Limit sets of cone manifolds

Geometric group theory of indiscrete groups of Möbius transforms

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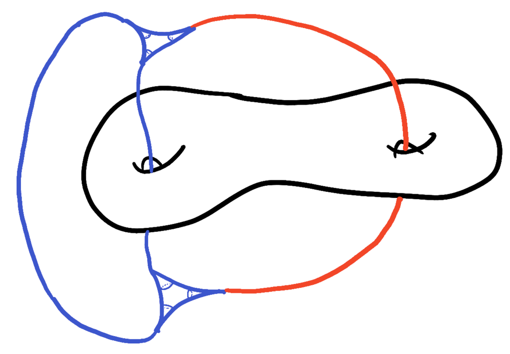


On the right side of the page, we show a deformation from one discrete group of conformal maps of the plane to another, through a family of indiscrete groups that uniformise cone surfaces. The black dots are the orbit of a single point.

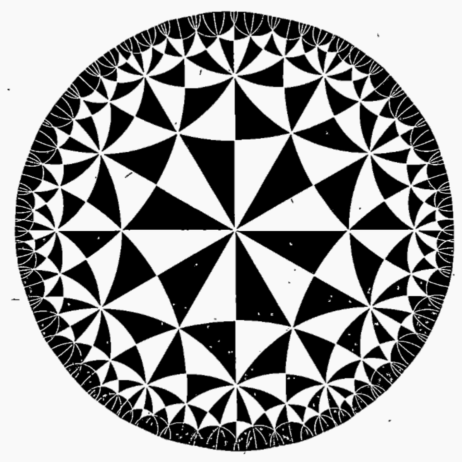
Associated arXiv preprint: Alex Elzenaar. *Changing topological type of compression bodies through cone manifolds.* 2024. arXiv: 2411.17940 [math.GT]

The hyperbolic plane

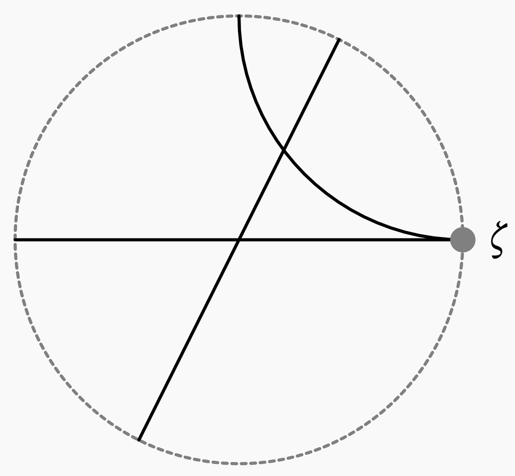
The first four of Euclid's five postulates now form the basis of modern metric geometry: they govern existence of geodesics, existence of circles, and well-definedness of angles. His fifth is harder to summarise:

> If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than two right angles [5].

Mathematicians spent over a thousand years trying to deduce it from his other four axioms. They failed for good reason: there exist metric geometries in which lines can converge without meeting.



Coxeter's tiling of of $\mathbb{H}^2$ [3] inspired many of Escher's most famous works.

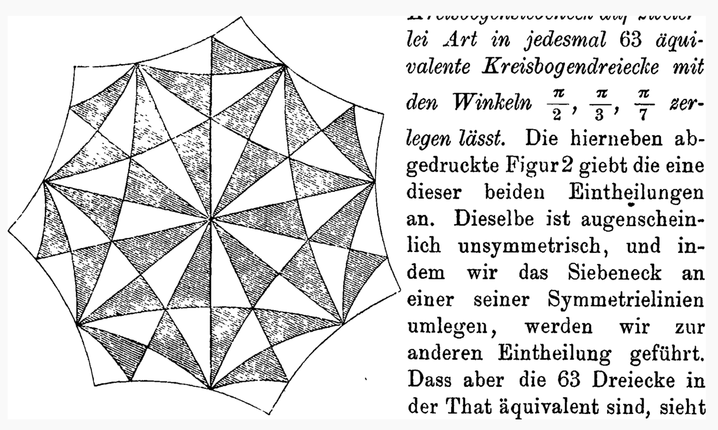


The two geodesics converging at ζ only meet at infinity.

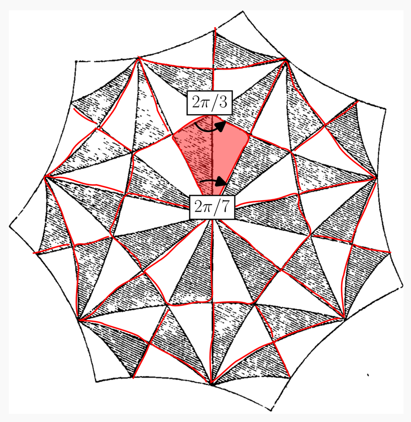
Hyperbolic geometry, $\mathbb{H}^2$, is a metric geometry on the disc. The boundary of the disc is a 'horizon' which is infinitely far away from every point in $\mathbb{H}^2$, angle measure is the usual angle measure that you get with a protractor, and geodesics are circular arcs that hit the horizon at a right angle.

Hyperbolic triangle groups

When p, q, and r are integers satisfying $p^{-1} + q^{-1} + r^{-1} < 1$ then there exists a hyperbolic triangle with angles π/p, π/q, π/r that tiles the hyperbolic plane.



Fricke drew the triangles with angles $\pi/2$, $\pi/3$, $\pi/7$ in 1893 [6].

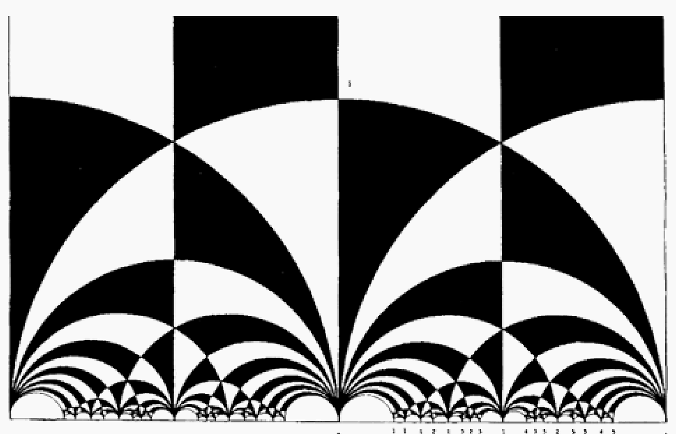


Rotations generate the kite tiling.

The symmetry group of a triangle tiling is generated by reflections in the sides of one triangle, and isn't orientation-preserving. Pair the triangles up to form kites: the new symmetry group is generated by two rotations, with presentation

$$\langle x, y : x^p = y^q = (xy)^r = 1 \rangle. \qquad (\blacktriangle)$$

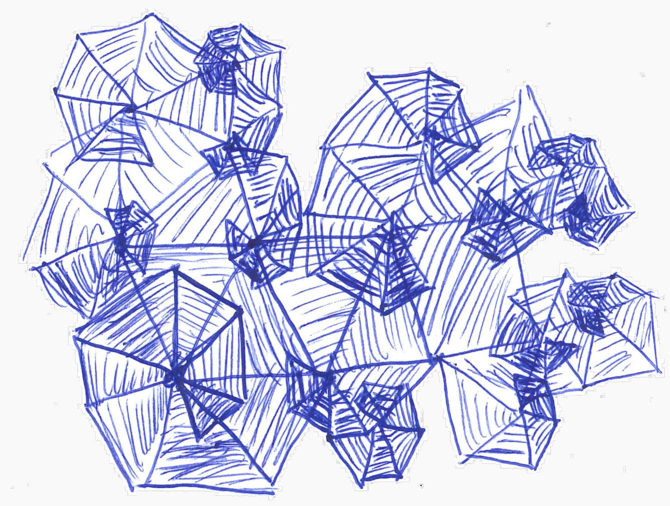
These *triangle groups* are actively studied: M. Conder and D. Young present papers in this meeting on their abstract group theory [2, 7].



The disc is now the upper half-plane so the $(\pi/2, \pi/3, 0)$ triangles stretch infinitely upwards. This tiling comes from $\mathrm{PSL}(2,\mathbb{Z})$ and is very important in number theory [1].

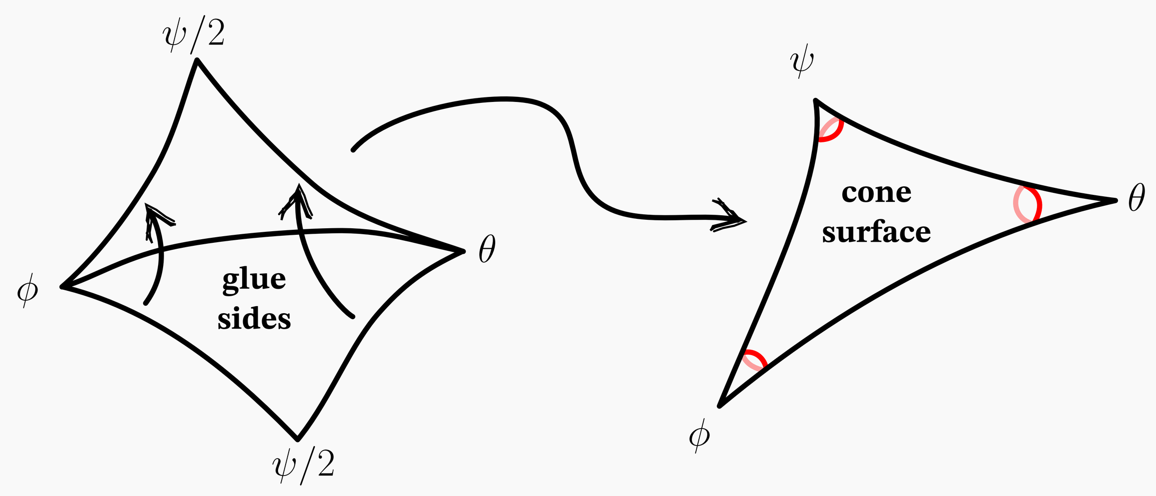
Indiscrete groups and cone manifolds

If we draw a triangle $\triangle$ in $\mathbb{H}^2$ with arbitrary angles, we can still take the group of reflections in its sides. But this group is no longer discrete and the triangle $\triangle$ no longer tiles $\mathbb{H}^2$ since if you keep adding copies around a single vertex, it won't 'close up' perfectly.



The metric space $\mathcal{X}_\triangle$ obtained by gluing copies of $\triangle$ together edge-to-edge is negatively curved and infinitely branched.

This infinitely branched space is a metric space where the group has a nicer action than $\mathbb{H}^2$. The reflection group acts to permute the copies of $\triangle$, and the orientation-preserving half G acts as rotations around the branch points.

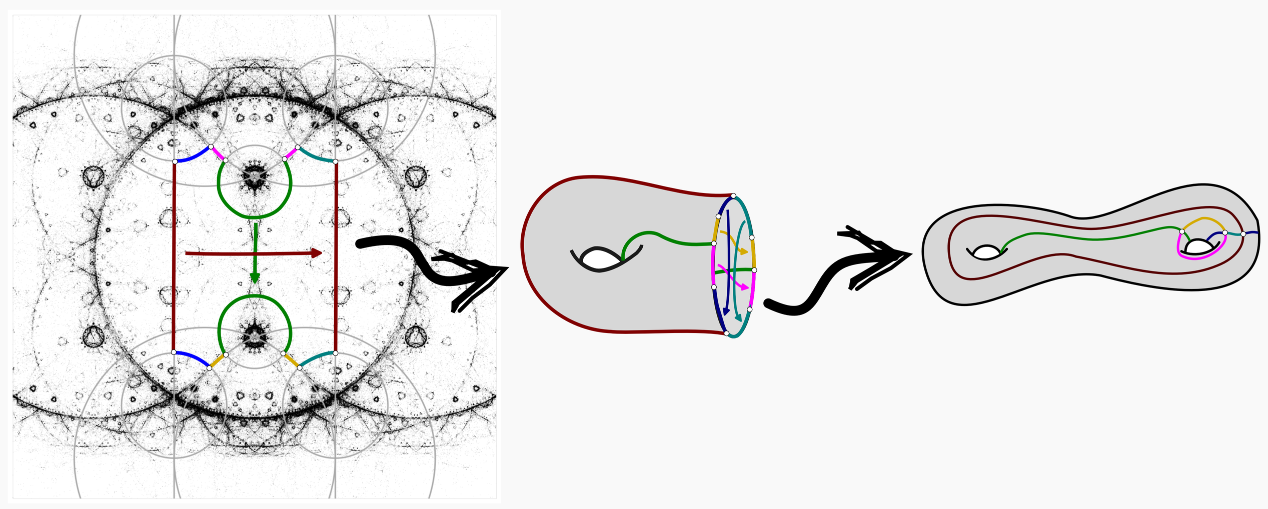


The quotient $\mathcal{X}_\triangle/G$ is a cone surface: it's a 2-sphere that's smooth everywhere except three singularities from the vertices of $\triangle$.

More generally we can take any polygons with side-pairings and use standard constructions like amalgamated products and HNN-extensions to package them all up in complexes of indiscrete groups that give more complicated cone surfaces. One dimension higher, in 3D, the picture is similar.

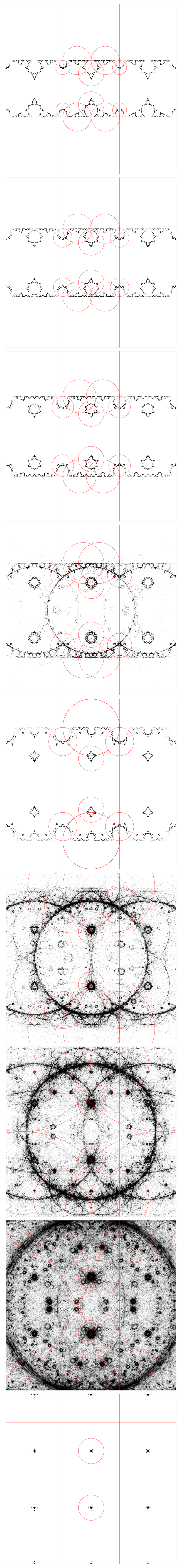
Deforming compression bodies

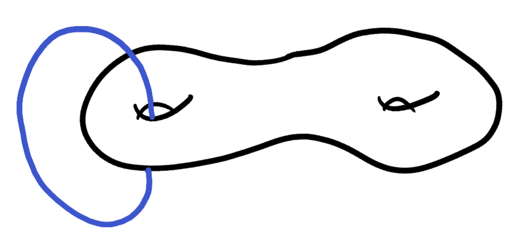
At the right running down the page, we show a family of groups which were constructed in [4]. At the top and the bottom, the groups are discrete. The intermediate groups correspond to cone manifolds where the red cusp arc in the top-most manifold becomes a singular arc of increasing cone angle until at the bottom (angle 2π) the singularity is gone.



The groups all have a fundamental domain with twelve sides that glues up to a genus 2 surface.

These groups all come from taking two triangle groups and gluing them together. Can you see the two hyperbolic triangles in the top few pictures on the right?





[1] Lieven Le Bruyn. *The Dedekind tessellation.* 2007. URL: http://www.neverendingbooks.org/the-dedekind-tessellation/ (visited on 11/21/2024).
[2] Marston Conder and Darius Young. *Soluble quotients of triangle groups.* 2024. arXiv: 2410.06571 [math.GR].
[3] H.S.M. Coxeter. "Crystal symmetry and its generalizations". In: *Proc. Trans. Roy. Soc. Canada* 51 (1957), pp. 1–13.
[4] Alex Elzenaar. *Changing topological type of compression bodies through cone manifolds.* 2024. arXiv: 2411.17940 [math.GT].
[5] Euclid. "The elements". In: *Greek mathematical works I: Thales to Euclid.* Trans. by Ivor Thomas. Loeb Classical Library 335. Harvard University Press, 1991, pp. 436–479.
[6] Robert Fricke. "Ueber den arithmetischen Charakter der zu den Verzweigungen (2, 3, 7) und (2, 4, 7) gehörenden Dreiecksfunctionen". In: *Math. Ann.* 41 (1893), pp. 443–468.
[7] Darius Young. *Quotient order density of ordinary triangle groups.* 2024. arXiv: 2408.02264 [math.GR].