Toric varieties

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Motivation

There are several motivations for the study of toric varieties:

Combinatorial: They are naturally related to lattice polytopes and related objects: results from the study of toric varieties may be used to study polytopes and convex bodies, and vice versa

Tropical: Results from tropical geometry may be used to study subvarieties and compactifications of toric varieties, in fact the tropicalisation of a projective toric variety is just (homeomorphic to) the associated polytope.

Motivation

There are several motivations for the study of toric varieties:

Algebraic An important source of examples in combinatorial commutative algebra; toric varieties come from a ring graded by an abelian group together with a squarefree monomial ring that is compatible with the grading. Essentially toric varieties are cut out by relations of the form $x_1^{a_1} \cdots x_n^{a_n} = x_1^{b_1} \cdots x_n^{b_n}$, i.e. monomial equalities; and morphisms are monomial, not polynomial.

Geometric Many classical algebro-geometric results (Riemann-Roch, Zariski's main theorem, study of the Hilbert variety, various cohomology theorems) have very simple proofs or descriptions in the toric case due to the relationships with other fields.

Historical notes 1/3

- Study originates with Demazure (1970), studying birational morphisms of $\mathbb{P}^n$.
- Kempf, Knudsen, Mumford, and Saint-Donat (1972): 'the power of [toric embedding techniques] has cropped up independently in the work of at least a dozen people'. Miyake and Oda were also studying these techniques independently around the same time (1973). It is at this time that the fundamental results (the orbit-cone correspondence theorem, the duality between fans and normal toric varieties) were first written down.
- In 1977 and 1978 respectively Khovanskii and Danilov wrote papers on the subject: Khovanskii studied the deep connections between torus embeddings and polytopes, while Danilov primarily studied cohomology. The origin of the name 'toric variety' is Reid's translation of the title of Danilov's paper from the Russian, the term became standard by the 1990s.

Historical notes 2/3

- Billera and Lee (1979) and Stanley (1980) used toric varieties to prove the McMullen g-conjecture on the face numbers of a simplicial polytope, a set of necessary and sufficient conditions for a sequence to count faces of such an object.
- In the 1990s, a quotient representation for toric varieties was given by Audin (1991); and toric varieties were applied to mirror symmetry in mathematical physics. Around this time a number of influential textbooks (Oda (1988); Fulton (1993); Ewald (1996)) were written.

Historical notes 3/3

- From the mid-1990s, applications to coding theory, integer programming, applied algebraic geometry, and other computational and applied fields became very important in the subject.
- Examples of more recent work include the relationship with tropical geometry, study of toric stacks, study of the ‘secondary fan’.

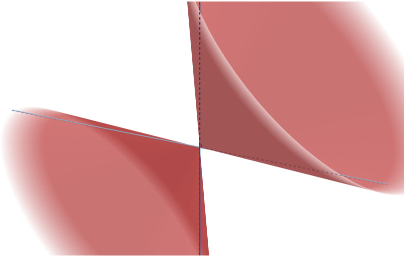
Definition

A **toric variety** is a variety V (that is, a finite collection of affine varieties glued on open subvarieties) together with an open subvariety $T \subseteq V$ such that $T \simeq (\mathbb{C}^*)^n$ (the rank n torus) for some n, with the property that the action of T on itself by multiplication extends to the entirety of V.

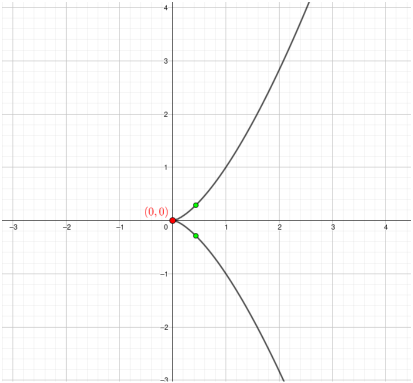
Examples

- Trivial examples: ${\mathbb{C}^*}^n$; $\mathbb{C}^n$; $\mathbb{P}^n$.
- The cone defined by $Y^2 = XZ$ with embedded torus T given by deleting the X and Z axes (send $(x, y, z) \mapsto (x, y)$).
- The cusped cubic K cut out by $X^3 = Y^2$ with torus $(\mathbb{C}^*)^2 \cap K \simeq \mathbb{C}^*$ (send $(x, y) = (t^2, t^3) \mapsto y/x$).
- The Hirzebruch surfaces $\mathbb{F}_a$.

Examples: cone



Examples: cusped cubic



Affine constructions 1/3

Geometric Let $\mathscr{A} := \{\mathrm{m}_1, ..., \mathrm{m}_r\} \subseteq \mathbb{Z}^n$ be a set of points; let $T \simeq (\mathbb{C}^*)^n$ be a torus. Define a map $\Phi_{\mathscr{A}} : T \to \mathbb{C}^r$ by

$$\Phi_{\mathscr{A}}(t) = (\chi^{\mathrm{m}_1}(t), ..., \chi^{\mathrm{m}_r}(t))$$

where $\chi^{\mathrm{m}} : T \to \mathbb{C}^*$ is the function defined by the polynomial $X_1^{m_1} \cdots X_r^{m_r}$; then the (Zariski) closure of $\Phi_{\mathscr{A}}(T)$ is an affine toric variety (and conversely). The torus is $\Phi_{\mathscr{A}}(T)$ itself.

Affine constructions 2/3

Algebraic Let $\mathfrak{b} \subseteq \mathbb{C}[X_1, ..., X_r]$ be an ideal generated by binomials. Then $\operatorname{Spec} \mathbb{C}[X_1, ..., X_r]/\mathfrak{b}$ is an affine toric variety (and conversely). The torus is $\operatorname{Spec} \mathbb{C}[\pm X_1, ..., \pm X_r]/\mathfrak{b}$.

Affine constructions 3/3

Combinatorial Let $\{a_1, \ldots, a_s\} \subseteq \mathbb{Z}^n$; then the set $\sigma = \mathbb{R}_{\geq 0} a_1 + \cdots + \mathbb{R}_{\geq 0} a_s \subseteq \mathbb{R}^n$ is a finitely generated polyhedral lattice cone. Further, the dual cone $\sigma^\vee$ is also a finitely generated polyhedral lattice cone, and the intersection $\sigma^\vee \cap \mathbb{Z}^n$ is a finitely generated semigroup with identity embedded in a lattice, S_σ. Define an algebra A_σ to be generated by $\{X^\alpha : \alpha \in S_\sigma\}$ modulo the relations induced by the semigroup. Then $U_\sigma = \operatorname{Spec} A_\sigma$ is an affine toric variety, and conversely. Further, open subvarieties invariant under the torus action are in 1-1 correspondence with the faces of the cone. The torus is the open subvariety corresponding to the minimal face containing 0.

Affine constructions: $\mathbb{C}^n$ and $(\mathbb{C}^*)^n$.

- We can construct $(\mathbb{C}^*)^n$ (which is affine, being isomorphic to $\mathrm{Z}(X_1 \cdots X_n Y - 1)$) by taking the cone $\varepsilon = \mathrm{pos}\{0\} \subseteq \mathbb{R}^n$; so $\varepsilon^\vee = \mathbb{R}^n$ and $U_\sigma = \operatorname{Spec} \mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \simeq (\mathbb{C}^*)^n$.
- On the other hand $\mathbb{C}^n = U_\sigma$ for $\sigma = \mathrm{pos}\{e_1, \ldots, e_n\}$.

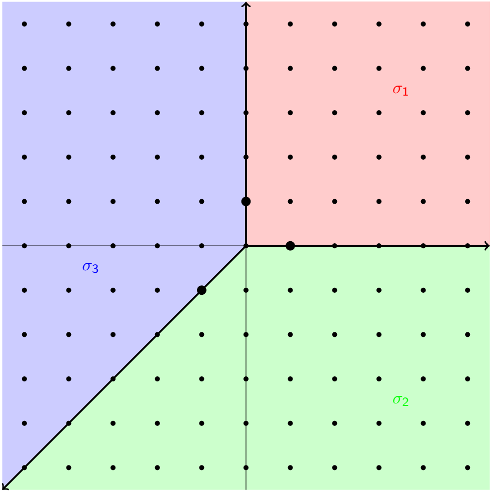
General construction

Take a set of cones, such that the intersection of any two cones in the set is a face of both, and such that the faces of every cone in the set are also cones in the set. Further ensure that no cone contains a line through the origin. The structure so obtained is a **fan**. Then there is a canonical way to glue all the affine toric varieties for the cones together, doing this we obtain a normal toric variety.

Theorem

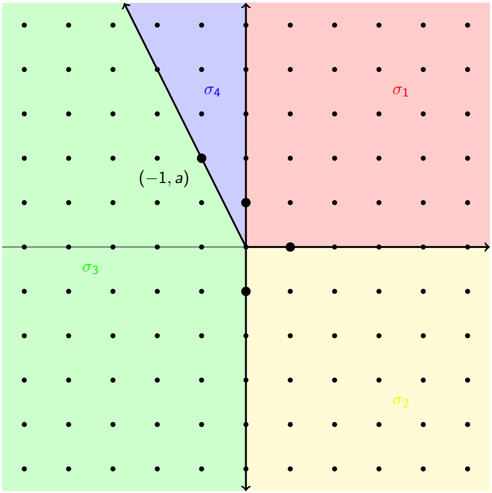
There is a bijection between fans and normal separated toric varieties; given a single normal separated toric variety there are bijections between cones in the related fan and orbits of the torus, and between cones in the fan and torus-invariant open subsets.

The fan of $\mathbb{P}^2$



(Gives the usual affine cover.)

The fan of the Hirzebruch surface $\mathbb{F}_2$



Geometric properties (easy)

Consider a normal separated toric variety of dimension n.

- The variety is compact (=proper over $\mathbb{C}$) in the usual topology iff the fan covers $\mathbb{R}^n$
- The variety is smooth iff every cone in the fan is generated by a set which can be extended to a basis of $\mathbb{Z}^n$
- The variety is an orbifold iff every cone in the fan is generated by a set linearly independent over $\mathbb{R}^n$
- The variety is affine iff the fan consists of a single cone and its faces

Geometric properties (scheme-theoretic)

Consider a normal separated toric variety of dimension n.

- Cohomology groups of nice sheaves tend to have strong vanishing properties
- Sheaves corresponding to divisors on the variety have cohomology groups closely related to simplicial homology groups (e.g. Hilbert polynomials for a variety correspond to Ehrhart polynomials for polytopes)
- These properties allow easy combinatorial(!) proofs of strong theorems for the case of nice enough toric varieties, e.g. Riemann-Roch for curves and surfaces, or even Hirzebruch-Riemann-Roch for arbitrary n.