then (e_s) is orthonormal with respect to B, and the obvious action of W by left multiplication on V is faithful and has the property that for all $s \in S$, $v \in V$,

$$s(v) = v - 2\,\mathrm{proj}^B_{e_s} v$$

where $\mathrm{proj}^B_{e_s} := \frac{B(e_s,v)}{B(e_s,e_s)} e_s = B(e_s, v)e_s$ is the orthogonal projection of v onto e_s with respect to B. In addition, this action induces an action of W on V^* by setting $\langle wv|wv^*\rangle = \langle v|v^*\rangle$ for all $v \in V$, $v^* \in V^*$.

*More informally, s acts as the reflection across the hyperplane $e_s^\perp$. The representation $W \to \mathrm{GL}(V)$ is the **canonical linear representation** of (W, S).*

Proof. Straightforward, see [AB08, sections 2.5.1 and 2.5.2].

II.4 Definition. If (W, S) is a Coxeter system with canonical linear representation V, we define:

1. $\Phi(W, S) := \{we_s : w \in W\}$ the set of **roots**
2. $\mathcal{H} := \{we_s^\perp : w \in W\}$ the set of **walls** (we write H_s for $e_s^\perp = \{v : \langle v|e_s\rangle = 0\}$)
3. $\mathcal{C} := \{wC : w \in W\}$ the set of **chambers** where $C = \cap_{s\in S}\{v \in V^* : \langle v|e_s\rangle > 0\}$ is the **fundamental chamber**.

Now we state some combinatorial properties without proof; these should be well-motivated by the standard theory of polyhedra and their face lattices (noting that each chamber is the interior of a polyhedron):-

II.5 Theorem.

For every wall wH_s and every chamber D, D lies on exactly one side of wH_s (write wH_s as the set $\langle\cdot|w^{-1}e_s\rangle = 0$ and require $\langle d|w^{-1}e_s\rangle = 0$ to have constant sign as d ranges over D).

Every chamber D may be addressed by a sequence $\{\varepsilon_{wH_s} : wH_s \in \mathcal{H}\}$ where ε_{wH_s} is $+$ or $-$ depending on whether D is on the positive or negative side of wH_s. There is a bijection between chambers and such sequences.

For any $s \in S$, H_s is the only wall separating C from sC. Any two chambers are separated by only finitely many walls. Equivalently, the sign sequences for any two chambers differ in only finitely many locations.

II.6 Example. Consider $D_6 = \langle s, t | s^2 = t^2 = (st)^3 = 1\rangle$. Its canonical linear representation is depicted as Fig. 4.

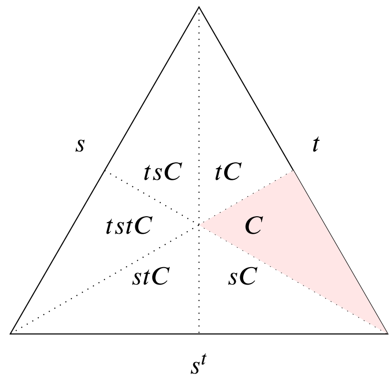

Figure 4: The canonical linear representation of D_6.

Let us now allow, in our 'addresses' for chambers, zero signs; then we may obtain the *faces* of the chambers as polyhedra in V. These faces are in correspondence with cosets of the form $w\langle J\rangle$ for subsets J of S: if A is a face of C, then its stabiliser is the set generated by all the elements of S whose hyperplanes contain it, and faces of every other chamber may be obtained by taking cosets; conversely, the fixed set of $w\langle J\rangle$ is a face of wC. Note that this correspondence reverses the face lattice.

II.7 Definition. A **standard coset** of (W, S) is a coset of the form $w\langle J\rangle$ for some $w \in W$, $J \subseteq S$. We define the **Coxeter complex** of (W, S), $\Sigma = \Sigma(W, S)$, to be the poset of standard cosets of (W, S) ordered by reverse inclusion; i.e. $A \preceq B$ in $\Sigma(W, S)$ iff $A \supseteq B$ in W. If $A \preceq B$ we say A is a **face** of B. Under this relation we obtain a simplicial complex; terminologically, the elements of Σ are **simplices**, maximal elements $w\langle 1\rangle$ are **chambers**, and simplices $w\langle s\rangle$ for $s \in S$ are **panels**. The **fundamental chamber** is $\langle 1\rangle$. There is a canonical action of W on Σ by left multiplication, which is simply transitive. Given $w \in W$ we may write $w = s_1 \cdots s_l$ with $s_i \in s$ for all I; then the sequence $wC, s_1 \cdots s_{l-1}C, ..., s_1C, C$ is a **gallery**. Note that we can put an incidence structure on the chambers of Σ by declaring $C \sim D$ iff there is a gallery of length 1 between C and D (i.e. $C = sD$ for some $s \in S$).

We now glue these simplicial complexes together:-

II.8 Definition. A **building** is a simplicial complex Δ that can be expressed as the union of subcomplexes called **apartments** satisfying the following axioms.

1. Each apartment is a Coxeter complex.
2. For any two simplices $A, B \in \Delta$ there is an apartment containing both.
3. If Σ and Σ' are two apartments containing A and B, then there is an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise.

A building is **thick** if every panel is a face of at least three chambers.

Note that a consequence of these axioms is that every apartment is isomorphic to a fixed Coxeter complex $\Sigma(W, S)$. We call $|S|$ the **rank** of the building, and $|S| - 1$ the **dimension**. In fact ([AB08, proposition 4.6]) every building is $|S|$-vertex-colourable.

In particular, every building is a **chamber complex**: that is, a simplicial complex, pure of some dimension $d < \infty$, such that the graph of maximal simplices is path-connected (in the sense of being 'connected through codimension 1', Definition I.12).

II.2 Hopefully some examples?

Let P be an incidence structure (i.e. a set with a binary relation $\sim$ that is reflexive and symmetric). A **flag** of P is a set of pairwise incident elements of P (i.e. $F \subseteq P$ is a flag if $x, y \in F \implies x \sim y$). For example, let V be a fdvs over a field k and define P to be the set of linear subspaces of V under inclusion; then define $x \sim y$ in P iff there is some inclusion chain $\emptyset \subseteq \cdots \subseteq V$ of subspaces including both x and y. (This is the origin of the term 'flag'!) The **flag complex** of P is the simplicial complex with P as vertex set and finite flags as simplices.

We now give some examples of buildings.

II.9 Example. Consider a building $\mathcal{B}$ of rank 1, so $W = \langle s | s^2 = 1 \rangle = C_2$. The Coxeter complex of $(W, \{s\})$ is the lattice

$$\begin{tikzcd}[column sep=small] \{1\} \arrow[dr, dash] & & \{s\} \arrow[dl, dash] & \text{(chambers)} \\ & \emptyset & & \end{tikzcd}$$

Hence an apartment of $\mathcal{B}$ is the simplicial complex consisting of two distinct vertices. Thus $\mathcal{B}$ is a simplical complex consisting of at least two distinct vertices with every pair of vertices forming an apartment.

II.10 Example. Consider a building $\mathcal{B}$ of rank 2, so $W = \langle s, t | s^2 = t^2 = (st)^m = 1 \rangle$. Then $W = D_{2m}$, and the Coxeter complex of $(W, \{s, t\})$ is the m-gon. If $m = 2$, we glue together 2-coloured quadrangles (the apartments) such that vertex-vertex, vertex-edge, or edge-edge pair appears in at last one square. To preserve colouring we must therefore have every vertex of type 1 joined by an edge to every vertex of type 2; i.e. we may view the building as an incidence geometry of points and lines such that every vertex lies on each line. Similarly if $m = 3$ we obtain an incidence geometry such that every pair of vertices lies on a unique line and vice versa, i.e. we obtain a combinatorial projective plane. The converse is also true: the flag complex of a projective plane is a building with triangles as apartments (the six points of the hexagon are three vertices and three edges distinguished by colouring).

For reference, we give the axioms and construction of finite projective planes over finite fields. A very good reference is [Wal88, chapter 5].

II.11 Definition. A **finite projective plane** is a finite set X which can be written as a disjoint union $X_0 \cup X_1$ where the elements of X_0 are **points** and the elements of X_1 are **lines**, together with an incidence relation $\sim$ such that:

1. For any pair $x, y \in X_0$ there is a unique $z \in X_1$ such that $x \sim z \sim y$;
2. For any pair $x, y \in X_1$ there is a unique $z \in X_0$ such that $x \sim z \sim y$;
3. We say a subset $A \subseteq X_0$ is **colinear** if there exists $\ell \in X_1$ such that $a \sim \ell$ for all $a \in A$. There exists a subset $Q \subseteq X_0$ such that $|Q| = 4$ and such that no 3-subset of Q is colinear.

In such an incidence structure there is a fixed constant n such that every line contains $n + 1$ points; then the structure is a balaced incomplete block design with parameters $(v, b, r, k, \lambda) = (n^2 + n + 1, n^2 + n + 1, n + 1, n + 1, 1)$.

A **finite projective geometry of dimension** d over GF(q), denoted by PG(d, q), is the set of non-zero vectors of GF$(q)^{d+1}$ modulo the relation $x \sim y \iff x = \lambda y$ for some $\lambda \in$ GF(q). Each PG$(2, q)$ is a finite projective plane.

Remark. Not every finite projective plane is constructed from a field!

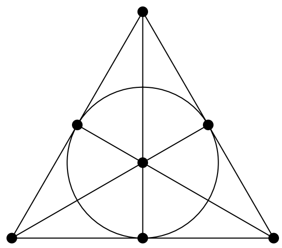

Figure 5: The Fano plane.

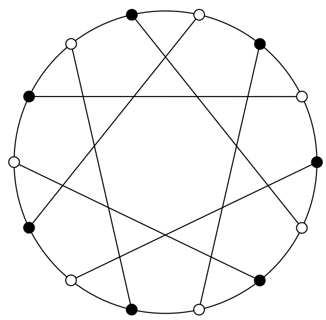

Figure 6: The building associated to the Fano plane.

II.12 Example. The **Fano plane** is the projective plane over GF(2) (a BIBD with parameters $(7, 7, 3, 3, 1)$), pictured in Fig. 5. The apartments of the corresponding building $\mathcal{B}$ are the triangles of the plane, made up of three vertices and three edges. There are therefore 14 vertices of $\mathcal{B}$, $\binom{7}{3} - 7 = 28$ apartments (all combinations of 3 vertices minus the colinear arrangements), and $7 \cdot 3 = 21$ edges (an edge exists between two vertices iff one end is incident with the other end, and each point is incident with exactly 3 lines so the number of pairs (x, y) with $x \sim y$ is $7 \cdot 3$). The building itself is depiced in Fig. 6.

Finally we give an example of a different flavour (c.f [Ser80] for details).

II.13 Example. Let T be an r-regular tree. Then T is a building with simplices the vertices and edges, and apartments infinite paths (that is, a sequence of vertices f indexed by $\mathbb{Z}$ such that $f(i) \sim f(i+1)$ and $f(i) \neq f(i+2)$ for all $i \in \mathbb{Z}$). The apartments are Coxeter complexes associated to D_∞.

II.3 BN-pairs

In this section and the next we follow a mixture of [AB08, chapter 6] and [Ser80, section II.1].

Our next goal is to try to identify a class of groups G with which we may associate a building Δ such that $G \leq \operatorname{Aut} \Delta$.

We say that an action of a group G on a building Δ is **strongly transitive** if it is transitive on the set of pairs (Σ, C) consisting of an apartment Σ of Δ and a chamber C of Σ. Pick an arbitrary pair (Σ, C) of this form, and let (W, S) be the Coxeter system of Δ. Then we introduce:

$$B = \operatorname{Stab}_G C$$
$$N = \operatorname{Stab}_G \Sigma$$
$$T = \cap_{\sigma \in \Sigma} \operatorname{Stab}_G \sigma.$$

(i.e. B is the stabiliser of the fundamental chamber, N the stabiliser of the fundamental apartment, and T the pointwise stabiliser of that apartment). Note also that there is a morphism $f : N \to W$ since every action on Σ induces an action on the whole building; this is surjective (every action on the building restricts to an action on Σ) and $\ker f = T$, so $W \simeq N/T$. Finally, $T = B \cap N$. It turns out that this is the general kind of structure we need on our group G in order to be able to *construct* a building:

II.14 Definition. A pair of subgroups (B, N) of a group G is a BN**-pair** if $G = \langle B, N \rangle$, $B \cap N$ is normal in G, and $W = G/(B \cap N)$ has a set S of generators satisfying the following conditions:

1. For all $s \in S$, $w \in W$, $sBw \subseteq BswB \cup BwB$;
2. For all $s \in S$, $sBs^{-1} \nsubseteq B$.

The quadruple (G, B, N, S) is a **Tits system**; the group W is the **Weyl group** associated to the system.

II.15 Theorem. *Given a BN-pair in G, the generating set S is uniquely determined and (W, S) is a Coxeter system. There is a thick building Δ and a strongly transitive G-action on Δ such that B is the stabiliser of a fundamental chamber and N stabilises a fundamental apartment and is transitive on its chambers.*

Conversely, if a group G acts transitively on a thick building Δ with fundamental apartment Σ and fundamental chamber $C \in \Sigma$ then (B, N) ($B = \mathrm{Stab}_G C$, $N \subseteq \mathrm{Stab}_G \Sigma$ transitive on the chambers of Σ). Then (B, N) is a BN-pair in G and Δ is canonically isomorphic to the building induced by it.

Proof. [AB08, theorem 6.56]

II.4 The Bruhat-Tits building

We will now survey the construction of buildings that are naturally acted upon by $\mathrm{SL}_n(K)$; we will do the construction in full for the case $n = 2$ and then sketch the case $n \geq 3$. We will also indicate the general theory obtained when $\mathrm{SL}_n(K)$ is replaced with any of the classical groups. A good source of motivation for the study of the interaction of $\mathrm{SL}_n(K)$ and buildings seems to be [Ji08].

A valuation v on a field K (Definition I.1) is **discrete** if $v(K^*) = \mathbb{Z}$. We will always assume such valuations to be surjective. The set $R = \{x \in K : v(x) \geq 0\}$ is the **valuation ring** of K, and any ring arising in such a way is a **discrete valuation ring**. The unit group R^* is precisely $v^{-1}(0)$, and $\mathfrak{m} = R \setminus R^*$ is the unique maximal ideal of R. If π is any element of R such that $v(\pi) = 1$ then $\mathfrak{m} = (\pi)$ and each $x \in K$ may be written uniquely in the form $x = \pi^n u$ for $n \in \mathbb{Z}$, $u \in R^*$. Thus $K = R[\pi^{-1}]$. We write $k := R/\mathfrak{m}$ and call this the **residue field** associated to v. Further, every nonzero ideal of R is of the form $(\pi)^n$ for some $n \in \mathbb{Z}_{\geq 0}$ and hence R is a PID.

II.16 Definition. Let V be a n-dimensional vector space over K. An R**-lattice** is an R-module $\Lambda \subseteq V$ such that Λ is generated by a basis for V; i.e. Λ is a free R-module of rank n.

II.17 Lemma. *If Λ and Λ' are two R-lattices, there exists a basis $(e_1, ..., e_n)$ for Λ and integers $r_1, ..., r_n$ such that $(\pi^{r_1} e_1, ..., \pi^{r_n} e_n)$ is a basis for Λ'; the integers $r_1, ..., r_n$ are unique up to ordering. $\Lambda \subseteq \Lambda'$ iff $r_i \geq 0$ for each i.*

The tree of $\mathrm{SL}_2(K)$. Consider the case where $n = 2$; define an equivalence relation $\sim$ on the set of R-lattices by $\Lambda \sim \Lambda'$ if there exists a basis (e_1, e_2) for Λ and a scalar $\lambda \in K^*$ such that $(\lambda e_1, \lambda e_2)$ is a basis for Λ'. Let the set of equivalence classes of lattices be denoted X. Consider two lattices Λ and Λ', and pick bases as in the lemma so $\Lambda = Re_1 \oplus Re_2$, $\Lambda' = R(\pi^{r_1} e_1) \oplus R(\pi^{r_2} e_2)$. Define a function d from the set of pairs of lattices to $\mathbb{Z}$ by sending $(\Lambda, \Lambda') \mapsto |r_1 - r_2|$; then if $s, t \in K^*$ give lattices $s\Lambda$, $t\Lambda$ we have bases (se_1, se_2) and $(t\pi^{r_1} e_1, t\pi^{r_2} e_2)$ so the second base may be written in terms of the first as

$$\left(\frac{t}{s} \pi^{r_1} se_1, \frac{t}{s} \pi^{r_2} se_2 \right)$$

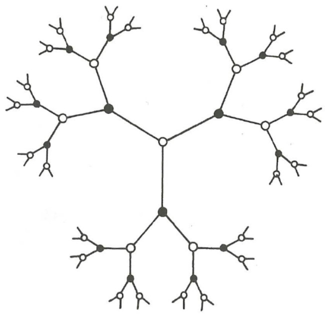

Figure 7: The tree X corresponding to lattices over $\mathbb{Q}_2$. Figure from [Ser80, p. 71].

so $d(s\Lambda, t\Lambda') = \left|(r_1 + v(t/s)) - (r_2 + v(t/s))\right| = \left|r_1 - r_2\right| = d(\Lambda, \Lambda')$; i.e. d induces a well-defined function on X. This gives a metric on X (i.e. $d(A, B) \geq 0$ with equality iff $A = B$; $d(A, B) + d(B, C) \geq d(A, C)$; and $d(A, B) = d(B, A)$).

We define a relation $\approx$ on X by $\Lambda \approx \Lambda' \iff d(\Lambda, \Lambda') = 1$. This defines an incidence structure on X. In fact ([Ser80, theorem 1 of II.1]) the structure is a tree. Further, this tree is regular: if Λ is a fixed vertex of X then the vertices Λ' such that $d(\Lambda, \Lambda') = \delta$ are in bijection with points of the projective line over $R/(\pi^\delta)$ since we may choose our representative Λ' such that (1) $\Lambda' \subseteq \Lambda$ and (2) $\Lambda/\Lambda' \simeq R/(\pi^\delta)$. (For example, if $K = \mathbb{Q}_2$ then each vertex has the same degree as the number of points on the projective line over $k = \mathrm{GF}(2)$; by standard considerations (c.f. Definition II.11) this is 3; see Fig. 7).

Note next that there is an action of $\mathrm{GL}_2(K)$ on this tree: Let $\Lambda = Re_1 \oplus Re_2$ be an R-lattice; then if $g \in \mathrm{GL}_2(K)$ we have $g\Lambda \coloneqq Rge_1 \oplus Rge_2$ (linear independence is preserved since g is invertible). If $\Lambda \sim \Lambda'$, write $\Lambda' = R\lambda e_1 \oplus R\lambda e_2$ so $g\Lambda' = R\lambda g e_1 \oplus R\lambda g e_2$ and thus $g\Lambda \sim g\Lambda'$: i.e. the action of g on lattices induces an action on X. (In fact it is clear that this action is transitive.) Further this action preserves the incidence structure: if $\Lambda, \Lambda' \in X$ such that $\Lambda \approx \Lambda$ then $\Lambda' = R\pi^{r_1} e_1 \oplus R\pi^{r_2} e_2$ so $g\Lambda' = R\pi^{r_1} g e_1 \oplus R\pi^{r_2} g e_2$ and thus $g\Lambda \approx g\Lambda'$ and so we have a homomorphism $G \to \mathrm{Aut}\, X$.

Fix a lattice class representative $\Lambda \in X$ and consider $\mathrm{Stab}_{\mathrm{GL}_2(K)} \Lambda$. We have that $g \in \mathrm{Stab}_{\mathrm{GL}_2(K)} \Lambda$ iff $g\Lambda \sim \Lambda$. Hence $g\Lambda = R\lambda e_1 \oplus R\lambda e_2 = \lambda(Re_1 \oplus Re_2) = \lambda\Lambda$ for some $\lambda \in K^*$; i.e. $\mathrm{Stab}_{\mathrm{GL}_2(K)} \Lambda = Z_2(K)$ ($Z_2(K)$ being the multiplicative 2-torus over K). But $\mathrm{GL}_2(K)/Z_2(K) \simeq \mathrm{SL}_2(K)$ (since if A is an arbitrary matrix we may decompose it uniquely as

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha/\det A & \beta/\det A \\ \gamma/\det A & \delta/\det A \end{pmatrix} \begin{pmatrix} \det A & \\ & \det A \end{pmatrix} = SD$$

where $S \in \mathrm{SL}_2(K)$, $D \in Z_2(K)$). In particular, while $\mathrm{GL}_2(K)$ does act on X it is more natural to consider the induced action of $\mathrm{SL}_2(K)$ as this action is *free*; note further that this action continues to be transitive on X and is in fact strongly transitive when we endow X with the building structure of Example II.13. Further, $\mathrm{SL}_2(X)$ acts without inversion (i.e. if $[\Lambda, \Lambda']$ is a 1-arc then $s[\Lambda, \Lambda'] \neq [\Lambda', \Lambda]$ for all $s \in \mathrm{SL}_2(K)$).

We now find a BN-pair of $\mathrm{SL}_2(K)$ which will have the associated building X (c.f. Theorem II.15). Pick a fundamental apartment/chamber pair (Σ, C) by taking $\Sigma = \{\Lambda_r = Re_1 \oplus R\pi^r e_2 : r \in \mathbb{Z}\}$ ((e_1, e_2) some R-basis for K^2) and $C = [\Lambda_0, \Lambda_1]$:

$$\Sigma = \qquad \cdots \;\text{———}\; \Lambda_{-1} \;\text{———}\; \Lambda_0 \;\overset{C}{\text{———}}\; \Lambda_1 \;\text{———}\; \Lambda_2 \;\text{———}\; \cdots$$

Then $B' \coloneqq \mathrm{Stab}_{\mathrm{GL}_2(K)} C$ is the set of matrices fixing the pair of lattice equivalence classes represented by Λ_0 and Λ_1. Consider an arbitrary member of B', say

$$M \coloneqq \begin{pmatrix} \pi^{r_1} u_1 & \pi^{r_2} u_2 \\ \pi^{r_3} u_3 & \pi^{r_4} u_4 \end{pmatrix}$$

where each $r_i \in \mathbb{Z}$ and $u_i \in R^*$. We have, considering the action of L on the basis of Λ_0, that

$$Me_1 = \pi^{r_1}u_1e_1 + \pi^{r_3}u_3e_2Me_2 = \pi^{r_2}u_2e_1 + \pi^{r_4}u_4e_1$$

and so (since both products lie in $Re_1 \oplus Re_2$ by assumption, so $\pi^{r_i}u_i \in R$ for each i) each r_i must be non-negative. Further, considering the action on the basis of Λ_1, we have

$$Me_1 = \pi^{r_1}u_1e_1 + \pi^{r_3}u_3e_2M\pi e_2 = \pi^{r_2+1}u_2e_1 + \pi^{r_4+1}u_4e_1$$

and so we obtain (in addition to the relations we derived above) $r_3 \geq 1$ and $r_2 + 1 \geq 0$ (which is redundant since we know that $r_2 \geq 0$). Finally from the relation $\det M = 1$ we have that

$$0 = v(1) = v(\pi^{r_1}\pi^{r_4}u_1u_4 - \pi^{r_2}\pi^{r_3}u_2u_3) \geq \min\{r_1 + r_4, r_2 + r_3\};$$

since $r_2 + r_3 \geq 1$ and each $r_i \geq 0$ we have that $r_1 = r_4 = 0$.

In summary, the matrix M must be of the form

$$M := \begin{pmatrix} \alpha & \beta \\ \pi\gamma & \delta \end{pmatrix} \tag{3}$$

where each of $\alpha, \beta, \gamma, \delta$ lies in R and α and δ are invertible. It is easy to see that, conversely, all matrices of this form do indeed fix the two lattices. Hence $\mathrm{Stab}_{\mathrm{SL}_2(K)} C = B' \cap \mathrm{SL}_2(K)$ is precisely the matrices of the form given as Eq. (3) that have determinant 1. We next find the subgroup $N = \mathrm{Stab}_{\mathrm{SL}_2(K)} \Sigma$. Each linear map fixing Σ must be a product of a permutation on $\{e_1, e_2\}$ and a diagonal matrix; hence N is the group of 2×2 matrices over K with determinant 1 such that every column contains exactly one non-zero element.

Given the discussion in Section II.3 we therefore have produced a BN-pair in $\mathrm{SL}_2(K)$ that corresponds to the tree X. Note that $B \cap N$ is the group of diagonal 2×2 matrices of determinant 1; we know that $W = \mathrm{SL}_2(K)/(B \cap N) = \mathrm{SL}_2(K)/(\mathrm{SL}_2(K) \cap \mathrm{Diag}_2(K))$ should (by the structure of X) just be $D_\infty \simeq \mathbb{Z} \rtimes \{\pm 1\}$; indeed we may embed $\mathbb{Z}$ into W using the standard $n \mapsto \begin{pmatrix} 1 & \\ n & 1 \end{pmatrix}$, and embed $-1 \mapsto \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, and this works.

The building of $\mathrm{SL}_n(K)$, $n \geq 3$. More generally the theory is similar to that of the $n = 2$ case (c.f. [AB08, section 6.9.3], [Ser80, exercise 4 to section II.1.1]). Note that in the $n = 2$ case we colour the lattices according to whether they are of odd or even distance to some fixed root point (c.f. Fig. 7); this is equivalent to colouring $\Lambda = Rf_1 \oplus Rf_2$ with the residue mod 2 of $v(\det(f_1, Xf_2))$ and this generalises to the case $n \geq 3$. Thus:

II.18 Definition. The **Bruhat-Tits building** associated to $\mathrm{SL}_n(K)$ for $n \geq 2$ is the flag complex associated with the classes of lattices of K^n modulo the relation $\Lambda \sim \Lambda' \iff \Lambda = \lambda\Lambda'$ for some $\lambda \in K^*$ and with adjacency relation $\Lambda \approx \Lambda'$ iff the representatives of the classes may be chosen so that $\pi\Lambda \leq \Lambda' \leq \Lambda$, and with the colouring of $\Lambda = Rf_1 \oplus \cdots \oplus Rf_n$ being the reduction of $v(\det(f_1, ..., f_n))$ modulo n.

One can show that the Weyl group associated to the Bruhat-Tits building of $\mathrm{SL}_n(K)$ is isomorphic to $\mathbb{Z}^{n-1} \rtimes S_n$ (where $\mathbb{Z}^{n-1}$ is invariant under the action of S_n on some vector space, see [AB08, p. 355]).

The general Bruhat-Tits theory. We will briefly outline, now, the general theory of the Bruhat-Tits buildings associated to the classical groups. More comprehensive studies are [AN02], [AB08, chapter 6], and [Gar97] (see in particular chapter 20).

As before, let K be a field with discrete valuation $v : K \to \mathbb{Z}$, local ring $(R, \mathfrak{m})$ with $\mathfrak{m} = (\pi)$, and residue field k; further let σ be an involution on K (i.e. an endomorphism of order 2) such that $R^\sigma = R$ and let $\epsilon = \pm 1$. A (σ, ϵ)-**hermitian form** on a vector space V over K is a function $h : V \times V \to K$ such that for all $u, v, w \in V$, $\alpha \in K$:

1. $h(u, v) = \epsilon h(v, u)^\sigma$
2. $h(\alpha u, v) = \alpha h(u, v)$
3. $h(u + v, w) = h(u, w) + h(v, w)$.

Two hermitan forms h, k on V are **equivalent** if there exists $\phi \in \mathrm{GL}(V)$ such that $h(\phi^{-1}u, \phi^{-1}v) = k(u, v)$ for all $u, v \in V$.

We say h is **nondegenerate** if $f(u, v) = 0$ for all $u \in V$ implies $v = 0$; in this case we say (V, h) is a **nondegenerate inner product space**. An **isometry** of a nondegenerate inner product space (V, h) is a linear transformation $f : V \to V$ such that $h(fu, fv) = h(u, v)$ for all $u, v \in V$. The set of all isometries, $\mathrm{Isom}(V, h)$, is a subgroup of $\mathrm{GL}(V)$. A subspace $W \leq V$ is **totally isotropic** if $h(u, v) = 0$ for all $u, v \in W$.

II.19 Definition. The term ‘classical group’ is a term with no universally accepted meaning. (c.f. [Rot95, pp. 234–246], [Hum11]) For us, a **classical group** is one of the following:

The general and special linear groups $\mathrm{GL}_n(K)$ or $\mathrm{SL}_n(K)$ for a field K.

The orthogonal groups $O_n(K, h) \coloneqq \mathrm{Isom}(V, h)$ for a field K and a $(1, 1)$-hermitian form h.

The symplectic groups $\mathrm{Sp}_n(K, h) \coloneqq \mathrm{Isom}(V, h)$ for a field K and a $(1, -1)$-hermitian form h. (In this case, n is even.)

The unitary groups $U_n(K, h) \coloneqq \mathrm{Isom}(V, h)$ for a field K and a $(\sigma, 1)$-hermitian form h ($\sigma \neq 1$).

A **primitive lattice** for (K, h) is an R-lattice such that h reduces to a nondegenerate form on the residue field k. Now define an incidence geometry to have as vertices the set of R-lattices Λ of K, modulo the same equivalence relation as before ($\Lambda \sim \Lambda' \iff \exists_{\lambda \in K^*}(\Lambda = \lambda\Lambda')$), which satisfy the following additional axioms:

1. The representative Λ may be chosen so there exists a lattice Λ_0 such that $\mathfrak{m}^{-1}\Lambda_0$ is primitive and $\Lambda_0 \subseteq \Lambda \subseteq \mathfrak{m}^{-1}\Lambda_0$;
2. $h(\Lambda, \Lambda) \subseteq \mathfrak{m}$.

(That is, Λ/Λ_0 is a totally isotropic k-subspace of $\mathfrak{m}^{-1}\Lambda_0/\Lambda_0$ with respect to the reduced form on the quotient space.) We define our incidence relation $\approx$ by $\Lambda \approx \Lambda'$ iff the representatives can be chosen so there exists Λ_0 such that:

1. $\mathfrak{m}^{-1}\Lambda_0$ is primitive;
2. $\Lambda \subseteq \Lambda'$ or $\Lambda' \subseteq \Lambda$;
3. $\Lambda_0 \subseteq \Lambda \subseteq \mathfrak{m}^{-1}\Lambda_0$; and
4. $\mathfrak{m}\Lambda_0 \subseteq \Lambda' \subseteq \mathfrak{m}^{-1}\Lambda_0$.

Then the **Bruhat-Tits building** of (K, h) is the flag complex associated to this incidence structure; it has a natural action (via a BN-pair, as above) by the relevant classical group.

Talk III: Convexity

We shall repeat for convenience:

III.1 Definition (Definition II.18)**.** The **Bruhat-Tits building** of $SL_n(K)$ is the flag simplicial complex $\mathcal{B}_n(K)$ whose vertices are equivalence classes of lattices in K^d and whose edges are the adjacent pairs of lattices.

Remark. The **link** of a vertex v in a simplicial compex Σ is the subcomplex with vertex set $\{w \in F_0(\Sigma) : w \sim v, w \neq v\}$ and simplices $\{\sigma \in \Sigma : v \notin \sigma, \{v\} \cup \sigma \in \Sigma\}$. The links of Λ in $\mathcal{B}_n(K)$ may be identified with the flag complex of chains of subspaces in the residue vector space k^3.

We shall now apply tropical geometry to the study of Bruhat-Tits buildings; let $\mathcal{B}_n(K)$ denote the building of $\mathrm{SL}_n(K)$ as in the previous talk. We shall focus on convexity, as in [JSY07]; however one can also use tropical methods to study the compactifications of buildings, as in [Wer11]. In order to motivate the definition of convexity in $\mathcal{B}_n(K)$ as it does not exactly match the usual definition (e.g. as found in [AB08]), we will summarise a motivating paper from the theory of abelian varieties [Fal01].

III.1 Summary of Faltings' paper

We shall begin by giving some vague background to the paper at hand [Fal01]. Roughly speaking, a **deformation problem** (c.f. [Har10]) is a study of a morphism of schemes $f : X \to T$, where we regard the fibres through f, X_t for $t \in T$, as a 'family' of schemes; generally one would pick a 'favourite' fibre X_0 and then consider infinitesimal neighbourhoods of this base fibre. Relatedly, we often study **moduli problems** (c.f. [Har10, chapter 4]): here we have a class of schemes with some fixed property $\mathcal{P}$, and we seek a morphism $f : X \to T$ where T is some parametrising **moduli scheme** and such that every isomorphism class of schemes with property $\mathcal{P}$ appears exactly once as a fibre of f in X. In general, the 'useful' points of a moduli scheme or of a deformation problem are singular.

The paper of Faltings considers some moduli spaces of abelian varieties; an abelian variety (see [BL04], [Oda78, section 11]) is a complex torus that admits a positive definite line bundle; more precisely,

III.2 Definition. A **lattice** in $\mathbb{C}^g$ is a discrete additive subgroup of maximal rank; this is a lattice in the sense of earlier definitions of rank $2g$. A **complex torus** is a quotient $X = \mathbb{C}^g/\Lambda$ for Λ a lattice in $\mathbb{C}^g$. A **morphism of tori** $f : X \to X'$ is a holomorphic map that preserves the group structure; it is an **isogeny** if it has finite kernel.

We say that a **polarisation** of X is the first Chern class of a positive definite line bundle L on X: that is, we take an exact sequence $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$ and make it into a long cohomology sequence; the map c_1 of interest is then depicted here (recalling that line bundles on X can be viewed as elements of $H^1(X, \mathcal{O}_X^*)$):

$$\cdots \longrightarrow H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) \longrightarrow \cdots$$

The first Chern class of L is a Hermitian form on Λ which may be written as a block matrix $\begin{bmatrix} & D \\ -D & \end{bmatrix}$, D a diagonal matrix. If $D = I_g$ then the polarisation is **principal**.

An **abelian variety** is a complex torus that admits a polarisation; a morphism of polarised abelian varieties $f : (X, L) \to (Y, M)$ is a morphism $f : X \to Y$ such that $f^*c_1(M) = c_1(L)$.

We consider a moduli space $\phi : A \to B$ of abelian varieties where ϕ is an isogeny, A and B are g-dimensional principally polarised abelian varieties, and ϕ is of degree p^g compatible with the polarisations. We further ask that ϕ may be factored into a series of g isogenies of degree p:

$$A = A_0 \to A_1 \to \cdots \to A_g = B.$$

We may study this situation profitably over $\mathbb{Z}_p$; in order to study the local singularities we must use some crystalline cohomology; to give a vague idea of this theory we follow briefly the first part of the introductory note [Cas15].

If X is some scheme, we define the nth ℓ-adic cohomologies (ℓ a prime) to be

$$H^n_{\text{ét}}(X, \mathbb{Z}_\ell) := \varprojlim H^n_{\text{ét}}(X, \mathbb{Z}/\ell^m\mathbb{Z})$$
$$H^n_{\text{ét}}(X, \mathbb{Q}_\ell) := H^n_{\text{ét}}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

This has various nice properties. If X is smooth, irreducible, and proper over a field k of characteristic $p \neq \ell$, the dimension $\dim_{\mathbb{Q}_\ell} H^n_{\text{ét}}(X, \mathbb{Q}_\ell)$ is independent of ℓ; however, if $p = \ell$ then this is no longer true (i.e. the p-adic cohomology of schemes over fields of characteristic p is pathological). The **crystalline cohomology** is a cohomology theory that behaves like ℓ-adic cohomology but for $p = \ell$. The coefficients of this cohomology theory come from the Witt ring of k:

III.3 Theorem ([Ser79, chapter II, theorem 3])**.** *For every perfect field k of characteristic p, there exists a unique (up to unique isomorphism) complete discrete valuation ring that is absolutely unramified and has k as its residue field. This is the **Witt ring** of k, $W(k)$.*

Denote the first crystalline cohomology of an abelian variety A by $M(A)$; the functor M is contravariant and the properties of the cohomology allow us to produce from the moduli family

$$A = A_0 \to A_1 \to \cdots \to A_g = B$$

a second family

$$M(B) = M(A_g) \to M(A_{g-1}) \to \cdots \to M(A_0) = M(A)$$

such that $\operatorname{coker}(M(A_i) \to M(A_{i-1}))$ is locally generated by a single element and so locally the sequence looks like a complete flag

$$N_g = pN \subset N_{g-1} \subset \cdots \subset N_0 = N \tag{4}$$

where N is a free $\mathbb{Z}_p$ module of rank g.

We now try to define a scheme X (the **Deligne-scheme**) parameterising the systems of direct summands $F_i \subseteq N_i \otimes R$ of some fixed rank a such that F_0 and F_g are identified under the isomorphism $N_g = pN_0$ and such that maps $F_i \to F_{i-1}$ are induced. This scheme will be (locally) a toric resolution of singularities for some interesting object (that is, there is a resolution of singularities $Y \to X$ with Y a toric variety; more precisely, see [Fal01, p. 170]).

To fix notation again, we have a discrete valuation ring (R, v) with field of fractions K, $\pi \in R$ an element with $v(\pi) = 1$, and residue field k; so the building $\mathcal{B}_n(K)$ can be viewed as parameterising the lattices of K^n. We may also (more usefully for our purposes) view $\mathcal{B}_n(K)$ as parameterising the *additive norms* of the vector space up to equivalence:

III.4 Definition. An **integral additive norm** on K^n is a map $\|\cdot\| : K^n \to \mathbb{Z} \cup \{\infty\}$ such that

1. $\|\lambda x\| = v(\lambda) + \|x\|$ for all $\lambda \in K$, $x \in K^n$;
2. $\|x + y\| \geq \min\{\|x\|, \|y\|\}$ for all $x, y \in K^n$;
3. $\|x\| = \infty$ iff $x = 0$.

To set up the parameterisation, if Λ is a R-lattice in K^n then we define

$$\|x\|_\Lambda := \sup\{r \in \mathbb{Z} : \pi^{-r}x \in \Lambda\}$$

and if $\|\cdot\|$ is an additive norm we define

$$\Lambda_{\{}\|\cdot\|\} = \{x \in K^n : \|x\| \geq 0\}.$$

If $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms, we define the distance $d(\|\cdot\|_1, \|\cdot\|_2)$ to be the variation

$$(\sup - \inf)\{\|\|x\|_1 - \|x\|_2\| : x \in K^n\}.$$

III.5 Theorem. *The function d is a integer-valued metric on $\mathcal{B}_n(K)$. The geodesic with respect to d between $\|\cdot\|_1$ and $\|\cdot\|_2$ is the set of norms*

$$\inf\{\|\cdot\|_1, x + \|\cdot\|_2\}$$

for $x \in \mathbb{R}$. A collection of lattices forms a simplex in $\mathcal{B}_n(K)$ if and only if the distance between any two lattices in the collection is 1.

We say a subset of $\mathcal{B}_n(K)$ is **convex** if it contains the geodesic between any two elements.

III.6 Theorem. *If $\{M_i\}$ is a finite collection of R-lattices, then the convex hull* $\operatorname{conv}\{M_i\}$ *is the union of all simplices associated to flags of the form Eq.* (4)*, where each vertex N_j is a linear combination of the M_i with coefficients in K^*. The convex hull is finite.*

Proof. [Fal01, lemma 3].

We now define the Deligne-scheme, which will parameterise a convex collection of lattices. To do this, we recall

III.7 Definition. For a moduli problem involving a class $\mathcal{M}$ of objects (closed schemes, line bundles, etc.) over K, parameterised by schemes over K, we define a functor $\mathcal{F} : \mathsf{Sch}(K) \to \mathsf{Set}$ that assigns to each S/K the set of equivalence classes of families of elements of $\mathcal{M}$ with respect to K. To solve the problem, then, is to construct a 'universal scheme' M/K together with a morphism of functors $\mathcal{F} \to \operatorname{Hom}(\cdot, M)$ satisfying some suitable universal property. If $\mathcal{F}$ is actually an *isomorphism* of functors we say that $\mathcal{F}$ **represents** the scheme M.[*]

(See [Wat79, section 1.2] and [Har10, section 23].)

and so we have for a convex collection of lattices $\{\Lambda_i\}$

III.8 Definition. The **Deligne-scheme** X represents the functor on R-algebras A such that an R-valued point of X is a collection of quotient line bundles $L(\Lambda_i)$ of $\Lambda_i \otimes_R A$ (modulo equivalence of lattices) such that each inclusion $\Lambda \subseteq M$ maps $L(\Lambda)$ to $L(M)$.

III.2 Tropical convexity and lattices

We now follow [JSY07].

Let Λ_1 and Λ_2 be R-lattices over K. Then $\Lambda_1 + \Lambda_2$ and $\Lambda_1 \cap \Lambda_2$ are lattices. We say that a vertex subset of $\mathcal{B}_d(K)$ closed under finite sums is **max-convex**; we say a vertex subset closed under finite intersection is **min-convex**. The latter notion is precisely the notion of convexity in the previous section; and the former arises naturally:

III.9 Lemma. *For Λ_1 and Λ_2 R-lattices in K^d, $(\Lambda_1 + \Lambda_2)^* = \Lambda_1^* \cap \Lambda_2^*$.*

[*]The j-line of Lukas' talk yesterday is a different example of what is known as a **coarse moduli scheme**: that is, a scheme satisfying all but the requirement that $\mathcal{F}$ be an isomorphism.

By Theorem III.6 both min-convex hulls and max-convex hulls are finite.

Given a finite subset $M = \{m_1, ..., m_k\}$ of K^d which spans K^d as a K-vector space, the set of all lattices of the form $R\pi^{u_1} m_1 + \cdots + R\pi^{u_k} m_k$ as $(u_1, ..., u_k)$ ranges over $\mathbb{Z}^k$ is the **membrane** spanned by M in $\mathcal{B}_d(K)$; if $k = n$ then the set is in fact an apartment of the building.

The goals are twofold:

1. Compute the min- and max-convex hulls of a finite set of lattices in $\mathcal{B}_d(K)$.

2. Compute the intersection of a finite set of apartments in $\mathcal{B}_d(K)$.

In [Jos20, sections 5 and 6] concrete algorithms are given for these problems. We will restrict ourselves to studying the theoretical relationship between these problems and tropical geometry. For some historical context and motivation for the following definitions, see [RST03].

III.10 Definition. Let $d \leq n$ be positive integers. A map $p : [n]^d \to \bar{\mathbb{R}}$ is a **valuated matroid** if

1. for any permutation $\pi \in \mathrm{Sym}[d]$, $p(a_1, ..., a_d) = p(a_{\pi(1)}, ..., a_{\pi(d)})$;

2. for all $(a_1, ..., a_d) \in [n]^d$, if there exist $i \neq j$ such that $a_i = a_j$ then $p(a_1, ..., a_d) = \infty$; and

3. for any $(d-1)$-subset σ and any $(d+1)$-subset τ of $[n]$, the minimum $\min\{p(\sigma \cup \{\tau_i\}) + p(\tau \setminus \{\tau_i\}) : i \in [d+1]\}$ is attained at least twice; put differently, the tropical polynomial
$$\bigoplus_{i=1}^{d+1} X_i \odot Y_i$$
has roots at $(X_i = p(\sigma \cup \{\tau_i\}) : i = 1, ..., d+1), (Y_i = p(\tau \setminus \{\tau_i\}) : i = 1, ..., d+1)$. (These are a tropical version of the quadratic **Plücker relations**, c.f. [MS05, section 14.2].)

For a fixed valuated matroid p, we define the **tropical linear space** of p to be the set of points $x \in \mathbb{PT}^{d-1}$ (recall Definition I.18) such that for any $(d+1)$-subset $\tau \subseteq [n]$, the polynomial
$$\bigoplus_{i=1}^{d+1} Y_{\tau_i} \odot X_{\tau_i}$$
has a root at $(Y_{\tau_i} = p(\tau \setminus \{\tau_i\}) : i = 1, ..., d+1)$ and $X = x$.

III.11 Theorem. *The tropical linear space L_p is a tropical lattice polytope; in fact, it is the tropical convex hull in $\mathbb{PT}^{d-1}$ of all the vectors of the form $p(\sigma\, *)$, where $p(\sigma\, *)_j = p(\sigma \cup \{j\})$ for all $j = 1, ..., d$.*

Proof. [YY06, theorem 16]

III.12 Theorem. *For every matrix $M \in \mathrm{Mat}_{d\times n}(K)$, we have a valuated matroid p_M defined by*
$$p_M(\omega) = v(\det M_\omega)$$
(where M_ω denotes the $d \times d$-submatrix of M indexed by the entries of d).

The lattice points of L_p are precisely the points $v(\mathrm{row}\, M)$ (valuation taken pointwise).

Proof. [SS03, theorem 2.1]

Define a metric on tropical projective space $\mathbb{PT}^{d-1}$ by $\delta(x, y) := \max_{1\leq i\leq j\leq n} \left|(x_i - x_j) + (y_i - y_j)\right|$. As in normal convex geometry we have a 'nearest point map':

III.13 Theorem. *Let $x \in \mathbb{PT}^{d-1}$ and P a tropically convex closed set in $\mathbb{PT}^{d-1}$. There is a unique point $\pi_P(x)$ that minimises the δ-distance from x to P.*

Proof. [JSY07, proposition 7]

Remark. The nearest point map has an easily computable formula in terms of the generators of P [JSY07, lemma 8].

We may define a graph Γ on the set of all lattice points of $\mathbb{PT}^{d-1}$ via the adjacency relation $x \sim y \iff \delta(x, y) = 1$.

III.14 Theorem. *The flag simplicial complex of Γ is a triangulation of $\mathbb{PT}^{d-1}$, and restricts to a triangulation of any tropical lattice polytope P, known as the* ***standard triangulation****.*

Proof. [JSY07, theorem 11]

Hence:

III.15 Theorem. *If $M \in \mathrm{Mat}_{d\times n}(K)$ has columns $f_1, ..., f_n$ and L_p is the associated tropical linear space, then*

$$\Psi_M \,:\, R\pi^{-u_1} f_1 + \cdots R\pi^{-u_n} f_n \mapsto \pi_{L_p}(u_1, ..., u_n)$$

is well-defined and induces an isomorphism of simplicial complexes between the standard triangulation of L_p and the membrane $[M]$.

III.16 Corollary. *Every lattice point of L_p uniquely represents a lattice in $[M]$.*

III.17 Corollary. *Every apartment $\mathcal{A}$ of $\mathcal{B}_d(K)$ may be identified with $\mathbb{PT}^{d-1}$.*

III.18 Theorem. *A finite set C of chambers in an apartment $\mathcal{A}$ of $\mathcal{B}_d(K)$ is convex (in the usual sense: that is, it contains every minimal gallery between two of its chambers) if and only if $F_0(C)$ is the set of lattice points in a tropical lattice polytope of the form*

$$\{w \in \mathbb{PT}^{d-1} \,:\, w_i - w_j \le u_{i,j} \text{ for all } i \ne j\} \tag{5}$$

for some sequence of $u_{i,j} \in \mathbb{R}$.

Proof. [JSY07, proposition 20]

III.19 Theorem (Alessandri's algorithm)**.** *The intersection of apartments $[M_1] \cap \cdots \cap [M_r]$ in $\mathcal{B}_d(K)$ is the standard triangulation of a particular tropical polytope of the form Eq.* (5)*.*

Proof. [JSY07, theorem 27]

We may also compute min-convex hulls (and hence by Lemma III.9 max-convex hulls).

III.20 Theorem. *Let $\Lambda_1, ..., \Lambda_r$ be lattices spanned by the columns of matrices $M_1, ..., M_r \in \mathrm{GL}_n(K)$. Let $[M]$ be any membrane in $\mathcal{B}_d(K)$. Let $r_M \,:\, \mathcal{B}_d(K) \to [M]$ be the retraction given by $\Lambda \mapsto (\Lambda \cap K f_1) + \cdots + (\Lambda \cap K f_n)$. Then the simplicial complex*

$$r_M \; \min - \mathrm{conv}\{\Lambda_1, ..., \Lambda_r\}$$

and the standard triangulation of the tropical polytope

$$\mathrm{tconv}\{\Psi_M(r_M(\Lambda_1)), ..., \Psi_M(r_M(\Lambda_r))\}$$

coincide.

Proof. [JSY07, proposition 22]

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